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The Zero Attractor of Perturbed Chebyshev Polynomials and Sums of Taylor Polynomials

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Table of Contents

List of Tables

List of Figures

Abstract

The Zero Attractor of Perturbed Chebyshev Polynomials
and Sums of Taylor Polynomials
Joseph L. Erickson

Defining $s_n(z)$ to be the nth degree Taylor polynomial at 0 for the exponential function, we employ methods from complex analysis to study the limiting behavior of the zero distribution of polynomials in the sequence $As_{an}(\alpha nz) + Bs_{bn}(\beta nz)$ as $n \to \infty$. Invariably the zero distribution approaches one or more fixed piecewise smooth curves in the complex plane which we call the "zero attractor" of the sequence. Also we determine the zero attractor of the sequence $T_n(z) - z^{\ell n}$ for fixed integer $\ell \geq 2$ and nth degree Chebyshev polynomial of the first kind $T_n(z)$.

1. Introduction

Given a sequence of polynomial functions $(p_n(z))_{n=1}^{\infty}$, does the distribution of zeros, or roots, of $p_n(z)$ approach any particular set of points $\mathcal{A}$ in the complex plane $\mathbb{C}$ as n approaches infinity?

A simple example is the sequence $(z^n - 1)_{n=1}^{\infty}$. It is well known that each polynomial $z^n - 1$ possesses n zeros that are uniformly distributed on the unit circle $\mathbb{S}$ with center 0. In Figure 1 are shown the zeros of $p_n(z) = z^{100} - 1$, for instance. As $n \to \infty$ we observe that the zeros of $z^n - 1$ appear to "fill in" the unit circle. Indeed, given any $\epsilon > 0$ and point $\omega \in \mathbb{S}$, we can find a sufficiently large integer n_0 such that, for all $n > n_0$, there is at least one zero of $z^n - 1$ in the disc $D_\epsilon(\omega)$ with center ω and radius ϵ. Thus, if $Z(p_n(z))$ denotes the set of zeros of $p_n(z)$, then $\mathbb{S}$ is the set of limit points of $\bigcup_{n=1}^{\infty} Z(p_n(z))$; but more than that, we find the set $Z(p_n(z))$ "approaches" $\mathbb{S}$ in some sense as $n \to \infty$, and it is in this sense—formally defined in the next section—that we refer to $\mathbb{S}$ as the "zero attractor" $\mathcal{A}$ of the sequence $z^n - 1$.

As another somewhat less trivial example we consider the sequence

$$q_n(z) = z^n - \left(1 - \frac{1}{\sqrt{n}}\right)^n.$$

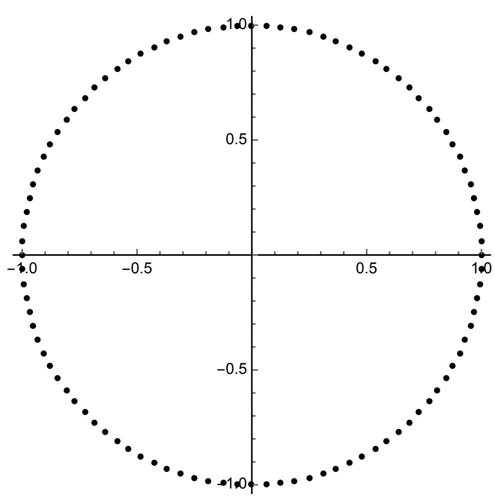

FIGURE 1. The roots of $z^{100} = 1$.

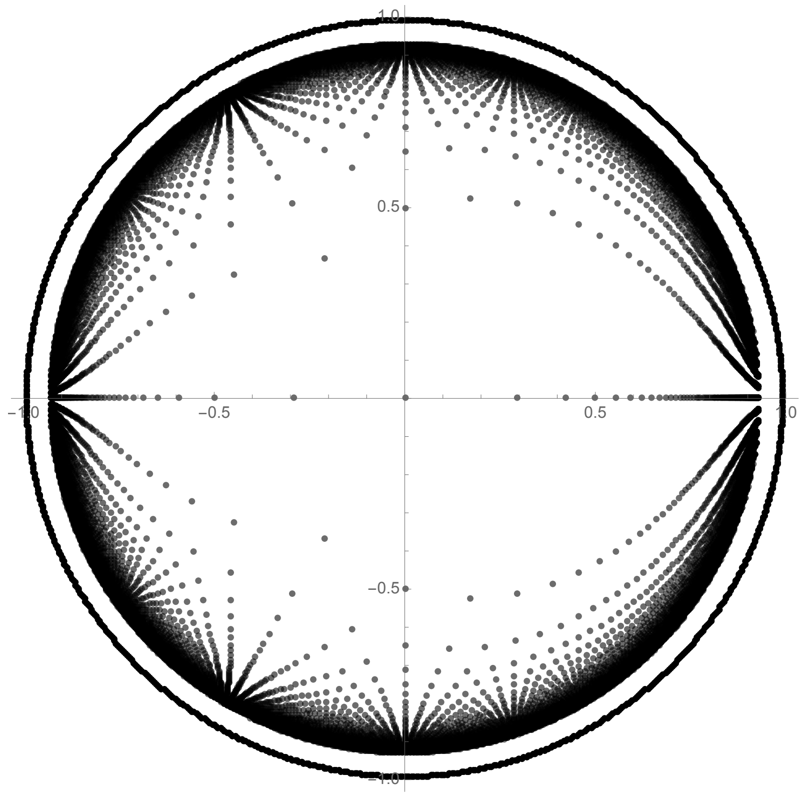

FIGURE 2. The roots of $z^n = (1 - n^{-1/2})^n$ for $1 \leq n \leq 200$ and $n = 1000$.

The polynomial $q_1(z)$ has only root 0, while for $n \geq 2$ we find that $q_n(z)$ has n zeros uniformly distributed on the circle $|z| = 1 - n^{-1/2}$. For instance $q_4(z)$ has zeros corresponding to the roots of $z^4 = (1/2)^4$, which are $\pm 1/2$, $\pm i/2$ and lie on the circle $|z| = 1/2$. Of course $1 - n^{-1/2}$ increases monotonically to 1 as n increases, so that the zeros of $q_n(z)$ draw inexorably nearer to the circle $\mathbb{S}$ with each successive value of n. Here, truly, we can see how $\mathbb{S}$ appears to be "attracting" the zeros of $q_n(z)$ as $n \to \infty$, with the result being that $\mathbb{S}$ is indeed the zero attractor of the sequence. In Figure 2 are shown the zeros of $q_n(z)$ for $1 \leq n \leq 200$, with the zeros of $q_{1000}(z)$ comprising the additional ring of points on the circle $|z| \approx 0.968$. In contrast to the previous example, here none of the elements of $\bigcup_{n=1}^{\infty} Z(q_n(z))$ lie on the zero attractor itself.

The two examples entertained thus far might lull one into thinking that the zero attractor $\mathcal{A}$ of a sequence $p_n(z)$ is simply the set of limit points of $\mathcal{Z} = \bigcup_{n=1}^{\infty} Z(p_n(z))$. While this is quite often the case, the formal definition given in §2 allows for the inclusion

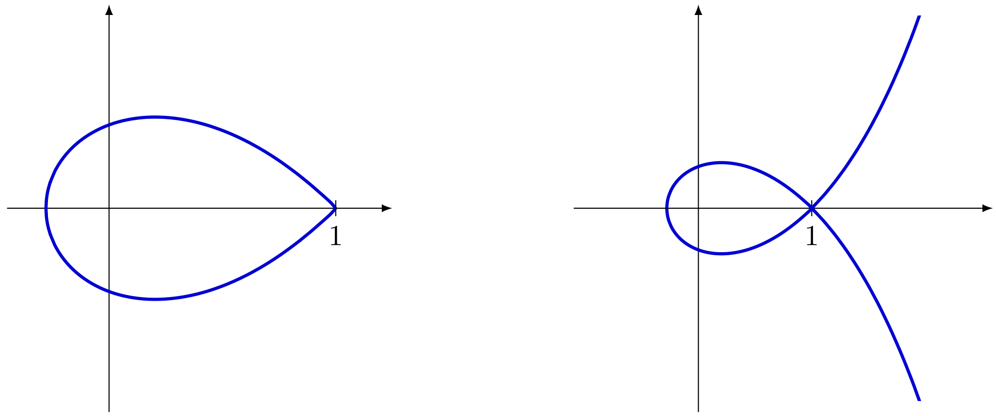

FIGURE 3. Left: the zero attractor for $s_n(nz)$. Right: the classic Szegő curve $|ze^{1-z}| = 1$.

of isolated points of $\mathcal{Z}$. For example the zero attractor of the sequence $r_n(z) = zq_n(z)$, where $q_n(z)$ is defined as above, has as its zero attractor the set $\mathbb{S} \cup \{0\}$.

The term "zero attractor" is relatively new, so that in the literature one frequently encounters other terms such as "limit curve" or "asymptotic zero distribution." However, as one might guess, a "limit curve" generally excludes isolated points whereas an "asymptotic zero distribution" does not.

Over the past century much research has been devoted to ascertaining the zero attractors (or limit curves) of various sequences of Taylor polynomials, with special attention given to partial sums of the Maclaurin series for e^z. Defining

$$s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}, \tag{1.1}$$

in [10] Gábor Szegő investigated the behavior as $n \to \infty$ of the zeros of $s_n(nz)$, the "normalized" partial sums of the series. Whereas the moduli of the zeros of $s_n(z)$ grow without bound as $n \to \infty$, Szegő found that the zero distribution of $s_n(nz)$ asymptotically approaches the set

$$\mathcal{A} = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } |ze^{1-z}| = 1\}, \tag{1.2}$$

shown at left in Figure 3, which is a portion of the so-called **Szegő curve** $|ze^{1-z}| = 1$ shown at right in the figure. That $\mathcal{A}$ is in some sense "attracting" the zeros of $s_n(nz)$ as

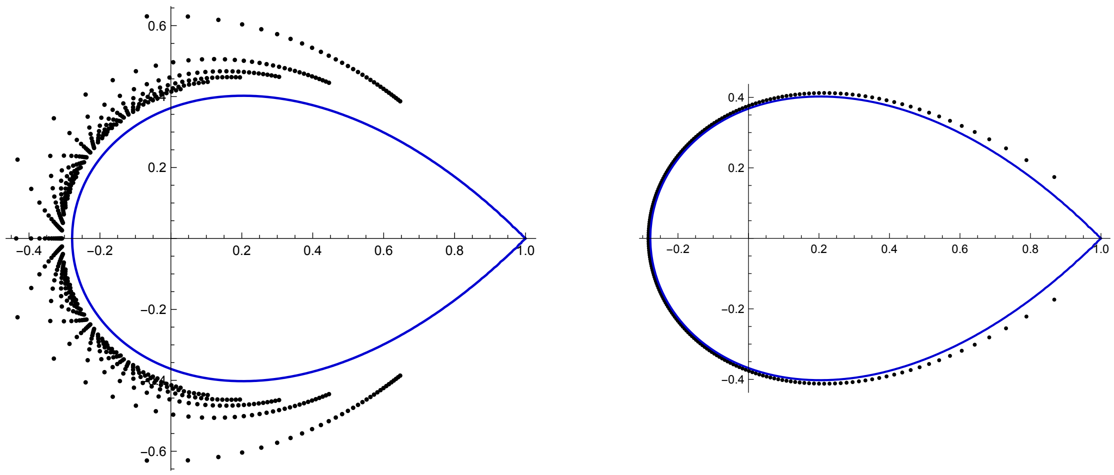

FIGURE 4. Left: the zeros of $s_n(nz)$ for $4 \leq n \leq 39$. Right: the zeros of $s_{220}(220z)$.

$n \to \infty$ can be seen in Figure 4, where $Z(s_n(nz))$ is plotted for all $4 \leq n \leq 39$ at left, and $Z(s_{220}(220z))$ is plotted at right.

In §3 we will obtain the same result as Szegő using a different technique involving the notion of a zero attractor, and then in later sections we will apply our technique to increasingly generalized two-term linear combinations $As_{an}(\alpha nz) + Bs_{bn}(\beta nz)$ of partial sums of the exponential function's Maclaurin series. Finally, in §13 the same method will be applied to ascertain at least a portion of the zero attractor for sequences of perturbed Chebyshev polynomials of the first kind of the form $T_n(z) - z^{\ell n}$ for fixed integer $\ell \geq 2$, with the remaining portion of the zero attractor determined by other means.

In the literature there is no shortage of inquiries into the zero attractors of a wide variety of sequences of functions. For fixed $c_j, \lambda_j \in \mathbb{C}$, exponential sums of the form

$$f(z) = \sum_{j=1}^{M} c_j e^{\lambda_j z}$$

have an associated Taylor series expression

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and in [2] the large n asymptotics of the zero distribution of

$$f_n(nz) = \sum_{k=0}^{n} a_k (nz)^k$$

is studied. In [11] Vargas analyzed the zero attractors of (normalized) partial sums of power series for Bessel functions of the first kind as well as a class of entire functions definable by integrals of the form

$$\int_{-a}^{b} \varphi(t) e^{zt} \, dt.$$

Finally, Boyer and Goh in [4] examine the zero attractors of Appell polynomials.

Section 2: The Zero Attractor of a Sequence

Let (X,d) be a metric space, and let $\mathcal{K}$ be the collection of all nonempty compact subsets of X. The **Hausdorff distance** on $\mathcal{K}$ is the function $d_H : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ given by

$$d_H(A,B) = \max\left\{ \sup_{a\in A}\left(\inf_{b\in B} d(a,b)\right),\ \sup_{b\in B}\left(\inf_{a\in A} d(a,b)\right)\right\}$$

for all $A, B \in \mathcal{K}$, which makes $(\mathcal{K}, d_H)$ a metric space in its own right. Define

$$D_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$$

for any $x \in X$ and $\epsilon > 0$, and declare the **ϵ-neighborhood** of any $S \subseteq X$ to be the set

$$S_\epsilon = \bigcup_{x\in S} D_\epsilon(x).$$

Then it is known that

$$d_H(A,B) = \inf\left\{\epsilon > 0 : B \subseteq A_\epsilon \text{ and } A \subseteq B_\epsilon\right\} \tag{2.1}$$

for any $A, B \in \mathcal{K}$.

Let $(p_n(z))$ be a sequence of polynomial functions such that $\deg(p_{n+1}) > \deg(p_n)$. For each n the set $Z(p_n)$ of zeros of p_n is finite, and so $(Z(p_n))$ is a sequence of subsets of the metric space $(\mathcal{K}, d_H)$, where $\mathcal{K}$ now is the collection of nonempty compact subsets of $\mathbb{C}$. If $(Z(p_n))$ is d_H-convergent to some $\mathcal{A} \in \mathcal{K}$ as $n \to \infty$, then $\mathcal{A}$ is called the **zero attractor** of $(p_n(z))$, and we write

$$\mathcal{A} = \lim_{n\to\infty} Z(p_n). \tag{2.2}$$

In explicit terms (2.2) holds if and only if

$$\forall\epsilon > 0\ \exists n_0\ \forall n \geq n_0\big(d_H(Z(p_n), \mathcal{A}) < \epsilon\big) \tag{2.3}$$

holds. In light of (2.1), we find that (2.3) holds if and only if for each $\epsilon > 0$ there exists some n_0 such that $\mathcal{A} \subseteq Z(p_n)_\epsilon$ and $Z(p_n) \subseteq \mathcal{A}_\epsilon$ whenever $n \geq n_0$.

The following derives from [9]. Let $(f_n(z))$ be a sequence of analytic functions on a region $\Omega \subseteq \mathbb{C}$, let $\mathcal{F}$ and $\mathcal{I}$ be the collection of finite and infinite subsets of $\mathbb{Z}$, respectively, and for each $z \in \Omega$ let $\mathcal{N}_z$ be the collection of open neighborhoods of z. Now define

$$\liminf Z(f_n) = \{z \in \Omega : \forall U \in \mathcal{N}_z \; \exists F \in \mathcal{F} \; \forall n \notin F \big(U \cap Z(f_n) \neq \varnothing\big)\} \tag{2.4}$$

and

$$\limsup Z(f_n) = \{z \in \Omega : \forall U \in \mathcal{N}_z \; \exists I \in \mathcal{I} \; \forall n \in I \big(U \cap Z(f_n) \neq \varnothing\big)\}. \tag{2.5}$$

It is clear that

$$\liminf Z(f_n) \subseteq \limsup Z(f_n), \tag{2.6}$$

and we purpose to show that, under certain conditions, the zero attractor of a polynomial sequence $(p_n(z))$ is equal to $\liminf Z(p_n)$. First we must establish that (2.4) and (2.5) are compact sets for a polynomial sequence whose zeros all lie in some bounded region.

Proposition 2.1. *Let $(p_n(z))$ be a sequence of polynomials such that $\bigcup_n Z(p_n) \subseteq K$ for some compact set K. Then $\liminf Z(p_n)$ and $\limsup Z(p_n)$ are compact.*

Proof. If $z \notin K$, then there exists $r > 0$ such that $D_r(z) \subseteq \mathbb{C} \setminus K$, and so $D_r(z) \cap Z(p_n) = \varnothing$ for all n. Since $D_r(z)$ is a neighborhood of z, it follows that $z \notin \liminf Z(p_n)$, which makes clear that $\liminf Z(p_n) \subseteq K$ and hence $\liminf Z(p_n)$ is a bounded set.

Let z' be a limit point for $\liminf Z(p_n)$, and let U be a neighborhood of z'. Let $\epsilon > 0$ be such that $D_\epsilon(z') \subseteq U$, define $V = D_\epsilon(z') \setminus \{z'\}$, and let $z_0 \in V \cap \liminf Z(p_n)$. Since V is a neighborhood of z_0, we have $V \cap Z(p_n) \neq \varnothing$ for all but finitely many n, and then $U \cap Z(p_n) \neq \varnothing$ for all but finitely many n since $U \supseteq V$. Thus every neighborhood of z' has nonempty intersection for all but finitely many of the sets $Z(p_n)$, implying that $z' \in \liminf Z(p_n)$, and hence $\liminf Z(p_n)$ is closed. Therefore $\liminf Z(p_n)$ is compact. That $\limsup Z(f_n)$ is compact is shown by a similar argument. ■

Proposition 2.2. *Let $(p_n(z))$ be a sequence of polynomials such that $\bigcup_n Z(p_n) \subseteq K$ for some compact K. Then a set $\mathcal{A}$ is the zero attractor of $(p_n(z))$ if and only if*

$$\mathcal{A} = \liminf Z(p_n) = \limsup Z(p_n). \tag{2.7}$$

Proof. Suppose $\mathcal{A}$ is the zero attractor of $(p_n(z))$. Let $z_0 \in \mathcal{A}$. Let U be a neighborhood of z_0, and let $\epsilon > 0$ be such that $D_\epsilon(z_0) \subseteq U$. There exists n_0 such that $\mathcal{A} \subseteq Z(p_n)_\epsilon$ for all

$n \geq n_0$, so that $z_0 \in Z(p_n)_\epsilon$ for $n \geq n_0$, and hence

$$U \cap Z(p_n) \supseteq D_\epsilon(z_0) \cap Z(p_n) \neq \varnothing$$

for $n \geq n_0$. Thus every neighborhood of z_0 has nonempty intersection with all but finitely many $Z(p_n)$, so that $z_0 \in \liminf Z(p_n)$, and hence $\mathcal{A} \subseteq \liminf Z(p_n)$.

Next, suppose that $z_0 \notin \mathcal{A}$. Since $\mathcal{A}$ is compact by Proposition 2.1, there exists $\epsilon > 0$ such that $D_{2\epsilon}(z_0) \subseteq \mathbb{C} \setminus \mathcal{A}$, and so $U = D_\epsilon(z_0)$ is a neighborhood of z_0 for which $U \cap \mathcal{A}_\epsilon = \varnothing$. Now, for some n_0 be have $Z(p_n) \subseteq \mathcal{A}_\epsilon$ for all $n \geq n_0$, so $U \cap Z(p_n) = \varnothing$ for $n \geq n_0$ and it follows that $z_0 \notin \limsup Z(p_n)$. Hence $z_0 \in \limsup Z(p_n)$ implies $z_0 \in \mathcal{A}$, and we now have $\limsup Z(p_n) \subseteq \mathcal{A} \subseteq \liminf Z(p_n)$. This, together with (2.6), yields (2.7).

For the converse, suppose (2.7) holds. Let $\epsilon > 0$. Suppose for each k there exists $n_k \geq k$ such that $\mathcal{A} \nsubseteq Z(p_{n_k})_\epsilon$, and so there is a sequence (z_{n_k}) in $\mathcal{A}$ such that $z_{n_k} \notin Z(p_{n_k})_\epsilon$ for all k. Since $\mathcal{A}$ is compact, (z_{n_k}) has a subsequence $(z_{n_{k_m}})$ that converges to some $z^* \in \mathcal{A}$. Let $U = D_{\epsilon/2}(z^*)$. There exists m_0 such that $z_{n_{k_m}} \in U$ for all $m \geq m_0$, and now $z_{n_{k_m}} \notin Z(p_{n_{k_m}})_\epsilon$ implies that $U \cap Z(p_{n_{k_m}}) = \varnothing$ for all $m \geq m_0$. Since U is a neighborhood of z^*, it follows that $z^* \notin \liminf Z(p_n) = \mathcal{A}$, which is a contradiction. Therefore there must exist j_1 such that $\mathcal{A} \subseteq Z(p_n)_\epsilon$ for all $n \geq j_1$.

Now suppose for each k there exists $n_k \geq k$ such that $Z(p_{n_k}) \nsubseteq \mathcal{A}_\epsilon$, thereby giving rise to a sequence (z_{n_k}) with $z_{n_k} \in Z(p_{n_k})$ and $z_{n_k} \notin \mathcal{A}_\epsilon$ for all k. Since $(z_{n_k}) \subseteq K$, there exists a subsequence converging to some $z^* \in K$. It follows that any neighborhood of z^* has nonempty intersection with infinitely many of the sets $Z(p_{n_k})$, implying that $z^* \in \limsup Z(p_n)$. On the other hand $z_{n_k} \notin \mathcal{A}_\epsilon$ implies that $z^* \notin \mathcal{A}$. Since $\limsup Z(p_n) = \mathcal{A}$ by hypothesis, there is a contradiction, and therefore there must exist j_2 such that $Z(p_n) \subseteq \mathcal{A}_\epsilon$ for all $n \geq j_2$. We now find that $\mathcal{A} \subseteq Z(p_n)_\epsilon$ and $Z(p_n) \subseteq \mathcal{A}_\epsilon$ for $n \geq \max\{j_1, j_2\}$, and therefore $\mathcal{A}$ is the zero attractor of $(p_n(z))$. ■

The following theorem is essentially [9, Theorem 3.2]. It gives rise to a method, made explicit in Theorem 2.5 and put into practice in the sequel, for determining the zero attractor of certain sequences of polynomials.

Theorem 2.3 (Sokal). *Let Ω be a region in $\mathbb{C}$, and let $z_0 \in \Omega$. Let $(f_n(z))$ be a sequence of analytic functions on Ω such that $(|f_n(z)|^{1/n})$ is uniformly bounded on compact subsets of Ω. Suppose there does not exist a neighborhood U of z_0 and a function v on U that is either harmonic or identically $-\infty$ such that*

$$\liminf \ln |f_n(z)|^{1/n} \leq v(z) \leq \limsup \ln |f_n(z)|^{1/n}$$

for all $z \in U$. Then $z_0 \in \liminf Z(f_n)$.

The following proposition will nicely facilitate the proof of Theorem 2.5. We hold to the convention that $\overline{S}$ denotes the closure of a set S, and S° the interior.

Proposition 2.4. *Let $(f_n(z))$ be a sequence of functions that is compactly convergent to f on a region Ω. Suppose f is continuous and nonvanishing on Ω. If $Z(f_n)$ is finite for all n, then $\bigcup_{n=1}^{\infty} Z(f_n)$ has no limit points in Ω.*

Proof. Fix $z_0 \in \Omega$. Let $r > 0$ be such that $K := \overline{D}_r(z_0) \subseteq \Omega$. Since $|f| : \Omega \to \mathbb{R}$ is continuous on K, the extreme value theorem implies there exists $\hat{z} \in K$ such that

$$\alpha := |f(\hat{z})| = \inf_{z \in K} |f(z)|,$$

where $\alpha > 0$ since f is nonvanishing on K. Now, there exists N such that $|f_n(z) - f(z)| < \alpha/2$ for all $n > N$ and $z \in K$, implying $|f(z)| - |f_n(z)| < \alpha/2$, and hence

$$|f_n(z)| > |f(z)| - \frac{\alpha}{2} \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} > 0$$

for all $n > N$ and $z \in K$, and so $Z(f_n) \cap K = \varnothing$ for $n > N$. Hence

$$\mathcal{Z} \cap K = \bigcup_{n=1}^{\infty} \big(Z(f_n) \cap K\big) = \bigcup_{n=1}^{N} \big(Z(f_n) \cap K\big),$$

a finite set, and so there are only finite many elements of $\mathcal{Z}$ within a distance $r > 0$ of z_0. This implies that z_0 is not a limit point of $\mathcal{Z}$, and since $z_0 \in \Omega$ is arbitrary, we conclude that $\mathcal{Z}$ has no limit points in Ω. ∎

The next theorem is the culmination of the present section. Once proven, the theorem will be put to work throughout the remainder of this thesis in order to find the zero attractors of various polynomial sequences.

Theorem 2.5. *Let $(p_n(z))$ be a polynomial sequence such that $\bigcup_n Z(p_n)$ is a bounded set, and suppose $(|p_n|^{1/n})$ is uniformly bounded on compact sets. Suppose there exist mutually disjoint domains $\Omega_1, \ldots, \Omega_m$ (only one of which is unbounded), harmonic functions $v_j : \Omega_j \to \mathbb{R}$, and closed sets $F_j \subseteq \Omega_j$ such that the following hypotheses hold:*

1. $\mathbb{C} = \bigcup_{j=1}^m \overline{\Omega}_j$.
2. *For each $1 \leq j \leq m$,*
$$\lim_{n\to\infty} \ln |p_n(z)|^{1/n} = v_j(z)$$
uniformly on compact subsets of $\Omega_j' := \Omega_j \setminus F_j$.
3. *For every $z \in A := \bigcup_{j=1}^m \partial\Omega_j$ and neighborhood N of z, there exists no analytic $f : N \to \mathbb{C}$ such that $\mathrm{Re}(f) = v_j$ on each nonempty $N \cap \Omega_j$.*

Then for $F := \bigcup_{j=1}^m F_j$,
$$A \subseteq \liminf Z(p_n) \subseteq A \cup F.$$
In particular, if the zero attractor $\mathcal{A}$ of $(p_n(z))$ exists and $F = \varnothing$, then $\mathcal{A} = A$; and if $F \subseteq \mathcal{A}$, then $\mathcal{A} = A \cup F$.

Proof. Suppose $z_0 \in \partial\Omega_k$ for some $1 \leq k \leq m$. Let U be a neighborhood of z_0. Then $v \equiv -\infty$ cannot satisfy

$$\liminf \ln |p_n(z)|^{1/n} \leq v(z) \leq \limsup \ln |p_n(z)|^{1/n} \tag{2.8}$$

for all $z \in U$ since $\liminf \ln |p_n(z)|^{1/n}$ is real-valued on $U \setminus A$ by hypothesis (2) in the theorem. Suppose there is a harmonic function v on U such that (2.8) holds for all $z \in U$. Then for each $1 \leq j \leq m$ for which $U \cap \Omega_j' \neq \varnothing$ (and there must be at least two such values) we have

$$v(z) = \lim_{n\to\infty} \ln |p_n(z)|^{1/n} = v_j(z)$$

for all $z \in U \cap \Omega_j'$. Since U is open and each F_j is closed, there exists $r > 0$ such that $N := D_r(z_0) \subseteq U$ and $N \cap F_j = \varnothing$ for all j, and thus $N \cap \Omega_j' = N \cap \Omega_j$ for all j. Now, v is harmonic on the simply-connected set N, so there exists harmonic $w : N \to \mathbb{R}$ to make $f := v + iw$ analytic on N. However, we then have $\mathrm{Re}(f) = v = v_j$ on each nonempty $N \cap \Omega_j$, contradicting hypothesis (3). Thus $z_0 \in \liminf Z(p_n)$ by Theorem 2.3, implying that $A \subseteq \liminf Z(p_n)$.

Fix $1 \leq k \leq m$. By hypothesis (2), $|p_n|^{1/n} \to e^{v_k}$ uniformly on compact subsets of Ω'_k, and since e^{v_k} is continuous and nonvanishing on Ω'_k, Proposition 2.4 implies that $\bigcup_n Z(|p_n|^{1/n})$ has no limits points in Ω'_k, and hence neither does the set

$$\mathcal{Z} := \bigcup_n Z(p_n).$$

So if $z_0 \in \Omega'_k$, then there exists $\epsilon > 0$ such that $[D_\epsilon(z_0) \setminus \{z_0\}] \cap \mathcal{Z} = \varnothing$. Moreover, $z_0 \in Z(p_n)$ can hold only for at-most finitely many n, since otherwise $\lim_{n\to\infty} \ln |p_n(z_0)|^{1/n}$ cannot be real-valued as required by hypothesis (2). Therefore $D_\epsilon(z_0) \cap Z(p_n) = \varnothing$ for all but finitely many n, whence $z_0 \notin \liminf Z(p_n)$ follows. That is, $z_0 \notin A \cup F$ implies $z_0 \notin \liminf Z(p_n)$, and so $\liminf Z(p_n) \subseteq A \cup F$.

Finally, if $\mathcal{A}$ exists and $F = \varnothing$, then $\mathcal{A} = \liminf Z(p_n) = A$ by Proposition 2.2; and if $\mathcal{A}$ exists with $F \subseteq \mathcal{A}$, then $A \subseteq \mathcal{A} \subseteq A \cup F$ implies that $\mathcal{A} = A \cup F$. ■

Section 3: The Classic One-Term Case

Here we use Theorem 2.5 to solve the problem considered by Szegő almost a century ago. Recalling (1.1), in the present setting this is the problem of finding the zero attractor of the sequence $(s_n(nz))$. In fact we will generalize slightly and treat the sequence $(s_{an}(nz))$ for any fixed positive integer a. Letting

$$\varphi(z) = ze^{1-z},$$

define the region

$$L_w = \left\{ z : \left| \varphi\left(\frac{z}{w}\right) \right| < 1 \text{ and } |z| < |w| \right\} \tag{3.1}$$

for any $w \neq 0$, shown in Figure 5. To prove is the following.

Theorem 3.1. *The zero attractor of the sequence $s_{an}(nz)$ is ∂L_a.*

A series of lemmas will be proven which, taken together with Theorem 2.5, will furnish the proof of the theorem. The bulk of the work will be to determine the limit of $|s_{an}(nz)|^{1/n}$ as $n \to \infty$ in the two regions $\mathbb{C} \setminus \overline{L}_a$ and L_a.

First some convenient asymptotics for $s_{rn}(nz)$ are needed, where r is any positive integer. Defining

$$\sigma_{rn} = \sqrt{2\pi(rn+1)},$$

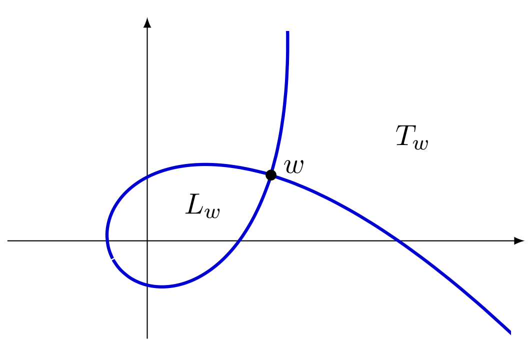

Figure 5. The regions L_w and T_w.

asymptotics presented in [4] are readily adapted to give, for some fixed $\nu < 0$, that

$$s_{rn}(nz) = \frac{e^{nz}\varphi^{rn+1}\left(\dfrac{nz}{rn+1}\right)}{\left(\dfrac{nz}{rn+1} - 1\right)\sigma_{rn}}[1 + O(n^{\nu})] \tag{3.2}$$

whenever $|z| > r + 1/n$, and

$$s_{rn}(nz) = e^{nz}\left[1 + \frac{\varphi^{rn+1}\left(\dfrac{nz}{rn+1}\right)}{\left(\dfrac{nz}{rn+1} - 1\right)\sigma_{rn}}[1 + O(n^{\nu})]\right] \tag{3.3}$$

whenever $\mathrm{Re}(z) < r + 1/n$. Each order term $O(n^{\nu})$ holds uniformly on compact sets.[1]

A pause to ponder notation seems timely here. As the overwhelming majority of the discs and annuli in $\mathbb{C}$ that we will encounter from now on will be centered at the origin, we establish the more economical notation

$$\mathbb{D}_s = \{z : |z| < s\}$$

and

$$\mathbb{A}_{s,t} = \{z : s < |z| < t\}.$$

We now commence with the construction of our aforementioned lemmas.

Lemma 3.2. *In the region* $\mathbb{C} \setminus \overline{L}_a$ *we have*

$$\lim_{n\to\infty} \frac{\ln|s_{an}(nz)|}{n} = a\ln\left|\frac{ez}{a}\right|$$

uniformly on compact sets.

Proof. Let K be a compact set in $\mathbb{C} \setminus \overline{L}_a$ such that $K \subseteq \mathbb{A}_{a,\infty}$, so that $z \in K$ implies $|z| > a$. Then (3.2) with $r = a$ gives

$$\lim_{n\to\infty} |s_{an}(nz)|^{1/n} = |e^z| \lim_{n\to\infty}\left|\frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1} - 1\right)\sigma_{an}}[1 + O(n^{\nu})]\right|^{1/n} \tag{3.4}$$

---

[1]These asymptotics are repeated in §10, where explicit steps are also given that show how one of them is derived from the appropriate asymptotic in [4].

The $O(n^\nu)$ term represents a sequence of functions $f_n(z)$ for which there exists a constant $C > 0$ and integer n_0 such that $|f_n(z)| < Cn^\nu$ holds for all $n \geq n_0$ and $z \in K$. We now write the limit in (3.4) as

$$|e^z| \lim_{n\to\infty} \left(\frac{\left|\varphi^a\left(\dfrac{nz}{an+1}\right)\right|}{\left|\dfrac{nz}{an+1} - 1\right|^{1/n} |\sigma_{an}|^{1/n}} \cdot \left|\varphi\left(\frac{nz}{an+1}\right)\right|^{1/n} \cdot |1 + f_n(z)|^{1/n} \right), \tag{3.5}$$

and consider the limit of each factor in the parentheses individually.

Let $\epsilon \in (0, 1]$. Since $\nu < 0$, there exists $n_1 > n_0$ such that $Cn_1^\nu < \epsilon$. Then $|f_n(z)| < Cn^\nu < \epsilon$ for any $n > n_1$ and $z \in K$, so that

$$1 + |f_n(z)| < 1 + \epsilon < (1 + \epsilon)^n,$$

and hence

$$|1 + f_n(z)|^{1/n} \leq \big(1 + |f_n(z)|\big)^{1/n} < 1 + \epsilon.$$

On the other hand,

$$\begin{aligned} |f_n(z)| < \epsilon \quad &\Rightarrow \quad \big|1 - |f_n(z)|\big| = 1 - |f_n(z)| > 1 - \epsilon \geq (1 - \epsilon)^n \\ &\Rightarrow \quad |1 + f_n(z)|^{1/n} \geq \big|1 - |f_n(z)|\big|^{1/n} > 1 - \epsilon \end{aligned}$$

Combining our results yields

$$\Big| |1 + f_n(z)|^{1/n} - 1 \Big| < \epsilon,$$

and hence

$$\lim_{n\to\infty} |1 + f_n(z)|^{1/n} = 1$$

uniformly on K.

Next, define

$$\psi_n(z) = \varphi^{1/n}\left(\frac{nz}{an+1}\right)$$

for each n, so that (ψ_n) is a sequence of nonvanishing analytic functions on $\mathbb{C} \setminus \overline{L}_a$. Let $E \subseteq \mathbb{C} \setminus \overline{L}_a$ be compact, and fix k. Define the sets

$$S = \left\{ \frac{z}{a + 1/n} : z \in E \text{ and } n \in \mathbb{N} \right\}$$

and

$$S_k = \left\{ \frac{z}{a + 1/k} : z \in E \right\}.$$

Clearly $S_k \subseteq S$ and $\overline{S}$ is compact. By the extreme value theorem there exists $M \in (0, \infty)$ such that

$$\max_{w \in \overline{S}} |\varphi(w)| = M,$$

and so

$$\|\psi_n\|_E = \sup_{z \in E} \left| \varphi^{1/n} \left(\frac{z}{a + 1/n} \right) \right| = \sup_{w \in S_n} |\varphi(w)|^{1/n} \leq \sup_{w \in \overline{S}} |\varphi(w)|^{1/n} = M^{1/n} < M + 1.$$

This implies that $\sup\{\|\psi_n\|_E : n \in \mathbb{N}\} \in \mathbb{R}$, and therefore (ψ_n) is a bounded subset of $\mathcal{A}(\mathbb{C} \setminus \overline{L}_a)$, the family of analytic functions on $\mathbb{C} \setminus \overline{L}_a$. By [1, Theorem 5.1.8], then, (ψ_n) is equicontinuous on $\mathbb{C} \setminus \overline{L}_a$, and since $\psi_n(z) \to 1$ pointwise on $\mathbb{C} \setminus \overline{L}_a$, [1, Theorem 5.1.9] implies that (ψ_n) converges uniformly to 1 on E. Now, because $K \subseteq \mathbb{C} \setminus \overline{L}_a$ is compact, we conclude that

$$\lim_{n \to \infty} \left| \varphi \left(\frac{nz}{an + 1} \right) \right|^{1/n} = 1$$

uniformly on K.

A similar argument will show that

$$\lim_{n \to \infty} \left| \varphi^a \left(\frac{nz}{an + 1} \right) \right| = \left| \varphi^a \left(\frac{z}{a} \right) \right| = \frac{e^a |z|^a}{a^a} |e^{-z}| \tag{3.6}$$

uniformly on K. Defining

$$\hat{\psi}_n(z) = \varphi^a \left(\frac{nz}{an + 1} \right),$$

we have

$$\|\hat{\psi}_n\|_E = \sup_{z \in E} \left| \varphi^a \left(\frac{z}{a + 1/n} \right) \right| = \sup_{w \in S_n} |\varphi(w)|^a \leq \sup_{w \in \overline{S}} |\varphi(w)|^a = M^a,$$

and thus $(\hat{\psi}_n)$ is a bounded subset of $\mathcal{A}(\mathbb{C} \setminus \overline{L}_a)$. Equicontinuity follows, and because (3.6) holds pointwise on $\mathbb{C} \setminus \overline{L}_a$, we conclude that it holds uniformly on compact subsets of $\mathbb{C} \setminus \overline{L}_a$.

Finally, as it is clear that

$$\lim_{n\to\infty} \frac{1}{|\sigma_{an}|^{1/n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\left|\dfrac{nz}{an+1} - 1\right|^{1/n}} = 1$$

both hold uniformly on K, from (3.5) we find that

$$\lim_{n\to\infty} |s_{an}(nz)|^{1/n} = |e^z| \left|\varphi^a\left(\frac{z}{a}\right)\right| = \frac{e^a|z|^a}{a^a} \tag{3.7}$$

uniformly on K.

If K is a compact set in $\mathbb{C} \setminus \overline{L}_a$ such that $K \subseteq \{z : \operatorname{Re} z < a\}$, then (3.3) with $r = a$ gives

$$\lim_{n\to\infty} |s_{an}(nz)|^{1/n} = |e^z| \lim_{n\to\infty} \left| 1 + \frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1} - 1\right)\sigma_{an}} [1 + O(n^\nu)] \right|^{1/n}, \tag{3.8}$$

and it can be shown that (3.7) again holds uniformly on K. Combining the results of our analyses on $|z| > a$ and $\operatorname{Re} z < a$, we conclude that (3.7) holds uniformly on any compact $K \subseteq \mathbb{C} \setminus \overline{L}_a$. Taking the logarithm then proves the lemma. ■

In the proof of Lemma 3.2 the argument that $|1 + f_n(z)|^{1/n} \to 1$ uniformly on compacta could have been accomplished quicker with application the following proposition, which will become a staple in future proofs.

Proposition 3.3. *Let $(f_n(z))$ be a sequence of functions on a compact set $K \subseteq \mathbb{C}$, and suppose there exists $\ell : K \to \mathbb{C}$ such that*

$$\lim_{n\to\infty} |f_n(z)|^{1/n} = \ell(z)$$

uniformly on K. If $\|\ell\|_K \in [0, 1)$, then

$$\lim_{n\to\infty} |1 + f_n(z)|^{1/n} = 1$$

uniformly on K.

Proof. Suppose $\|\ell\|_K \in [0,1)$, so there is some $\delta \in (0,1)$ such that $|\ell(z)| \leq 1-2\delta$ for all $z \in K$. Now, there exists n_0 such that

$$\Big||f_n(z)|^{1/n} - \ell(z)\Big| < \delta$$

for $n > n_0$ and $z \in K$, whence

$$|f_n(z)|^{1/n} < \delta + |\ell(z)| \leq 1-\delta,$$

and thus

$$1 + |f_n(z)| < 1 + (1-\delta)^n \tag{3.9}$$

for $n > n_0$ and $z \in K$. Let $\epsilon \in (0,1)$. Since $1+(1-\delta)^n \to 1$ and $(1+\epsilon)^n \to \infty$ as $n \to \infty$, there exists $n_1 > n_0$ such that

$$1 + (1-\delta)^n < (1+\epsilon)^n$$

for all $n > n_1$, and then

$$|1 + f_n(z)| \leq 1 + |f_n(z)| < 1 + (1-\delta)^n < (1+\epsilon)^n \tag{3.10}$$

for $n > n_1$ and $z \in K$. Also, from (3.9) we have $|f_n(z)| < (1-\delta)^n < 1$ for $n > n_0$ and $z \in K$, so there exists $n_2 > n_1$ such that $|f_n(z)| < \epsilon$ for $n > n_2$ and $z \in K$, and then

$$|1 + f_n(z)| \geq \big|1 - |f_n(z)|\big| \geq 1 - |f_n(z)| > 1 - \epsilon \geq (1-\epsilon)^n \tag{3.11}$$

for $n > n_2$ and $z \in K$. Combining (3.10) and (3.11), we finally obtain

$$\Big||1 + f_n(z)|^{1/n} - 1\Big| < \epsilon$$

for all $n > n_2$ and $z \in K$. Therefore $|1+f_n(z)|^{1/n} \to 1$ uniformly on K. ∎

Lemma 3.4. *In the region L_a we have*

$$\lim_{n\to\infty} \frac{\ln|s_{an}(nz)|}{n} = \operatorname{Re}(z)$$

uniformly on compact sets.

Proof. Let $K \subseteq L_a$ be compact. Equation (3.3) again gives (3.8). From the analysis starting with (3.4) and ending with (3.7), we know that

$$\lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1}-1\right)\sigma_{an}}[1+O(n^{\nu})] \right|^{1/n} = \left|\varphi^a\left(\frac{z}{a}\right)\right| = \frac{e^a|z|^a}{a^a}|e^{-z}|$$

uniformly on K. (The functions ψ_n and $\hat{\psi}_n$ are not nonvanishing on L_a, but we can assume $0 \notin K$ since the $z = 0$ case is easily treated separately.) Now, $|\varphi^a(z/a)| < 1$ holds for all $z \in L_a$, and since $z \mapsto \varphi^a(z/a)$ is continuous on K, the extreme value theorem implies that

$$\left\|\varphi^a\left(\frac{z}{a}\right)\right\|_K \in [0, 1).$$

By Proposition 3.3 it follows that the limit at right in (3.8) equals 1 uniformly on K, and therefore

$$\lim_{n\to\infty} |s_{an}(nz)|^{1/n} = |e^z|$$

uniformly on K. Since $|e^z| = e^{\operatorname{Re} z}$, taking logarithms finishes the proof. ∎

The sequence $s_{an}(nz)$ is now seen to satisfy hypothesis (2) in Theorem 2.5, and it is a relatively straightforward matter to verify the theorem's other hypotheses. Uniform boundedness on compacta is addressed next.

Lemma 3.5. *The sequence $|s_{an}(nz)|^{1/n}$ is uniformly bounded on compact sets.*

Proof. Suppose $K \subseteq \mathbb{C}$ is compact, and fix $n \geq 1$. Let $s \in (0, \infty)$ be such that $K \subseteq \mathbb{D}_s$. Now, for any $z \in K$,

$$\begin{aligned} |s_{an}(nz)|^{1/n} &= \left|\sum_{k=0}^{an} \frac{(nz)^k}{k!}\right|^{1/n} \leq \left(\sum_{k=0}^{an} \frac{|nz|^k}{k!}\right)^{1/n} \\ &\leq \left(\sum_{k=0}^{\infty} \frac{|nz|^k}{k!}\right)^{1/n} = (e^{n|z|})^{1/n} = e^{|z|} \leq e^s, \end{aligned}$$

and so e^s is an upper bound for $\{|s_{an}(nz)|^{1/n} : z \in K\}$. Hence

$$\||s_{an}(nz)|^{1/n}\|_K = \sup\left\{|s_{an}(nz)|^{1/n} : z \in K\right\} \leq e^s$$

for all $n \geq 1$, so that

$$\sup\left\{\||s_{an}(nz)|^{1/n}\|_K : n \geq 1\right\} \leq e^s,$$

and therefore $|s_{an}(nz)|^{1/n}$ is uniformly bounded on K. ■

Finally, from [7, p.106] we have the following result which will be used to verify the boundedness of $\bigcup_n Z(s_n(nz))$ that is required by Theorem 2.5.

Proposition 3.6. *For $c_1, \ldots, c_n \in \mathbb{C}$, let*

$$P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n.$$

If $P(z_0) = 0$, then

$$|z_0| \leq 2 \max_{1 \leq k \leq n} |c_k|^{1/k}.$$

Lemma 3.7. $\bigcup_n Z(s_n(nz))$ *is a bounded set.*

Proof. Fix $n \geq 0$. Recalling

$$s_{an}(nz) = \sum_{k=0}^{an} \frac{(nz)^k}{k!} = \frac{n^{an}}{(an)!} z^{an} + \cdots + \frac{n^2}{2!} z^2 + nz + 1,$$

factoring out the leading coefficient makes clear that $s_{an}(nz) = 0$ if and only if

$$z^{an} + \frac{an}{n} z^{an-1} + \frac{(an)(an-1)}{n^2} z^{an-2} + \frac{(an)(an-1)(an-2)}{n^3} z^{an-3} + \cdots$$
$$\cdots + \frac{(an)(an-1)(an-2)\cdots 2}{n^{an-1}} z + \frac{(an)!}{n^{an}} = 0,$$

which in turn implies, by Proposition 3.6, that

$$|z| \leq 2 \max\left\{a, \left[\frac{(an)(an-1)}{n^2}\right]^{\frac{1}{2}}, \ldots, \left[\frac{(an)(an-1)(an-2)\cdots 2}{n^{an-1}}\right]^{\frac{1}{an-1}}, \left[\frac{(an)!}{n^{an}}\right]^{\frac{1}{an}}\right\}.$$

However, for any integer $1 \leq k \leq an$,

$$\left[\frac{(an)(an-1)\cdots[an-(k-1)]}{n^k}\right]^{\frac{1}{k}} \leq \left[\frac{(an)^k}{n^k}\right]^{\frac{1}{k}} = a,$$

and thus we find that $|z| \leq 2a$. Therefore $Z(s_n(nz)) \subseteq \overline{\mathbb{D}}_{2a}$ for all $n \geq 0$, and $\bigcup_n Z(s_n(nz))$ is a bounded set. ■

Clearly $\overline{L}_a \cup (\mathbb{C} \setminus L_a) = \mathbb{C}$. Also, for any $z \in \partial L_a$, there is no neighborhood N of z for which $f : N \to \mathbb{C}$ is analytic, and yet $\operatorname{Re} f(w) = \operatorname{Re}(w)$ for $w \in L_a$ while $\operatorname{Re} f(w) = \operatorname{Re}[a \ln |(ew)/a|]$ for $w \in \mathbb{C} \setminus L_a$. Hence hypotheses (1) and (3) of Theorem 2.5 are satisfied, and therefore the zero attractor of the sequence $s_{an}(nz)$ is ∂L_a. Theorem 3.1 is proven.

Section 4: $\boldsymbol{As_{an}(nz) + Bs_{bn}(nz)}$ with $\boldsymbol{A, B \neq 0}$ and $\boldsymbol{B \neq -A}$

We conjecture that the zero attractor of $As_{an}(nz) + Bs_{bn}(nz)$ is the same for any nonzero $A, B \in \mathbb{C}$ such that $B \neq -A$. To show this, we note $As_{an}(nz) + Bs_{bn}(nz)$ and $s_{an}(nz) + (B/A)s_{bn}(nz)$ have the same zero attractor, and so it is sufficient to prove that

$$p_n(z) := s_{an}(nz) + Bs_{bn}(nz)$$

has the same zero attractor for any $B \neq -1$. The case when $B = 1$ and $a = 1$ was analyzed in [5], and so we carry out a similar (albeit streamlined) analysis here.

The zeros of $p_{800}(800z)$ in the case when $a = 1$, $b = 2$, and $B = 1$ are shown in Figure 6, along with the curves $|\varphi(z/a)| = 1$ and $|\varphi(z/b)| = 1$, and also the circle C_{ab} at the origin containing the points of intersection of these curves. Let D_{ab} denote the interior of the circle, and define the region

$$T_w = \left\{ z : \left| \varphi\left(\frac{z}{w}\right) \right| < 1 \text{ and } |z| > |w| \right\},$$

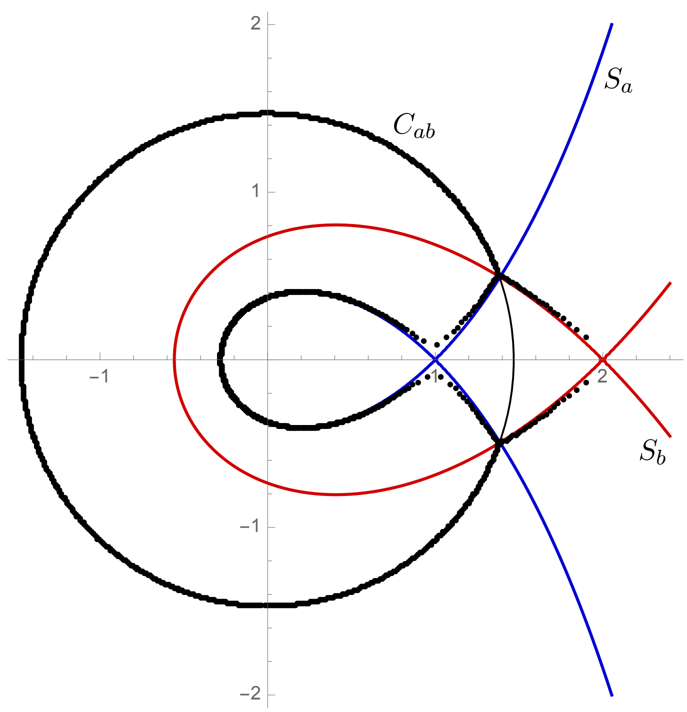

FIGURE 6. The zeros of $s_n(nz) + s_{2n}(nz)$ for $n = 800$, with S_a, S_b, C_{ab} for $(a, b) = (1, 2)$.

shown in Figure 5. Recalling (3.1), the figure strongly hints that the zero attractor of $p_n(z)$ is the union of the boundaries of the following four regions:

$$\Omega_1 = \mathbb{C} \setminus (\overline{D}_{ab} \cup \overline{L}_b),$$

$$\Omega_2 = D_{ab} \setminus (\overline{L}_a \cup \overline{T}_a),$$

$$\Omega_3 = L_b \cap T_a,$$

$$\Omega_4 = L_a.$$

Theorem 4.1. *The zero attractor of the sequence $p_n(z)$ is $\bigcup_{k=1}^4 \partial\Omega_k$.*

For convenience, Figure 7 illustrates the conjectured zero attractor together with the regions Ω_k. As with Theorem 3.1, the proof of Theorem 4.1 will be facilitated by a series of lemmas which affirm that various hypotheses of Theorem 2.5 are satisfied. But first we establish a lemma which will help resolve certain limits in this section and the next.

Lemma 4.2. *Let $\mu, \lambda \in \mathbb{C}$ with $|\mu| = |\lambda| = 1$. For $1 \le a < b$ set*

$$M = \frac{|e^{\lambda z}| \left|\varphi\left(\frac{\lambda z}{a}\right)\right|^a}{|e^{\mu z}| \left|\varphi\left(\frac{\mu z}{b}\right)\right|^b}.$$

If $z \in D_{ab} \setminus \{0\}$, then $M > 1$; and if $z \in \mathbb{C} \setminus \overline{D}_{ab}$, then $M < 1$.

Proof. Since

$$z \in D_{ab} \setminus \{0\} \;\Leftrightarrow\; |z| < \frac{a}{e}\left(\frac{b}{a}\right)^{\frac{b}{b-a}} \;\Leftrightarrow\; |z|^{b-a} < \frac{e^a b^b}{e^b a^a} \;\Leftrightarrow\; |z|^{a-b} > \frac{e^b a^a}{e^a b^b},$$

we have

$$M = \frac{|e^{\lambda z}| \left|\varphi\left(\frac{\lambda z}{a}\right)\right|^a}{|e^{\mu z}| \left|\varphi\left(\frac{\mu z}{b}\right)\right|^b} = \frac{|e^{\lambda z}| \left|\frac{\lambda z}{a} e^{1-\lambda z/a}\right|^a}{|e^{\mu z}| \left|\frac{\mu z}{b} e^{1-\lambda z/s}\right|^b} = \frac{e^a b^b}{e^b a^a} |z|^{a-b} > 1$$

for any nonzero $z \in D_{ab}$. Similarly we obtain $M < 1$ if $z \in \mathbb{C} \setminus \overline{D}_{ab}$. ■

To confirm hypothesis (2) in Theorem 2.5 we now evaluate the limit of $|p_n(z)|^{1/n}$ as $n \to \infty$ in each of the regions Ω_k.

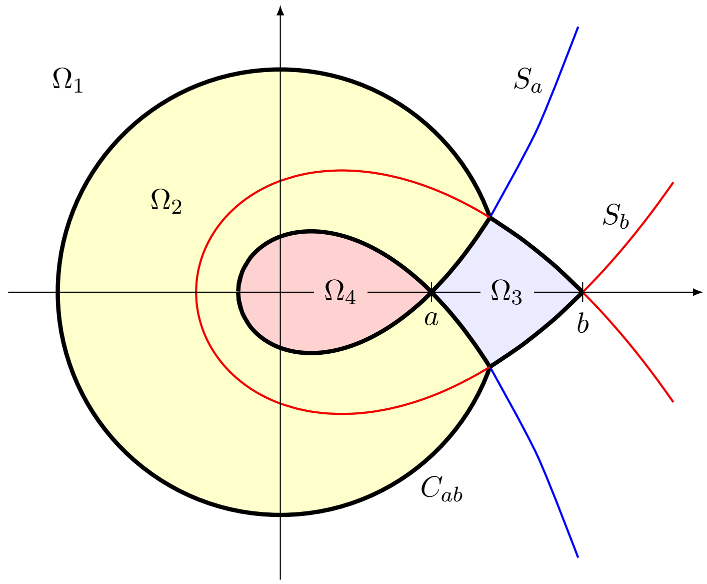

FIGURE 7. The regions Ω_k, with the zero attractor of $s_{an}(nz) + Bs_{bn}(nz)$ in bold.

Lemma 4.3. *In the region* Ω_1 *we have*

$$\lim_{n\to\infty} \frac{\ln|p_n(z)|}{n} = b \ln\left|\frac{ez}{b}\right|$$

uniformly on compact sets.

Proof. The analysis of Ω_1 can be broken into two cases: $|z| \le b$ and $|z| > b$. Assume $|z| \le b$, so that in particular $\operatorname{Re} z < b$. By (3.2) with $r = a$ and (3.3) with $r = b$,

$$\frac{|p_n(z)|^{1/n}}{|e^z|} = \left| \left[\frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1} - 1\right)\sigma_{an}} + \frac{B\varphi^{bn+1}\left(\dfrac{nz}{bn+1}\right)}{\left(\dfrac{nz}{bn+1} - 1\right)\sigma_{bn}} \right] [1 + O(n^\nu)] + B \right|^{1/n}. \tag{4.1}$$

Since $\nu < 0$ and $nz/(rn+1) \to z/r$ as $n \to \infty$ for $r \ne 0$, it is clear that

$$\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} = \lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\dfrac{z}{a}\right)}{\left(\dfrac{z}{a} - 1\right)\sigma_{an}} + \frac{B\varphi^{bn+1}\left(\dfrac{z}{b}\right)}{\left(\dfrac{z}{b} - 1\right)\sigma_{bn}} + B \right|^{1/n}.$$

Now, since $|\varphi(z/b)| > 1$, the second term above will dominate the constant term B. We thus may drop the constant term and further simplify the expression in the limit:

$$\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} = \lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)\sqrt{an+1}} + \frac{B\varphi^{bn+1}\left(\frac{z}{b}\right)}{\left(\frac{z}{b}-1\right)\sqrt{bn+1}} \right|^{1/n}.$$

We simplify still further by dropping the 1 in each radicand, drawing out $1/\sqrt{n}$ to obtain

$$\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} = \lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)\sqrt{a}} + \frac{B\varphi^{bn+1}\left(\frac{z}{b}\right)}{\left(\frac{z}{b}-1\right)\sqrt{b}} \right|^{1/n},$$

since $(1/\sqrt{n})^{1/n} \to 1$ as $n \to \infty$. Finally, let

$$A_z = \frac{\varphi\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)\sqrt{a}} \quad \text{and} \quad B_z = \frac{B\varphi\left(\frac{z}{b}\right)}{\left(\frac{z}{b}-1\right)\sqrt{b}},$$

so that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} &= \lim_{n\to\infty} \left| A_z\varphi^{an}\left(\frac{z}{a}\right) + B_z\varphi^{bn}\left(\frac{z}{b}\right) \right|^{1/n} \\
&= \lim_{n\to\infty} |B_z|^{1/n} \left|\varphi\left(\frac{z}{b}\right)\right|^b \left| \frac{A_z\varphi^{an}\left(\frac{z}{a}\right)}{B_z\varphi^{bn}\left(\frac{z}{b}\right)} + 1 \right|^{1/n} \\
&= \left|\varphi\left(\frac{z}{b}\right)\right|^b \lim_{n\to\infty} \left| \frac{A_z\varphi^{an}\left(\frac{z}{a}\right)}{B_z\varphi^{bn}\left(\frac{z}{b}\right)} + 1 \right|^{1/n}.
\end{aligned} \tag{4.2}$$

Since

$$\lim_{n\to\infty} \left| \frac{A_z\varphi^{an}\left(\frac{z}{a}\right)}{B_z\varphi^{bn}\left(\frac{z}{b}\right)} \right|^{1/n} = \frac{\left|\varphi\left(\frac{z}{a}\right)\right|^a}{\left|\varphi\left(\frac{z}{b}\right)\right|^b} < 1$$

by Lemma 4.2, from equation (4.2) we obtain

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |e^z| \left|\varphi\left(\frac{z}{b}\right)\right|^b = \left(\frac{e}{b}\right)^b |z|^b \tag{4.3}$$

by Proposition 3.3.

If $z \in \Omega_1$ is such that $|z| > b$, then by (3.2) with $r = a, b$ we have

$$\frac{|p_n(z)|^{1/n}}{|e^z|} = \left| \frac{\varphi^{an+1}\left(\frac{nz}{an+1}\right)}{\left(\frac{nz}{an+1} - 1\right)\sigma_{an}}[1 + O(n^\nu)] + \frac{B\varphi^{bn+1}\left(\frac{nz}{bn+1}\right)}{\left(\frac{nz}{bn+1} - 1\right)\sigma_{bn}}[1 + O(n^\nu)] \right|^{1/n},$$

which is handled in the same manner as (4.1) and again leads to (4.3). Thus we obtain (4.3) for all $z \in \Omega_1$, and since the $[1 + O(n^\nu)]$ factors in (3.2) and (3.3) hold uniformly on compact sets, we find that (4.3) holds uniformly on compact subsets of Ω_1. ■

Lemma 4.4. *In the region Ω_2 we have*

$$\lim_{n\to\infty} \frac{\ln |p_n(z)|}{n} = a \ln \left| \frac{ez}{a} \right|$$

uniformly on compact sets.

Proof. Let $z \in \Omega_2$ with $\operatorname{Re} z < a$. By (3.3) with $r = a, b$,

$$\frac{|p_n(z)|^{1/n}}{|e^z|} = \left| \left[\frac{\varphi^{an+1}\left(\frac{nz}{an+1}\right)}{\left(\frac{nz}{an+1} - 1\right)\sigma_{an}} + \frac{B\varphi^{bn+1}\left(\frac{nz}{bn+1}\right)}{\left(\frac{nz}{bn+1} - 1\right)\sigma_{bn}} \right] [1 + O(n^\nu)] + B + 1 \right|^{1/n}. \quad (4.4)$$

The constant terms B and 1 may be neglected since $|\varphi(z/a)| > 1$, giving

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} &= \lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\frac{z}{a}\right)}{\left(\frac{z}{a} - 1\right)\sigma_{an}} + \frac{B\varphi^{bn+1}\left(\frac{z}{b}\right)}{\left(\frac{z}{b} - 1\right)\sigma_{bn}} \right|^{1/n} \\
&= \lim_{n\to\infty} \left| \frac{\varphi^{an+1}\left(\frac{z}{a}\right)}{\left(\frac{z}{a} - 1\right)\sqrt{a}} + \frac{B\varphi^{bn+1}\left(\frac{z}{b}\right)}{\left(\frac{z}{b} - 1\right)\sqrt{b}} \right|^{1/n} \\
&= \lim_{n\to\infty} \left| A_z \varphi^{an}\left(\frac{z}{a}\right) + B_z \varphi^{bn}\left(\frac{z}{b}\right) \right|^{1/n} \\
&= \left| \varphi\left(\frac{z}{a}\right) \right|^a \lim_{n\to\infty} \left| 1 + \frac{B_z \varphi^{bn}\left(\frac{z}{b}\right)}{A_z \varphi^{an}\left(\frac{z}{a}\right)} \right|^{1/n} \qquad (4.5)
\end{aligned}$$

Since

$$\lim_{n\to\infty} \left| \frac{B_z \varphi^{bn}\left(\frac{z}{b}\right)}{A_z \varphi^{an}\left(\frac{z}{a}\right)} \right|^{1/n} = \frac{\left|\varphi\left(\frac{z}{b}\right)\right|^b}{\left|\varphi\left(\frac{z}{a}\right)\right|^a} < 1$$

by Lemma 4.2, from (4.5) we obtain

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |e^z| \left|\varphi\left(\frac{z}{a}\right)\right|^a = \left(\frac{e}{a}\right)^a |z|^a \tag{4.6}$$

by Proposition 3.3. A nearly identical analysis for $z \in \Omega_2$ with $|z| > a$ will again yield (4.6). ■

Lemma 4.5. *In the region Ω_3 we have*

$$\lim_{n\to\infty} \frac{\ln |p_n(z)|}{n} = \mathrm{Re}(z)$$

uniformly on compact sets.

Proof. For z in

$$\Omega_3 = \{z : |\varphi(z/a)| < 1\} \cap \{z : |\varphi(z/b)| < 1\} \cap \{z : a < \mathrm{Re}\, z < b\},$$

equation (3.2) with $r = a$ and (3.3) with $r = b$ yields (4.1). The nonzero constant term B dominates since $|\varphi(z/a)| < 1$ and $|\varphi(z/b)| < 1$ both hold, so that

$$\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} = \lim_{n\to\infty} |B|^{1/n} = 1,$$

and hence

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |e^z|. \tag{4.7}$$

■

Lemma 4.6. *In the region Ω_4 we have*

$$\lim_{n\to\infty} \frac{\ln |p_n(z)|}{n} = \mathrm{Re}(z)$$

uniformly on compact sets.

Proof. For $z \in \Omega_4$ equation (3.3) with $r = a, b$ yields (4.4). Now, $B \neq -1$ implies that the constant term $B + 1$ is nonzero, and since $|\varphi(z/a)| < 1$ and $|\varphi(z/b)| < 1$, this constant term dominates. From (4.4),

$$\lim_{n\to\infty} \frac{|p_n(z)|^{1/n}}{|e^z|} = \lim_{n\to\infty} |B+1|^{1/n} = 1,$$

so that (4.7) results once more. This finishes the proof of Lemma 4.3. ∎

We now have

$$\lim_{n\to\infty} \ln \left|p_n(z)\right|^{1/n} = \begin{cases} b\ln\left|\dfrac{ez}{b}\right|, & z\in\Omega_1 \\ a\ln\left|\dfrac{ez}{a}\right|, & z\in\Omega_2 \\ \ln|e^z|, & z\in\Omega_3\cup\Omega_4, \end{cases}$$

and so $\ln\left|p_n(z)\right|^{1/n}$ converges uniformly on compact sets to a harmonic function in each region. Hypotheses (1) and (3) of Theorem 2.5 being clear, it remains to verify that the sequence $|p_n(z)|^{1/n}$ is uniformly bounded on compact sets and $\bigcup_n Z(p_n(z))$ is a bounded set.

Lemma 4.7. *The sequence $|p_n(z)|^{1/n}$ is uniformly bounded on compact sets.*

Proof. Let K be a compact set, and fix n. Let $r>0$ be such that $K\subseteq \mathbb{D}_r$, and set $M=|B|+1$. For any $z\in K$,

$$\begin{aligned} |p_n(z)|^{1/n} &= \left|\sum_{j=0}^{an}\frac{(nz)^j}{j!} + B\sum_{j=0}^{bn}\frac{(nz)^j}{j!}\right|^{1/n} \le \left(\sum_{j=0}^{an}\frac{n^j|z|^j}{j!} + |B|\sum_{j=0}^{bn}\frac{n^j|z|^j}{j!}\right)^{1/n} \\ &\le M^{1/n}\left(\sum_{j=0}^{an}\frac{n^j|z|^j}{j!} + \sum_{j=0}^{bn}\frac{n^j|z|^j}{j!}\right)^{1/n} \le M^{1/n}\left(2\sum_{j=0}^{\infty}\frac{(n|z|)^j}{j!}\right)^{1/n} \\ &= \left(2Me^{n|z|}\right)^{1/n} = e^{|z|}\sqrt[n]{2M} \le e^r\sqrt[n]{2M} \le (2M+1)e^r := C, \end{aligned} \tag{4.8}$$

so $\{|p_n(z)|^{1/n} : z\in K\}$ has upper bound $C\in\mathbb{R}$. It follows that

$$\left\||p_n|^{1/n}\right\|_K = \sup\{|p_n(z)|^{1/n} : z\in K\}$$

exists in $\mathbb{R}$, with $\left\||p_n|^{1/n}\right\|_K \le C$ for all n, and hence

$$\sup_{n\in\mathbb{N}} \left\||p_n|^{1/n}\right\|_K$$

exists in $\mathbb{R}$. Therefore the sequence $|p_n|^{1/n}$ is uniformly bounded on K. ∎

Lemma 4.8. $\bigcup_n Z(p_n(z))$ *is a bounded set.*

Proof. We have

$$p_n(z) = B \sum_{k=an+1}^{bn} \frac{n^k z^k}{k!} + (B+1) \sum_{k=0}^{an} \frac{n^k z^k}{k!},$$

and so $p_n(z) = 0$ if and only if

$$\begin{aligned} &\left[z^{bn} + bz^{bn-1} + \frac{(bn)(bn-1)}{n^2} z^{bn-2} + \cdots + \frac{(bn)(bn-1)\cdots(an+2)}{n^{bn-an+1}} z^{an+1} \right] \\ &+ \frac{B+1}{B} \left[\frac{(bn)(bn-1)\cdots(an+1)}{n^{bn-an}} z^{an} + \frac{(bn)(bn-1)\cdots(an)}{n^{bn-an+1}} z^{an-1} + \cdots + \frac{(bn)!}{n^{bn}} \right] = 0. \end{aligned}$$

Thus if $p_n(z) = 0$, then, letting $B_0 = |B+1|/|B|$, Proposition 3.6 implies that $|z|$ is at most equal to

$$\begin{aligned} M = 2\max \Bigg\{ & b, \sqrt{\frac{(bn)(bn-1)}{n^2}}, \sqrt[3]{\frac{(bn)(bn-1)(bn-2)}{n^3}}, \ldots, \sqrt[bn-an-1]{\frac{(bn)(bn-1)\cdots(an+2)}{n^{bn-an-1}}}, \\ & \sqrt[bn-an]{\frac{B_0(bn)(bn-1)\cdots(an+1)}{n^{bn-an}}}, \sqrt[bn-an+1]{\frac{B_0(bn)(bn-1)\cdots(an)}{n^{bn-an+1}}}, \ldots, \\ & \sqrt[bn-1]{\frac{B_0(bn)(bn-1)\cdots 2}{n^{bn-1}}}, \sqrt[bn]{\frac{B_0(bn)!}{n^{bn}}} \Bigg\}. \end{aligned}$$

Now, since in general

$$(bn)(bn-1)\cdots[bn-(k-1)] \leq (bn)^k,$$

we find that

$$M \leq 2b \max \left\{ 1, M^{1/(bn-an)}, M^{1/(bn-an+1)}, \ldots, M^{1/(bn)} \right\},$$

and so $M < 3b$ for all sufficiently large n. That is, there exists some n_0 such that $Z(p_n(z))$ lies in the disc $\mathbb{D}_{3b}$ for all $n \geq n_0$, and therefore $\bigcup_n Z(p_n(z))$ is a bounded set. ■

We finally conclude by Theorem 2.5 that the zero attractor of $s_{an}(nz) + Bs_{bn}(nz)$ is $\bigcup_{k=1}^4 \partial\Omega_k$, proving Theorem 4.1.

Section 5: $As_{an}(nz) + Bs_{bn}(\omega nz)$ with $A, B \in \mathbb{C} \setminus \{0\}$ and $|\omega| = 1$

We now find the zero attractor of all sequences of the form $As_{an}(nz) + Bs_{bn}(\omega nz)$ with $A, B \in \mathbb{C} \setminus \{0\}$ and unimodular $\omega \in \mathbb{C}$. As is demonstrated in a more general setting at the beginning of the next section, it is enough to consider

$$p_n(z) := s_{an}(nz) + Bs_{bn}(e^{i\theta}nz).$$

for $B \neq 0$ and $0 \leq \theta < 2\pi$.

For any fixed $w \in \mathbb{C}$ let S_w denote the Szegő curve $|\varphi(z/w)| = 1$. The zeros of $p_{800}(800z)$ in the case when $a = 1$, $b = 2$, $B = 1$, and $\theta = \pi/5$ are shown in Figure 8, along with the curves S_a, $S_{ae^{-i\theta}}$, $S_{be^{-i\theta}}$, and the circle C_{ab} at the origin that contains the intersection points of the latter two curves. In addition to portions of the aforementioned Szegő curves and circle, the zero attractor would seem to also include a line segment with endpoints being the elements of $S_a \cap S_{ae^{-i\theta}} \cap \mathbb{D}_a$.

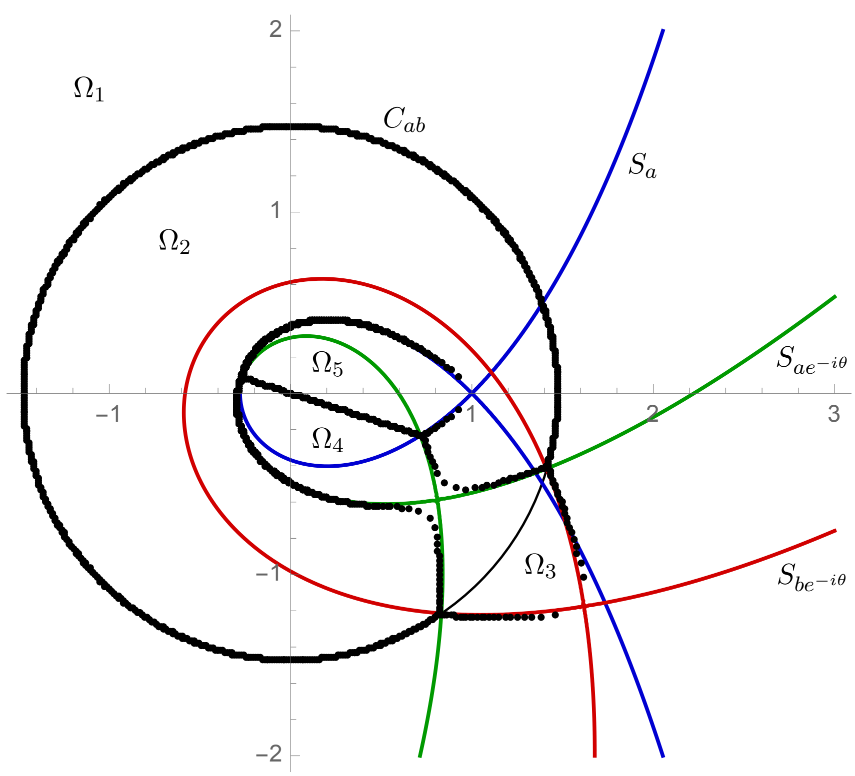

FIGURE 8. Zeros of $s_n(nz) + s_{2n}(e^{i\pi/5}nz)$ for $n = 800$, with S_a, $S_{ae^{-i\theta}}$, $S_{be^{-i\theta}}$, C_{ab} for $(a, b, \theta) = (1, 2, \pi/5)$.

For any $\theta \in \mathbb{R}$ the angle this segment makes with the positive real axis is $-\theta/2$. To see this, suppose $z \in S_a \cap S_{ae^{-i\theta}}$, so that

$$\left|\frac{z}{a}e^{1-z/a}\right| = \left|\frac{z}{ae^{-i\theta}}e^{1-z/(ae^{-i\theta})}\right| = 1.$$

Assuming $z = re^{-i\theta/2}$ for some $r > 0$, we obtain

$$\left|e^{-re^{-i\theta/2}/a}\right| = \left|e^{-re^{i\theta/2}/a}\right| = \frac{a}{er},$$

then

$$\operatorname{Re}\Big(\frac{r}{a}e^{-i\theta/2}\Big) = \operatorname{Re}\Big(\frac{r}{a}e^{i\theta/2}\Big) = \ln\Big(\frac{er}{a}\Big).$$

The first equality is in fact an identity that puts no constraints on r, and so from the second equality we arrive at the single equation

$$\cos\frac{\theta}{2} = \frac{a}{r}\ln\frac{er}{a}.$$

We need only confirm that there must exist $r > 0$ that satisfies this equation, which amounts to showing the function

$$f(r) = \frac{a}{r}\ln\frac{er}{a}$$

has range containing $[-1, 1]$. But this follows from the continuity of f and the observation that $f(a) = 1$ and $f(a/e^2) = -e^2 < -1$.

Let D_{ab} denote the interior of the circle C_{ab}, so $\partial D_{ab} = C_{ab}$. For

$$\mathbb{H} = \{z : \operatorname{Re}(z) > 0\}$$

and fixed $\theta \in \mathbb{R}$ define the open half-plane

$$H_\theta = e^{i\theta}\mathbb{H} = \{e^{i\theta}z : z \in \mathbb{H}\}.$$

Designating the regions

$$\Omega_1 = \mathbb{C} \setminus (\overline{D}_{ab} \cup \overline{L}_{be^{-i\theta}}),$$

$$\Omega_2 = D_{ab} \setminus (\overline{L}_a \cup \overline{L}_{ae^{-i\theta}} \cup \overline{T}_{ae^{-i\theta}}),$$

$$\Omega_3 = L_{be^{-i\theta}} \cap T_{ae^{-i\theta}},$$

$$\Omega_4 = L_{ae^{-i\theta}} \setminus \overline{H}_{(\pi-\theta)/2},$$

$$\Omega_5 = L_a \cap H_{(\pi-\theta)/2},$$

which are displayed in Figure 8, we make the following theorem.

Theorem 5.1. *For any $\theta \in [0, 2\pi)$, the zero attractor of $s_{an}(nz)+Bs_{bn}(e^{i\theta}nz)$ is $\bigcup_{k=1}^{5} \partial\Omega_k$.*

To prove this theorem with the use of Theorem 2.5 will require some asymptotic results for $s_{bn}(e^{i\theta}nz)$. From (3.2) and (3.3) we have

$$s_{bn}(e^{i\theta}nz) = \frac{e^{e^{i\theta}nz}\varphi^{bn+1}\left(\dfrac{e^{i\theta}nz}{bn+1}\right)}{\left(\dfrac{e^{i\theta}nz}{bn+1}-1\right)\sigma_{bn}}\left[1+O(n^{\nu})\right] \tag{5.1}$$

for $|z| > b + 1/n$, and

$$s_{bn}(e^{i\theta}nz) = e^{e^{i\theta}nz}\left[1+\frac{\varphi^{bn+1}\left(\dfrac{e^{i\theta}nz}{bn+1}\right)}{\left(\dfrac{e^{i\theta}nz}{bn+1}-1\right)\sigma_{bn}}\left[1+O(n^{\nu})\right]\right] \tag{5.2}$$

for $\text{Re}(e^{i\theta}z) < b + 1/n$, or equivalently for $z \in \mathbb{C} \setminus \overline{e^{-i\theta}[(b+1/n)+\mathbb{H}]}$.

We now apply our asymptotic results to evaluate the limit $\lim_{n\to\infty} \ln|p_n(z)|^{1/n}$ in each of the regions Ω_k, as required by Theorem 2.5. Here we will make use of the symbol $\sim$ to denote asymptotic equivalence or "equality of the limits."

Lemma 5.2. *In the region $\Omega_1 = \mathbb{C} \setminus (\overline{D}_{ab} \cup \overline{L}_{be^{-i\theta}})$ we have*

$$\lim_{n\to\infty} \frac{\ln|p_n(z)|}{n} = b\ln\left|\frac{ez}{b}\right|.$$

uniformly on compact sets.

Proof. Let $z \in \Omega_1$ such that $\text{Re}(e^{i\theta}z) < b$. From (3.2) with $r = a$ and (5.2) we have

$$\frac{\ln|p_n(z)|}{n} \sim \frac{1}{n}\ln\left|\frac{e^{nz}\varphi^{an+1}\left(\dfrac{z}{a}\right)}{\left(\dfrac{z}{a}-1\right)\sigma_{an}} + Be^{e^{i\theta}nz}\left(1+\frac{\varphi^{bn+1}\left(\dfrac{e^{i\theta}z}{b}\right)}{\left(\dfrac{e^{i\theta}z}{b}-1\right)\sigma_{bn}}\right)\right|$$

$$\sim \frac{1}{n}\ln\left|e^{nz}\varphi^{an}\left(\frac{z}{a}\right)+Be^{e^{i\theta}nz}\left[1+\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right]\right|, \tag{5.3}$$

and then $|\varphi(e^{i\theta}z/b)|>1$ implies that

$$\begin{aligned}\frac{\ln|p_n(z)|}{n} &\sim \frac{1}{n}\ln\left|e^{nz}\varphi^{an}\left(\frac{z}{a}\right)+Be^{e^{i\theta}nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right|\\ &= \frac{1}{n}\ln\left|Be^{e^{i\theta}nz}\right|\left|\varphi\left(\frac{e^{i\theta}z}{b}\right)\right|^{bn}\left|\frac{e^{nz}\varphi^{an}\left(\frac{z}{a}\right)}{Be^{e^{i\theta}nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}+1\right|\\ &\sim b\ln\left|\frac{ez}{b}\right|+\frac{1}{n}\ln\left|\frac{e^{nz}\varphi^{an}\left(\frac{z}{a}\right)}{Be^{e^{i\theta}nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}+1\right|. \end{aligned}\tag{5.4}$$

Since

$$\lim_{n\to\infty}\left|\frac{e^{nz}\varphi^{an}\left(\frac{z}{a}\right)}{Be^{e^{i\theta}nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}\right|^{1/n} = \frac{|e^{z}|\left|\varphi\left(\frac{z}{a}\right)\right|^{a}}{|e^{e^{i\theta}z}|\left|\varphi\left(\frac{e^{i\theta}z}{b}\right)\right|^{b}}<1$$

by Lemma 4.2, the desired conclusion follows from (5.4) and Proposition 3.3. If $z\in\Omega_1$ is such that $|z|>b$, then a similar argument follows using (5.1). ∎

Lemma 5.3. *In the region $\Omega_2 = D_{ab}\setminus(\overline{L}_a\cup\overline{L}_{ae^{-i\theta}}\cup\overline{T}_{ae^{-i\theta}})$ we have*

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = a\ln\left|\frac{ez}{a}\right|.$$

uniformly on compact sets.

Proof. Let $z\in\Omega_2$ such that $\operatorname{Re} z<a$. We have, from (3.3) with $r=a$ and (5.2),

$$\begin{aligned}\frac{\ln|p_n(z)|}{n} &\sim \frac{1}{n}\ln\left|e^{nz}\left(1+\frac{\varphi^{an+1}\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)\sigma_{an}}\right)+Be^{e^{i\theta}nz}\left(1+\frac{\varphi^{bn+1}\left(\frac{e^{i\theta}z}{b}\right)}{\left(\frac{e^{i\theta}z}{b}-1\right)\sigma_{bn}}\right)\right|\\ &\sim \frac{1}{n}\ln\left|e^{nz}+e^{nz}\varphi^{an}\left(\frac{z}{a}\right)+Be^{e^{i\theta}nz}+Be^{e^{i\theta}nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right|\\ &= \frac{1}{n}\ln\left|e^{e^{i\theta}nz}\right|\left|e^{(1-e^{i\theta})nz}+e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right)+B+B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right|. \end{aligned}\tag{5.5}$$

Since

$$e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) = (e^{-i\theta})^{an}\varphi^{an}\left(\frac{e^{i\theta}z}{a}\right) \tag{5.6}$$

and $|\varphi(e^{i\theta}z/a)| > 1$, the constant term B in (5.5) is dominated by the preceding term, and hence

$$\begin{aligned}\frac{\ln|p_n(z)|}{n} &\sim \frac{1}{n}\ln\left|e^{e^{i\theta}nz}\right|\left|e^{(1-e^{i\theta})nz} + e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) + B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right| \\ &\sim \frac{1}{n}\ln|e^{nz}|\left|1 + \varphi^{an}\left(\frac{z}{a}\right) + \frac{B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}{e^{(1-e^{i\theta})nz}}\right|.\end{aligned}$$

We have $|\varphi(z/a)| > 1$, so that

$$\begin{aligned}\frac{\ln|p_n(z)|}{n} &\sim \frac{1}{n}\ln|e^{nz}|\left|\varphi^{an}\left(\frac{z}{a}\right) + \frac{B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}{e^{(1-e^{i\theta})nz}}\right| \\ &= \frac{1}{n}\ln|e^{nz}|\left|\varphi\left(\frac{z}{a}\right)\right|^{an}\left|1 + \frac{B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}{e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right)}\right|,\end{aligned}$$

and since

$$\lim_{n\to\infty}\left|\frac{B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)}{e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right)}\right|^{1/n} = \frac{\left|e^{e^{i\theta}z}\right|\left|\varphi\left(\frac{e^{i\theta}z}{b}\right)\right|^{b}}{|e^{z}|\left|\varphi\left(\frac{z}{a}\right)\right|^{a}} < 1$$

by Lemma 4.2, Proposition 3.3 implies that

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = \lim_{n\to\infty}\frac{1}{n}\ln|e^{nz}|\left|\varphi\left(\frac{z}{a}\right)\right|^{an} = \ln|e^{z}|\left|\varphi\left(\frac{z}{a}\right)\right|^{a} = a\ln\left|\frac{ez}{a}\right|.$$

If $z \in \Omega_2$ is such that $|z| > a$, then a similar argument follows using (3.2). ∎

Lemma 5.4. *In the region* $\Omega_3 = L_{be^{-i\theta}} \cap T_{ae^{-i\theta}}$ *we have*

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = \ln\left|e^{e^{i\theta}z}\right|.$$

uniformly on compact sets.

Proof. As in the proof of Lemma 5.2 we use (3.2) with $r = a$ and (5.2) to obtain (5.3), whereupon (5.6) leads us to

$$\frac{\ln |p_n(z)|}{n} \sim \frac{1}{n} \ln \left|e^{e^{i\theta} nz}\right| \left|(e^{-i\theta})^{an} \varphi^{an}\left(\frac{e^{i\theta} z}{a}\right) + B + B\varphi^{bn}\left(\frac{e^{i\theta} z}{b}\right)\right|.$$

Since $|\varphi(e^{i\theta} z/a)| < 1$ and $|\varphi(e^{i\theta} z/b)| < 1$, the constant term B dominates, so

$$\lim_{n\to\infty} \frac{\ln |p_n(z)|}{n} = \lim_{n\to\infty} \frac{1}{n} \ln \left|e^{e^{i\theta} nz}\right| |B| = \ln \left|e^{e^{i\theta} z}\right|$$

as desired. ■

To carry out the analysis in the remaining regions Ω_4 and Ω_5 necessitates use of the following lemma, as these two regions lie on either side of the line segment discussed at the beginning of the section.

Lemma 5.5. *Suppose* $\theta \in [0, 2\pi)$, *and let* $\zeta = (e^{i\theta} - 1)z$. *Then* $\mathrm{Re}(\zeta) < 0$ *if* $z \in H_{(\pi-\theta)/2}$, *and* $\mathrm{Re}(-\zeta) < 0$ *if* $z \in \mathbb{C} \setminus \overline{H}_{(\pi-\theta)/2}$.

Proof. Fix $z \in H_{(\pi-\theta)/2}$. Then $z = e^{i(\pi-\theta)/2} w = ie^{-i\theta/2} w$ for some $w \in \mathbb{H}$. Now,

$$\zeta = (e^{i\theta} - 1)ie^{-i\theta/2} w = iw(e^{i\theta/2} - e^{-i\theta/2}) = -2w \sin\frac{\theta}{2},$$

and since $\mathrm{Re}(w) > 0$ we have

$$\mathrm{Re}(\zeta) = -2\,\mathrm{Re}(w) \sin\frac{\theta}{2} < 0.$$

If $z \in \mathbb{C} \setminus \overline{H}_{(\pi-\theta)/2}$, then $z = ie^{-i\theta/2} w$ for w such that $\mathrm{Re}(w) < 0$, whereupon

$$\mathrm{Re}(-\zeta) = 2\,\mathrm{Re}(w) \sin\frac{\theta}{2} < 0$$

obtains. ■

Lemma 5.6. *In the region* $\Omega_4 = L_{ae^{-i\theta}} \setminus \overline{H}_{(\pi-\theta)/2}$ *we have*

$$\lim_{n\to\infty} \frac{\ln |p_n(z)|}{n} = \ln \left|e^{e^{i\theta} z}\right|.$$

uniformly on compact sets.

Proof. As in the proof of Lemma 5.3 we use (3.3) with $r = a$ and (5.2) to obtain

$$\frac{\ln|p_n(z)|}{n} \sim \frac{1}{n}\ln\left|e^{e^{i\theta}nz}\right|\left|e^{(1-e^{i\theta})nz} + e^{(1-e^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) + B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right) + B\right|$$
$$= \ln\left|e^{e^{i\theta}z}\right| + \frac{1}{n}\ln\left|e^{(1-e^{i\theta})nz} + (e^{-i\theta})^{an}\varphi^{an}\left(\frac{e^{i\theta}z}{a}\right) + B\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right) + B\right|. \quad (5.7)$$

Since $|\varphi(e^{i\theta}z/a)| < 1$ and $|\varphi(e^{i\theta}z/b)| < 1$, and $\mathrm{Re}[(1-e^{i\theta})z] < 0$ by Lemma 5.5, the constant term B in (5.7) dominates the others as $n \to \infty$, and so

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = \lim_{n\to\infty}\left(\ln\left|e^{e^{i\theta}z}\right| + \frac{1}{n}\ln|B|\right) = \ln\left|e^{e^{i\theta}z}\right|.$$

■

Lemma 5.7. *In the region $\Omega_5 = L_a \cap H_{(\pi-\theta)/2}$ we have*

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = \ln|e^z|.$$

uniformly on compact sets.

Proof. As in the proof of Lemma 5.6 the equations (3.3) and (5.2) give the asymptotics, leading to

$$\frac{\ln|p_n(z)|}{n} \sim \frac{1}{n}\ln|e^{nz}|\left|1 + \varphi^{an}\left(\frac{z}{a}\right) + Be^{(e^{i\theta}-1)nz} + Be^{(e^{i\theta}-1)nz}\varphi^{bn}\left(\frac{e^{i\theta}z}{b}\right)\right|$$
$$= \ln|e^z| + \frac{1}{n}\ln\left|1 + \varphi^{an}\left(\frac{z}{a}\right) + Be^{(e^{i\theta}-1)nz} + B(e^{i\theta-1})^{bn}\varphi^{bn}\left(\frac{z}{b}\right)\right|. \quad (5.8)$$

Since $|\varphi(z/a)| < 1$ and $|\varphi(z/b)| < 1$, and $\mathrm{Re}[(e^{i\theta}-1)z] < 0$ by Lemma 5.5, the constant term 1 in (5.8) dominates the others, so that

$$\lim_{n\to\infty}\frac{\ln|p_n(z)|}{n} = \lim_{n\to\infty}\left(\ln|e^z| + \frac{1}{n}\ln|1|\right) = \ln|e^z|.$$

■

That the sequence $|p_n(z)|^{1/n}$ is uniformly bounded on compact sets and $\bigcup_n Z(p_n(z))$ is a bounded set is argued along lines similar to the proofs of Lemmas 4.7 and 4.8. We therefore conclude by Theorem 2.5 that the zero attractor of $s_{an}(nz) + Bs_{bn}(e^{i\theta}nz)$ is $\bigcup_{k=1}^{5}\partial\Omega_k$, proving Theorem 5.1.

Section 6: Introducing the General Two-Term Case

We undertake to find the zero attractors of all sequences $(p_n(z))_{n=1}^{\infty}$ for which

$$p_n(z) = As_{an}(\alpha nz) + Bs_{bn}(\beta nz)$$

for fixed integers $1 \le a < b$ and nonzero constants $\alpha, \beta, A, B \in \mathbb{C}$. To do this it will be sufficient to consider only sequences of the form

$$\hat{p}_n(z) = s_{an}(nz) + Cs_{bn}(\gamma nz) \tag{6.1}$$

for nonzero $\gamma, C \in \mathbb{C}$. To see this, we note that

$$\hat{p}_n(z) = \frac{1}{A} p_n\left(\frac{z}{\alpha}\right)$$

if we choose $\gamma = \beta/\alpha$ and $C = B/A$, and so if $Z(p_n(z))$ is the set of zeros of $p_n(z)$, then the set of zeros of $\hat{p}_n(z)$ is $\alpha Z(p_n(z))$. Therefore

$$Z(p_n(z)) = \frac{1}{\alpha} Z(\hat{p}_n(z))$$

for all n, and so if $\hat{\mathcal{A}}$ is the zero attractor of $(\hat{p}_n(z))$, then the zero attractor of $(p_n(z))$ is $\frac{1}{\alpha}\hat{\mathcal{A}}$.

Setting $\gamma = re^{i\theta}$ for constants $r > 0$ and $\theta \in \mathbb{R}$, we recast the family of sequences (6.1) as

$$P_n(z) = s_{an}(nz) + Cs_{bn}(re^{i\theta} nz) \tag{6.2}$$

with C taking the place of B. In [3] the exceptional case in which $r = 1$, $\theta = 0$, and $C = -1$ was treated. (Strictly speaking that paper kept a fixed at 1, but the technique employed would be the same for $a > 1$.) The case $r = 1$, $\theta = 0$, and $C \neq -1$ is addressed in §4, while all other cases in which $r = 1$ are addressed in §5. It thus remains to consider the cases when $r < 1$ and $r > 1$.

The value of C will be seen to have no effect on the zero attractor except in the aforementioned special case when $C = -1$ for $r = 1$ and $\theta = 0$.

As in the past we define $\varphi : \mathbb{C} \to \mathbb{C}$ by

$$\varphi(z) = ze^{1-z},$$

and use this function to define the following four Szegő curves:

$$\begin{aligned}
\mathcal{S}_1 : &\quad \left|\varphi\left(\frac{z}{a}\right)\right| = 1, \\
\mathcal{S}_2 : &\quad \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right| = 1, \\
\mathcal{S}_3 : &\quad \left|\varphi\left(\frac{re^{i\theta}z}{a}\right)\right| = r, \\
\mathcal{S}_4 : &\quad \left|\varphi\left(\frac{z}{b}\right)\right| = \frac{1}{r}.
\end{aligned}$$

For any Szegő curve $\mathcal{S}$ given by $|\varphi(wz)| = s$ for constants $w \in \mathbb{C}$ and $s \in \mathbb{R}$, it will be convenient to define the "interior of $\mathcal{S}$" to be the open region

$$\mathcal{S}^{<} = \{z : |\varphi(wz)| < s\}, \tag{6.3}$$

and the "exterior of $\mathcal{S}$" to be

$$\mathcal{S}^{>} = \{z : |\varphi(wz)| > s\}.$$

The symbols $\overline{\mathcal{S}}^{<}$ and $\overline{\mathcal{S}}^{>}$ will denote the closures of regions $\mathcal{S}^{<}$ and $\mathcal{S}^{>}$, so for instance

$$\overline{\mathcal{S}}^{<} := \overline{\mathcal{S}^{<}} = \{z : |\varphi(wz)| \leq s\}.$$

We will discover in the next section that the points in $\mathcal{S}_2 \cap \mathcal{S}_3$ and $\mathcal{S}_1 \cap \mathcal{S}_4$ lie on an "intersection circle" $\mathcal{C}_r$ of special significance, with center at the origin and radius

$$\rho_r = \frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}}, \tag{6.4}$$

called the "intersection radius." All points in $\mathcal{S}_1 \cap \mathcal{S}_3$, moreover, we will find from Proposition 7.1 lie on an "intersection line"

$$\mathcal{L}_{r\theta} = \{z : \mathrm{Arg}(\pm z) = \ell_{r\theta}\},$$

where

$$\ell_{r\theta} = \arctan\left(\frac{r\cos\theta - 1}{r\sin\theta}\right) \tag{6.5}$$

if $\theta \neq k\pi$ for any $k \in \mathbb{Z}$. If $\theta = k\pi$ we (quite arbitrarily) set

$$\ell_{r\theta} = \begin{cases} \pi/2, & \text{if } \theta = 2k\pi \\ -\pi/2, & \text{if } \theta = (2k+1)\pi. \end{cases}$$

In any case a suitable parametrization for $\mathcal{L}_{r\theta}$ would be

$$t \mapsto te^{i\ell_{r\theta}}, \quad t \in \mathbb{R}.$$

A useful formula for the future is

$$\cos \ell_{r\theta} = \frac{r|\sin\theta|}{\sqrt{r^2 - 2r\cos\theta + 1}}. \tag{6.6}$$

It will be convenient to give notation to the half planes with common boundary $\mathcal{L}_{r\theta}$, letting

$$\mathcal{H}^+_{r\theta} = e^{(\ell_{r\theta}+\pi/2)i}\mathbb{H} \quad \text{and} \quad \mathcal{H}^-_{r\theta} = -\mathcal{H}^+_{r\theta}.$$

Also

$$\mathcal{D}_r = \left\{ z : |z| < \frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} \right\}$$

will be the disc with boundary $\mathcal{C}_r$.

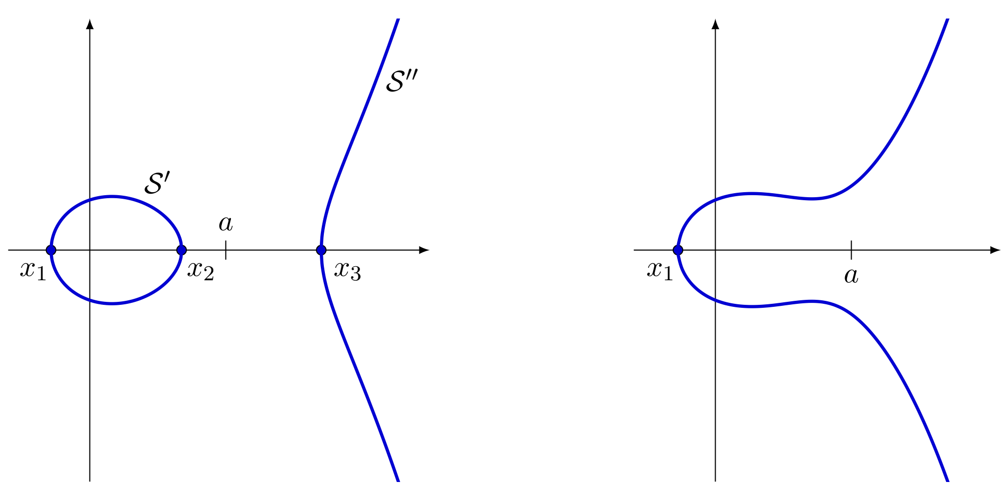

FIGURE 9. Left: an example of $\mathcal{S}$ for $r < 1$. Right: an example of $\mathcal{S}$ for $r > 1$.

For fixed values a, b, θ, and r, the zero attractor of (P_n) will always be a subset of the four Szegő curves, circle, and line defined above. To prove this it will be necessary to first establish some basic facts about Szegő curves. The proposition below does this with the aid of the Lambert-W function. The function $E : \mathbb{R} \to [-1/e, \infty)$ given by $E(x) = xe^x$ defines a bijection $[-1, \infty) \to [-1/e, \infty)$ with inverse the increasing function $W_0 : [-1/e, \infty) \to [-1, \infty)$ (which forms the principal branch), and another bijection $(-\infty, -1] \to [-1/e, 0)$ with inverse the decreasing function $W_{-1} : [-1/e, 0) \to (-\infty, -1]$. In particular we have

$$W_0(-1/e) = W_{-1}(-1/e) = -1. \tag{6.7}$$

The various parts of the next proposition, and especially the last two parts, are tailored to provide the minimum that is required to push the proofs of certain future results across the finish line.

Proposition 6.1 (**Properties of Szegő Curves**). *For $a, r > 0$ let $\mathcal{S}$ be the Szegő curve*

$$\left|\varphi\left(\frac{rz}{a}\right)\right| = r,$$

which is symmetric about $\mathbb{R}$.

(1) *If $r < 1$, $\mathcal{S} \cap \mathbb{R}$ consists of the points*

$$x_1 = -\frac{a}{r}W_0\left(\frac{r}{e}\right), \quad x_2 = -\frac{a}{r}W_0\left(-\frac{r}{e}\right), \quad x_3 = -\frac{a}{r}W_{-1}\left(-\frac{r}{e}\right),$$

where $-a < x_1 < 0 < x_2 < a < a/r < x_3$. Moreover $\mathcal{S} \cap \mathbb{R} = \{x_1, a\}$ when $r = 1$, and $\mathcal{S} \cap \mathbb{R} = \{x_1\}$ when $r > 1$ (where again $-a < x_1 < 0$).

(2) *If $r < 1$, the graph of $\mathcal{S}$ consists of two components: a simple closed curve $\mathcal{S}'$ in $\mathbb{D}_a$ and a simple unbounded curve $\mathcal{S}''$ in $x_3 + \overline{\mathbb{H}}$.*

(3) *If $r \geq 1$, the graph of $\mathcal{S}$ is connected and lies in $x_1 + \overline{\mathbb{H}}$. Also $\mathcal{S}$ is simple if $r > 1$.*

(4) *If $r < 1$, then $|\varphi(rz/a)| < r$ for all z inside $\mathcal{S}'$ and on the side of $\mathcal{S}''$ containing (x_3, ∞). Otherwise $|\varphi(rz/a)| > r$ holds.*

(5) *If $r > 1$, then $|\varphi(rz/a)| < r$ in the region bounded by $\mathcal{S}$ that contains (x_1, ∞), otherwise $|\varphi(rz/a)| > r$ holds. If $r = 1$, then $|\varphi(z/a)| < 1$ in the region bounded by $\mathcal{S}$ containing the origin, and also the region bounded by $\mathcal{S}$ containing (a, ∞).*

(6) *For* $0 < r \leq 1$ *the portion of* $\mathcal{S}$ *lying in the upper half-plane where* $x_1 \leq \operatorname{Re} z \leq x_2$ *is concave.*

(7) *When* $r = 1$ *define* $\mathcal{S}'$ *to be the region enclosed by the portion of* $\mathcal{S}$ *in* $\overline{\mathbb{D}}_a$*. Then* $\mathcal{S}'$ *is convex for all* $0 < r \leq 1$.

Figure 9 illustrates the essential shape of the curve $\mathcal{S}$ when $r < 1$ (at left) and $r > 1$ (at right). The shape of $\mathcal{S}$ when $r = 1$ is exactly as shown at right in Figure 3, with a in place of 1, and x_1 the leftmost point on the curve. We now commence with the proof of the proposition.

Proof.

Proof of (1). For $z = x + iy$,

$$\left|\varphi\left(\frac{rz}{a}\right)\right| = r \quad \Leftrightarrow \quad \left|\frac{rz}{a}e^{1-rz/a}\right| = r \quad \Leftrightarrow \quad \left|e^{rz/a}\right| = \frac{e|z|}{a}$$

$$\Leftrightarrow \quad e^{rx/a} = \frac{e\sqrt{x^2+y^2}}{a} \quad \Leftrightarrow \quad y^2 = a^2e^{2rx/a-2} - x^2, \tag{6.8}$$

and so the graph of the level curve $|\varphi(rz/a)| = r$ in $\mathbb{C}$ may be identified with the graphs of

$$v(x) = \sqrt{a^2e^{2rx/a-2} - x^2} \tag{6.9}$$

and $-v(x)$ in $\mathbb{R}^2$. Thus $\mathcal{S}$ is symmetric about the real axis, and of particular importance is the fact that $\mathcal{S}$ consists of two continuous *simple* curves that are reflections of each other about $\mathbb{R}$. No vertical line intersects $\mathcal{S}$ at more than two points.

Define

$$f_r(x) = \frac{x}{a}e^{1-rx/a}.$$

If $z \in \mathbb{R}$, then $a^2e^{2rx/a-2} - x^2 = 0$ by (6.8), giving

$$0 = \left(ae^{rx/a-1} - x\right)\left(ae^{rx/a-1} + x\right) = ae^{rx/a-1}\left[1 - f_r(x)\right]\left[1 + f_r(x)\right],$$

and thus $f_r(x) = 1$ or $f_r(x) = -1$. Of special import are the observations that

$$f_r(x) = -1 \quad \Leftrightarrow \quad -\frac{rx}{a}e^{-rx/a} = \frac{r}{e}, \tag{6.10}$$

and

$$f_r(x) = 1 \quad \Leftrightarrow \quad -\frac{rx}{a}e^{-rx/a} = -\frac{r}{e}. \tag{6.11}$$

Suppose $r < 1$. Since $r/e \in (0, 1/e)$, it is exclusively in the domain of W_0 and so (6.10) yields the unique solution

$$x_1 = -\frac{a}{r}W_0\Big(\frac{r}{e}\Big).$$

In contrast $-r/e \in (-1/e, 0)$ is in the domain of both W_0 and W_{-1}, and so (6.11) yields two solutions:

$$x_2 = -\frac{a}{r}W_0\Big(-\frac{r}{e}\Big) \quad \text{and} \quad x_3 = -\frac{a}{r}W_{-1}\Big(-\frac{r}{e}\Big).$$

It is immediate from the definitions of W_0 and W_{-1} that $x_1 < 0 < x_2 < x_3$, noting in particular that $W_0 : [-1/e, 0] \to [-1, 0]$ is a bijection. Next, since $E(x) = xe^x$ is increasing on $[-1, 0]$, we find

$$e^{-1} < e^{-r} \;\;\Rightarrow\;\; -\frac{r}{e} > -re^{-r} \;\;\Rightarrow\;\; E\Big(W_0\Big(-\frac{r}{e}\Big)\Big) > E(-r) \;\;\Rightarrow\;\; W_0\Big(-\frac{r}{e}\Big) > -r,$$

and hence $x_2 < a$. A similar argument shows $-a < x_1$ for any $r > 0$, and $-\infty < W_{-1}(-r/e) < -1$ implies $x_3 > a/r > a$.

If $r = 1$, then $x_2 = x_3 = a$ results from (6.7), implying $\mathcal{S} \cap \mathbb{R} = \{x_1, a\}$.

Finally, if $r > 1$ then $-r/e$ falls outside the domain of W_{-1} and W_0, and so only the solution x_1 deriving from (6.10) results. That $-a < x_1 < 0$ still holds for $r \geq 1$ is argued similarly as in the $r < 1$ setting. In particular, since $E(x) = xe^x$ is increasing on $(0, \infty)$,

$$e^{-1} < e^{r} \;\;\Rightarrow\;\; \frac{r}{e} < re^{r} \;\;\Rightarrow\;\; E\Big(W_0\Big(\frac{r}{e}\Big)\Big) > E(r) \;\;\Rightarrow\;\; W_0\Big(\frac{r}{e}\Big) < r,$$

and so $x_1 > -a$.

Proof of (2). Suppose $r < 1$. From (6.8) it is clear that $|\varphi(rz/a)| = r$ admits a solution $z = x + iy$ if and only if $|x| \leq ae^{rx/a-1}$, or equivalently $f_r(x) \in [-1, 1]$. Calculating

$$f_r'(x) = \frac{1}{a}\Big(1 - \frac{rx}{a}\Big)e^{1-rx/a},$$

we find that $f_r' > 0$ on $(-\infty, a/r)$ and $f_r' < 0$ on $(a/r, \infty)$, and so f_r has a maximum at a/r with $f_r(a/r) = 1/r > 1$. Now, setting $f_r(x) = 1$ yields the solutions $x_2 < a/r$ and

$x_3 > a/r$ found before, so that the graph of $\mathcal{S}$ is disjoint from the strip $x_2 < \operatorname{Re} z < x_3$, and at least some portion of the graph lies in $x_3 + \overline{\mathbb{H}}$. Indeed, because $f_r(x_3) = 1$, f_r decreases on $[x_3, \infty)$, and $f_r(x) = -1$ has only solution $x_1 \notin [x_3, \infty)$, it follows that $f_r(x) \in [-1, 1]$ for all $x \geq x_3$, and hence the graph of $\mathcal{S}$ in $x_3 + \overline{\mathbb{H}}$ consists of the points

$$x \pm iv(x), \quad x \geq x_3.$$

Since the function v is continuous with $v(x_3) = 0$, we conclude that $x_3+\overline{\mathbb{H}}$ contains precisely one component of the level curve $|\varphi(rz/a)| = r$, and it is both simple and unbounded.

Since $f_r(x)$ decreases on $(-\infty, a/r)$ as $x \to -\infty$ and $f(x) = -1$ has solution $x_1 < 0$, we find that another portion of the graph of $|\varphi(rz/a)| = r$ is confined to the strip $x_1 \leq \operatorname{Re} z \leq x_2$, and in addition the graph intersects every vertical line in the strip. Because $x_2 < a$, if $z = x + iy$ is a solution to $|\varphi(rz/a)| = r$ such that $x_1 \leq x \leq x_2$, then $x < a$ follows, and

$$x < a \quad \Rightarrow \quad a^2 e^{2rx/a - 2} < a^2 \quad \Rightarrow \quad x^2 + y^2 < a^2$$

by (6.8). This clearly indicates that the portion of the graph of $|\varphi(rz/a)| = r$ in the strip $x_1 \leq \operatorname{Re} z \leq x_2$ must lie within $\mathbb{D}_a$. The graph is generated by the graphs of the continuous functions $\pm v$, which are symmetric about the real axis and join at x_1 and x_2. Therefore the graph of $\mathcal{S}$ possesses a component in $\mathbb{D}_a$ that is a simple closed curve.

Proof of (3). Recalling that $x_2 = x_3$ when $r = 1$, in the $r \geq 1$ case the strips $x_1 \leq \operatorname{Re} z \leq x_2$ and $x_2 < \operatorname{Re} z < x_3$ of part (2) join to form the half-plane $x_1+\overline{\mathbb{H}}$, and by similar arguments we find that the graph of $|\varphi(rz/a)| = r$ is comprised of the graphs of the continuous functions $\pm v$ for all $x \geq x_1$. As $x \to \infty$, the functions proceed from the common starting point x_1 symmetrically into the upper and lower half planes to form a single connected level curve. The graph of $\mathcal{S}$ is also simple when $r > 1$, since in this case $\mathcal{S} \cap \mathbb{R}$ consists of a single point by part (1).

Proof of (4). This follows from part (2) and the observations that $|\varphi(0)| = 0 < r$ and

$$\lim_{\operatorname{Re} z \to \infty} \left|\varphi\left(\frac{rz}{a}\right)\right| = \frac{er}{a} \lim_{x \to \infty} e^{-rx/a} \sqrt{x^2 + y^2} = 0 < r$$

for any $r < 1$.

Proof of (5). By part (3), the same observations made in the proof of part (4) hold here for any $r \geq 1$.

Proof of (6). Fix $0 < r \leq 1$. In the proof of parts (1) and (2) it was found that the portion of $\mathcal{S}$ in question is given by (6.9) and lies in the strip $x_1 \leq \operatorname{Re} z \leq x_2$. To show is that $v''(x) < 0$ for all $x \in (x_1, x_2)$. From

$$v''(x) = \frac{e^{2rx/a-2}(a^2r^2e^{2rx/a-2} - a^2 + 2arx - 2r^2x^2)}{(a^2e^{2rx/a-2} - x^2)^{3/2}}$$

it's seen that, for $x \in (x_1, x_2)$, $v''(x) < 0$ if and only if

$$h(x) := 2r^2x^2 - 2arx + a^2 - \frac{a^2r^2}{e^2}e^{2rx/a} > 0.$$

To start, we observe that

$$\exp\Big(-2W_0\Big(-\frac{r}{e}\Big)\Big) = \left[\frac{W_0(-r/e)e^{W_0(-r/e)}}{W_0(-r/e)}\right]^{-2} = \left(\frac{-r/e}{W_0(-r/e)}\right)^{-2} = \frac{e^2}{r^2}W_0^2\Big(-\frac{r}{e}\Big),$$

and so

$$\begin{aligned}
h(x_2) &= 2r^2 \cdot \frac{a^2}{r^2}W_0^2\Big(-\frac{r}{e}\Big) + 2ar \cdot \frac{a}{r}W_0\Big(-\frac{r}{e}\Big) + a^2 - \frac{a^2r^2}{e^2}\exp\Big(-\frac{2r}{a} \cdot \frac{a}{r}W_0\left(-\frac{r}{e}\right)\Big) \\
&= 2a^2W_0^2\Big(-\frac{r}{e}\Big) + 2a^2W_0\Big(-\frac{r}{e}\Big) + a^2 - a^2W_0^2\Big(-\frac{r}{e}\Big) \\
&= a^2\left[W_0^2\Big(-\frac{r}{e}\Big) + 2W_0\Big(-\frac{r}{e}\Big) + 1\right] \\
&= a^2\left[W_0\Big(-\frac{r}{e}\Big) + 1\right]^2.
\end{aligned}$$

In particular $h(x_2) \geq 0$, and so $h(x) > 0$ for $x_1 < x < x_2$ if $h' < 0$ on (x_1, x_2). We have

$$h'(x) = 4r^2x - 2ar - \frac{2ar^3}{e^2}e^{2rx/a},$$

and since

$$\begin{aligned}
h'(x_2) &= -4r^2 \cdot \frac{a}{r}W_0\Big(-\frac{r}{e}\Big) - 2ar - \frac{2ar^3}{e^2}\exp\Big(-\frac{2r}{a} \cdot \frac{a}{r}W_0\left(-\frac{r}{e}\right)\Big) \\
&= -4arW_0\Big(-\frac{r}{e}\Big) - 2ar - 2arW_0^2\Big(-\frac{r}{e}\Big)
\end{aligned}$$

$$= -2ar\left[W_0\left(-\frac{r}{e}\right) - 1\right]^2 \le 0,$$

$h' < 0$ on (x_1, x_2) will obtain if $h'' > 0$ on (x_1, x_2). Now,

$$h''(x) = 4r^2 - \frac{4r^4}{e^2}e^{2rx/a},$$

and so $h'' > 0$ on (x_1, x_2) will obtain if

$$e^{2rx/a} < \frac{e^2}{r^2} \tag{6.12}$$

for all $x < x_2$. Since (6.12) is equivalent to $x < (a/r)\ln(e/r)$, it remains only to show that $x_2 \le (a/r)\ln(e/r)$. Recalling that W_0 maps $[-1/e, 0)$ onto $[-1, 0)$, so $W_0(-r/e) < 0$ in particular, we have

$$x_2 \le \frac{a}{r}\ln\left(\frac{e}{r}\right) \;\Leftrightarrow\; W_0\left(-\frac{r}{e}\right) \ge \ln\left(\frac{r}{e}\right) \;\Leftrightarrow\; \frac{r}{e} \le e^{W_0(-r/e)} = -\frac{r}{eW_0(-r/e)},$$

or equivalently $W_0(-r/e) \ge -1$, which is of course true. The proof is done.

Proof of (7). By parts (1) and (6), the boundary of $\mathcal{S}'$ consists of the graphs of the concave function $v(x)$ in the upper half-plane and the convex function $-v(x)$ in the lower half-plane for $x \in [x_1, x_2]$, which immediately implies that $\mathcal{S}'$ is convex. ■

The curve $\mathcal{S}_3$ is merely $\mathcal{S}$ of Proposition 6.1 rotated about the origin clockwise by θ, and so the properties of $\mathcal{S}$ are easily adapted to suit $\mathcal{S}_3$. Moreover, the curve $\mathcal{S}_1$ is $\mathcal{S}$ with $r = 1$, while $\mathcal{S}_2$ is obtained by rotating $\mathcal{S}$ clockwise by θ and replacing r with 1 and a with b/r. Finally, if r is replaced by $1/r$ and a by b/r, we find that $\mathcal{S}$ becomes $\mathcal{S}_4$. These observations readily imply the following several corollaries, the first and third of which will occasionally be called upon in some of the proofs carried out in §9. The second and fourth corollaries are given for the sake of completeness, but will not be used in any proofs.

Corollary 6.2. *The curve $\mathcal{S}_1$ is symmetric about $\mathbb{R}$ and has the following properties.*

(1) *The set $\mathcal{S}_1 \cap \mathbb{R}$ consists of the points*

$$x_{11} = -aW_0\left(\frac{1}{e}\right)$$

and a, where $-a < x_{11} < 0$.

(2) *The graph of* $\mathcal{S}_1$ *is connected and lies in* $x_{11} + \overline{\mathbb{H}}$.

(3) $\mathcal{S}_1^<$ *consists of the region bounded by* $\mathcal{S}_1$ *containing the origin, and also the region bounded by* $\mathcal{S}_1$ *containing* (a, ∞).

(4) *The region enclosed by the portion of* $\mathcal{S}_1$ *in* $\overline{\mathbb{D}}_a$ *is convex.*

Corollary 6.3. *The curve* $\mathcal{S}_2$ *is symmetric about the line* $\mathrm{e}^{-i\theta}\mathbb{R}$ *and has the following properties.*

(1) *The set* $\mathcal{S}_2 \cap \mathrm{e}^{-i\theta}\mathbb{R}$ *consists of the points*

$$x_{21} = -\frac{b}{r} W_0\left(\frac{1}{e}\right) e^{-i\theta}$$

and $(b/r)e^{-i\theta}$, *where* $-b/r < -(b/r)W_0(1/e) < 0$.

(2) *The graph of* $\mathcal{S}_2$ *is connected and lies in* $x_{21} + \mathrm{e}^{-i\theta}\overline{\mathbb{H}}$.

(3) $\mathcal{S}_2^<$ *consists of the region bounded by* $\mathcal{S}_2$ *containing the origin, and also the region bounded by* $\mathcal{S}_2$ *containing the open ray* $\mathrm{e}^{-i\theta}(b/r, \infty)$.

(4) *The region enclosed by the portion of* $\mathcal{S}_2$ *in* $\overline{\mathbb{D}}_{b/r}$ *is convex.*

The following corollary is the most important of the four. In its statement we omit the rather obvious analogues to parts (6) and (7) of Proposition 6.1.

Corollary 6.4. *The curve* $\mathcal{S}_3$ *is symmetric about the line* $\mathrm{e}^{-i\theta}\mathbb{R}$ *and has the following properties.*

(1) *If* $r < 1$, $\mathcal{S}_3 \cap \mathrm{e}^{-i\theta}\mathbb{R}$ *consists of the points*

$$x_{31} = -\frac{a}{r} W_0\left(\frac{r}{e}\right) e^{-i\theta}, \quad x_{32} = -\frac{a}{r} W_0\left(-\frac{r}{e}\right) e^{-i\theta}, \quad x_{33} = -\frac{a}{r} W_{-1}\left(-\frac{r}{e}\right) e^{-i\theta}.$$

Moreover $\mathcal{S}_3 \cap \mathrm{e}^{-i\theta}\mathbb{R} = \{x_{31}, ae^{-i\theta}\}$ *when* $r = 1$, *and* $\mathcal{S}_3 \cap \mathrm{e}^{-i\theta}\mathbb{R} = \{x_{31}\}$ *when* $r > 1$.

(2) *If* $r < 1$, *the graph of* $\mathcal{S}_3$ *consists of two components: a simple closed curve* $\mathcal{S}_3'$ *in* $\mathbb{D}_a$ *and a simple unbounded curve* $\mathcal{S}_3''$ *in* $x_{33} + \mathrm{e}^{-i\theta}\overline{\mathbb{H}}$.

(3) *If* $r \geq 1$, *the graph of* $\mathcal{S}_3$ *is connected and lies in* $x_{31} + \mathrm{e}^{-i\theta}\overline{\mathbb{H}}$. *Also* $\mathcal{S}_3$ *is simple if* $r > 1$.

(4) *If* $r < 1$, *then* $\mathcal{S}_3^<$ *consists of the region inside* $\mathcal{S}_3'$, *and also the region bounded by* $\mathcal{S}_3''$ *that contains the open ray* $\mathrm{e}^{-i\theta}(x_{33}, \infty)$.

(5) *If* $r > 1$*, then* $\mathcal{S}_3^<$ *consists of the region bounded by* $\mathcal{S}_3$ *that contains* $\mathrm{e}^{-i\theta}(x_{31}, \infty)$*. If* $r = 1$*, then* $\mathcal{S}_3^<$ *consists of the region bounded by* $\mathcal{S}_3$ *containing the origin, and also the region bounded by* $\mathcal{S}_3$ *containing* $\mathrm{e}^{-i\theta}(a, \infty)$*.*

Corollary 6.5. *The curve* $\mathcal{S}_4$ *is symmetric about* $\mathbb{R}$ *and has the following properties.*

(1) *If* $r > 1$*,* $\mathcal{S}_4 \cap \mathbb{R}$ *consists of the points*

$$x_{41} = -bW_0\left(\frac{1}{er}\right), \quad x_{42} = -bW_0\left(-\frac{1}{er}\right), \quad x_{43} = -bW_{-1}\left(-\frac{1}{er}\right),$$

where $-b/r < x_{41} < 0 < x_{42} < b/r < b < x_{43}$*. Moreover* $\mathcal{S}_4 \cap \mathbb{R} = \{x_{41}, b\}$ *when* $r = 1$*, and* $\mathcal{S}_4 \cap \mathbb{R} = \{x_{41}\}$ *when* $r < 1$ *(where again* $-b/r < x_{41} < 0$*).*

(2) *If* $r > 1$*, the graph of* $\mathcal{S}_4$ *consists of two components: a simple closed curve* $\mathcal{S}_4'$ *in* $\mathbb{D}_{b/r}$ *and a simple unbounded curve* $\mathcal{S}_4''$ *in* $x_{43} + \overline{\mathbb{H}}$*.*

(3) *If* $r \leq 1$*, the graph of* $\mathcal{S}_4$ *is connected and lies in* $x_{41} + \overline{\mathbb{H}}$*. Also* $\mathcal{S}_4$ *is simple if* $r < 1$*.*

(4) *If* $r > 1$*, then* $\mathcal{S}_4^<$ *consists of the region inside* $\mathcal{S}_4'$ *and the region bounded by* $\mathcal{S}_4''$ *that contains* (x_{43}, ∞)*.*

(5) *If* $r < 1$*, then* $\mathcal{S}_4^<$ *consists of the region bounded by* $\mathcal{S}_4$ *that contains* (x_{41}, ∞)*. If* $r = 1$*, then* $\mathcal{S}_4^<$ *consists of the region bounded by* $\mathcal{S}_4$ *containing the origin, and also the region bounded by* $\mathcal{S}_4$ *containing* (a, ∞)*.*

Section 7: Coordinates of Key Points

Here we determine precise coordinates for the intersection points of certain pairs of Szegő curves. In the case of the points in $\mathcal{S}_1 \cap \mathcal{S}_3$, which we consider first, we will once again have need of the Lambert-W function. As usual we assume a and b are positive integers with $a < b$.

Proposition 7.1. *If $\theta \neq k\pi$ for any $k \in \mathbb{Z}$, then the intersection points of $\mathcal{S}_1$ and $\mathcal{S}_3$ are*

$$p_1 = \frac{a}{\cos \ell_{r\theta}} W_0\left(\frac{\cos \ell_{r\theta}}{e}\right) e^{i(\ell_{r\theta}+\pi)}, \quad p_2 = -\frac{a}{\cos \ell_{r\theta}} W_0\left(-\frac{\cos \ell_{r\theta}}{e}\right) e^{i\ell_{r\theta}},$$

and

$$p_3 = -\frac{a}{\cos \ell_{r\theta}} W_{-1}\left(-\frac{\cos \ell_{r\theta}}{e}\right) e^{i\ell_{r\theta}}.$$

If $\theta = k\pi$ (with $r \neq 1$ if k is even), then the only intersection points are $p_1 = (a/e)i$ and $p_2 = -(a/e)i$. Moreover the points p_1, p_2, and p_3 always lie on the line $\mathcal{L}_{r\theta}$.

Proof. Suppose $\theta \neq k\pi$. Any $z \in \mathcal{S}_1 \cap \mathcal{S}_3$ must satisfy

$$\left|\frac{z}{a} e^{1-re^{i\theta}z/a}\right| = \left|\frac{z}{a} e^{1-z/a}\right| = 1. \tag{7.1}$$

Setting $z = se^{i\ell}$ for $s > 0$, the first equality in particular gives

$$rs\cos(\theta + \ell) = s\cos \ell. \tag{7.2}$$

If $\cos \ell = 0$ then (7.2) becomes $\sin\theta = 0$, so our assumption that $\theta \neq k\pi$ implies $\cos \ell \neq 0$. Now, from (7.2) we obtain

$$r(\cos\theta - \sin\theta \tan \ell) = 1,$$

and hence

$$\tan \ell = \frac{r\cos\theta - 1}{r\sin\theta}.$$

There are two solutions:

$$\ell_1 = \arctan\left(\frac{r\cos\theta - 1}{r\sin\theta}\right) \quad \text{and} \quad \ell_2 = \pi + \arctan\left(\frac{r\cos\theta - 1}{r\sin\theta}\right).$$

We note that ℓ_1 in particular is precisely $\ell_{r\theta}$ as given by (6.5), thus showing that all points in the set $\mathcal{S}_1 \cap \mathcal{S}_3$ lie on the line $\mathcal{L}_{r\theta}$.

From the second equality in (7.1) we obtain

$$\frac{s}{a} e^{1-(s/a)\cos\ell} = 1,$$

and hence

$$-\frac{s\cos\ell}{a} e^{-(s/a)\cos\ell} = -\frac{\cos\ell}{e},$$

and so if $E(z) = ze^z$ it follows that

$$E\left(-\frac{s\cos\ell}{a}\right) = -\frac{\cos\ell}{e} \tag{7.3}$$

for $\ell \in \{\ell_1, \ell_2\}$.

We consider first the $\ell = \ell_1$ case. Since $\ell_1 \in (-\pi/2, \pi/2)$ implies $\cos\ell_1 > 0$, we have $-(1/e)\cos\ell_1 \in [-1/e, 0)$. This makes clear that the value at right in (7.3) is in the domain of both W_0 and W_{-1}, and there exist $s_2, s_3 > 0$ such that $-(s_2/a)\cos\ell_1 \in [-1, 0)$ and $-(s_3/a)\cos\ell_1 \in (-\infty, -1]$ with

$$-\frac{s_2\cos\ell_1}{a} = W_0\left(-\frac{\cos\ell_1}{e}\right) \quad \text{and} \quad -\frac{s_3\cos\ell_1}{a} = W_{-1}\left(-\frac{\cos\ell_1}{e}\right).$$

Now we have solutions for s given by

$$s_2 = -\frac{a}{\cos\ell_1} W_0\left(-\frac{\cos\ell_1}{e}\right) \quad \text{and} \quad s_3 = -\frac{a}{\cos\ell_1} W_{-1}\left(-\frac{\cos\ell_1}{e}\right)$$

for $\ell = \ell_1$, and therefore we have points $p_2 = s_2 e^{i\ell_1}$ and $p_3 = s_3 e^{i\ell_1}$ in $\mathcal{S}_1 \cap \mathcal{S}_3$.

Next we consider $\ell = \ell_2$. Since $\cos\ell_2 < 0$ we have $-(1/e)\cos\ell_1 \in (0, 1/e]$, implying the value at right in (7.3) is in the domain of W_0. Indeed, because $W_0 : (0,\infty) \to (0,\infty)$ is a bijection, there exists a unique $s_1 > 0$ such that $-(s_1/a)\cos\ell_2 \in (0,\infty)$ and

$$-\frac{s_1\cos\ell_2}{a} = W_0\left(-\frac{\cos\ell_2}{e}\right).$$

Now we have a solution for s given by

$$s_1 = -\frac{a}{\cos\ell_2} W_0\left(-\frac{\cos\ell_2}{e}\right)$$

for $\ell = \ell_2$, and therefore the point $p_1 = s_1 e^{i\ell_2}$ is in $\mathcal{S}_1 \cap \mathcal{S}_3$. This is the same formulation of p_1 as in the lemma, since $\ell_2 = \ell_1 + \pi$ and $\cos \ell_2 = -\cos \ell_1$.

Suppose $\theta = k\pi$. We seek all $z = x + iy$ that satisfy (7.1). The first equality in (7.1) yields $|e^{\pm rz/a}| = |e^{-z/a}|$, so that $x/a \pm rx/a = 0$ and either $x = 0$ or $r = 1$. If $x = 0$, then the second equality in (7.1) implies $|y| = a/e$, and so $z = \pm(a/e)i$ are the only solutions. The result is the same if $r = 1$ for odd k. ■

In the foregoing proof there exists at each stage the technical necessity of additionally solving the equation

$$\left| \frac{z}{a} e^{1 - re^{i\theta} z/a} \right| = 1$$

deriving from (7.1), but the work is similar and the outcome identical to the treatment of the second equality in (7.1). We discount the case when $\theta = 2k\pi$ for $r = 1$ since we then find that $\mathcal{S}_3 = \mathcal{S}_1$.

The manner in which p_1 and p_2 are defined in Proposition 7.1 ensures that, for $\theta \neq k\pi$, the point p_1 lies always in $\mathbb{C} \setminus \overline{\mathbb{H}}$ and p_2 lies always in $\mathbb{H}$. Thus, for fixed a, b, r, the manner in which these points move as θ increases is not continuous. As $\theta \to \infty$, the segment $[p_1, p_2]$ that is a subset of $\mathcal{L}_{r\theta}$ will rotate in clockwise fashion until it passes through the imaginary axis, whereupon the points will switch places to remain in their specified half planes. This is to say the designations $p_1 = (a/e)i$ and $p_2 = -(a/e)i$ in the case when $\theta = k\pi$ are arbitrary.

Lemma 7.2. *Let $r > 0$.*

(1) $|p_1| \leq a/e$ *for all* $\theta \in \mathbb{R}$.

(2) $|p_1| < |p_2|$ *if* $\theta \neq k\pi$, *with equality if* $\theta = k\pi$.

(3) $|p_2| < a$ *for all* $\theta \in \mathbb{R}$ *such that* $\cos\theta \neq 1/r$, *with* $p_2 = a$ *if* $\cos\theta = 1/r$.

(4) $|p_3| > a$ *if* $\theta \neq k\pi$ *and* $\cos\theta \neq 1/r$, *with* $p_3 = a$ *if* $\theta \neq k\pi$ *and* $\cos\theta = 1/r$.

Proof.

Proof of (1). Proposition 7.1 makes clear that for any θ the point p_1 lies on the portion of $\mathcal{S}_1$ in $\mathbb{C} \setminus \mathbb{H}$, and so it's sufficient to show that $|z| \leq a/e$ for any $z \in \mathcal{S}_1$ with $\mathrm{Re}(z) \leq 0$.

Indeed, from $|\varphi(z/a)| = 1$ we obtain

$$|\varphi(z/a)| = 1 \quad\Rightarrow\quad |z||e^{1-z/a}| = a \quad\Rightarrow\quad |z| = \frac{a}{e}e^{\mathrm{Re}(z)/a},$$

and so $|z| \leq a/e$ follows whenever $\mathrm{Re}(z) \leq 0$.

Proof of (2). Suppose $\theta \neq k\pi$, and fix $x \in (0, 1/e]$. Then there exist $u, v > 0$ such that $W_0(x) = u$ and $W_0(-x) = -v$, and so $ue^u = x$ and $-ve^{-v} = -x$. Adding these results gives $ue^u - ve^{-v} = 0$, so that

$$\frac{v}{u} = e^{u+v} > 1$$

holds, and therefore $v > u$. This shows that $|W_0(-x)| > |W_0(x)|$, and since $(1/e)\cos\ell_{r\theta} \in (0, 1/e]$ it follows that

$$|p_1| = \frac{a}{\cos\ell_{r\theta}}\left|W_0\left(\frac{\cos\ell_{r\theta}}{e}\right)\right| < \frac{a}{\cos\ell_{r\theta}}\left|W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right)\right| = |p_2|$$

by Proposition 7.1.

If $\theta = k\pi$, it is immediate from Proposition 7.1 that $|p_1| = |p_2| = a/e$.

Proof of (3). That $|p_2| < a$ if $\theta = k\pi$ is clear, so assume $\theta \neq k\pi$. Suppose $\cos\theta \neq 1/r$. By (6.5) it follows that $\ell_{r\theta} \in (-\pi/2, \pi/2)$ with $\ell_{r\theta} \neq 0$, and thus $0 < \cos\ell_{r\theta} < 1$. Now $-e^{-1}\cos\ell_{r\theta} \in (-1/e, 0)$, and since W_0 maps $(-1/e, 0)$ onto $(-1, 0)$ we have

$$0 < 1 + W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right) < 1.$$

Using Proposition 7.1 and the property $W_0(x) = xe^{-W_0(x)}$,

$$|p_2| = -\frac{a}{\cos\ell_{r\theta}}W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right) = \left(\frac{a}{e}\right)e^{-W_0(-e^{-1}\cos\ell_{r\theta})} = \frac{a}{e^{1+W_0(-e^{-1}\cos\ell_{r\theta})}} < a.$$

If $\cos\theta = 1/r$, then $\ell_{r\theta} = 0$ by (6.5), so that $p_2 = -aW_0(-1/e) = a$ by Proposition 7.1.

Proof of (4). Suppose $\theta \neq k\pi$ and $\cos\theta \neq 1/r$. From (6.5) we find that the latter condition ensures $\ell_{r\theta} \neq 0$, and so $0 < \cos\ell_{r\theta} < 1$. Now we have $-e^{-1}\cos\ell_{r\theta} \in (-1/e, 0)$, and since W_0 maps $(-1/e, 0)$ onto $(-1, 0)$ while W_{-1} maps $(-1/e, 0)$ onto $(-\infty, -1)$, Proposition 7.1 and part (2) give $|p_3| > |p_2| > a$. If $\cos\theta = 1/r$, then $\ell_{r\theta} = 0$ holds and Proposition 7.1 gives $p_3 = -aW_{-1}(-1/e) = a$. ■

Lemma 7.3. *Let r_5 be the value of r for which $\rho_r = \min\{|z| : z \in \mathcal{S}_1\}$. Then*

$$r_5 = \frac{b}{a}\left[eW_0\left(\frac{1}{e}\right)\right]^{a/b-1} > 1. \tag{7.4}$$

Proof. Corollary 6.2(1) implies that $x_1 = -aW_0(1/e) \in \mathcal{S}_1$. Suppose $z = x + iy$ is such that $|z| < aW_0(1/e)$. Then $|z|/a < W_0(1/e)$, and because $E(w) = we^w$ is increasing on $[-1, \infty)$, we have

$$\frac{|z|}{a}e^{|x|/a} \leq \frac{|z|}{a}e^{|z|/a} = E(|z|/a) < E(W_0(1/e)) = 1/e,$$

and hence

$$\frac{|z|}{a}e^{1-|x|/a} \leq \frac{|z|}{a}e^{1+|x|/a} < 1.$$

If $x < 0$, then

$$\left|\varphi\left(\frac{z}{a}\right)\right| = \frac{|z|}{a}e^{1-x/a} = \frac{|z|}{a}e^{1+|x|/a} < 1,$$

and so $z \notin \mathcal{S}_1$. If $x \geq 0$, then

$$\left|\varphi\left(\frac{z}{a}\right)\right| = \frac{|z|}{a}e^{1-|x|/a} < 1,$$

and again $z \notin \mathcal{S}_1$ results. Therefore $\min\{|z| : z \in \mathcal{S}_1\} = aW_0(1/e)$.[2] Now setting ρ_r equal to $\min\{|z| : z \in \mathcal{S}_1\}$ gives rise to the equation

$$\frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} = aW_0\left(\frac{1}{e}\right),$$

and solving for r yields the expression at right in (7.4). This by definition is r_5.

Finally, using the identity $\ln[W_0(x)] = \ln x - W_0(x)$ for $x > 0$,

$$W_0(1/e) > 0 \;\Leftrightarrow\; 1 + \ln(1/e) - W_0(1/e) < 0 \;\Leftrightarrow\; \ln\bigl(eW_0(1/e)\bigr) < 0,$$

and so $eW_0(1/e) < 1$. Thus $[eW_0(1/e)]^{a/b-1} > 1$ since $a/b < 1$, and the inequality in (7.4) follows. ∎

[2]In fact $x_1 = -aW_0(1/e)$ is the *unique* point on $\mathcal{S}_1$ that is closest to 0: putting $z = x_1e^{it}$ into $|\varphi(z/a)| = 1$ readily gives $t = \pi$.

The motivation for denoting the special value in Lemma 7.3 by the symbol r_5 will be revealed early in §9.

Proposition 7.4. *The intersection points of $\mathcal{S}_2$ and $\mathcal{S}_3$ are*

$$q_1 = \rho_r \exp\left[\left(-\theta - \arccos\left(\frac{b}{r\rho_r} \ln \frac{er\rho_r}{b}\right)\right) i\right]$$

and

$$q_2 = \rho_r \exp\left[\left(-\theta + \arccos\left(\frac{b}{r\rho_r} \ln \frac{er\rho_r}{b}\right)\right) i\right]$$

for all $r \in (0, r_5]$. Moreover $q_1, q_2 \in \mathcal{C}_r$.

Proof. We have $z \in \mathcal{S}_2 \cap \mathcal{S}_3$ if and only if

$$\frac{1}{r}\left|\varphi\left(\frac{re^{i\theta}z}{a}\right)\right| = \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right| = 1, \tag{7.5}$$

so that in particular

$$\frac{1}{r^a}\left|\varphi\left(\frac{re^{i\theta}z}{a}\right)\right|^a = \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right|^b$$

holds, and hence

$$\frac{e^a|z|^a}{a^a} = \frac{e^b r^b |z|^b}{b^b}. \tag{7.6}$$

Solving for $|z|$ then yields

$$|z| = \frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} = \rho_r,$$

which indicates that all intersection points of the curves $\mathcal{S}_2$ and $\mathcal{S}_3$ lie on the circle $\mathcal{C}_r$.

Set $z = \rho_r e^{it}$ and solve for t with the second equality in (7.5). This gives

$$\left|e^{r\rho_r e^{i(\theta+t)}}\right| = \left(\frac{er\rho_r}{b}\right)^b,$$

whence

$$\cos(\theta + t) = \frac{b}{r\rho_r}\ln\left(\frac{er\rho_r}{b}\right) = e\left(\frac{ar}{b}\right)^{\frac{a}{b-a}}\ln\left(\frac{b}{ar}\right)^{\frac{a}{b-a}} = \frac{ea}{a-b}\left(\frac{ar}{b}\right)^{\frac{a}{b-a}}\ln\left(\frac{ar}{b}\right).$$

Provided that the rightmost expression is in $[-1, 1]$, there are generally two solutions for t,

$$t_1 = -\theta - \arccos\left(\frac{b}{r\rho_r}\ln\frac{er\rho_r}{b}\right) \quad \text{and} \quad t_2 = -\theta + \arccos\left(\frac{b}{r\rho_r}\ln\frac{er\rho_r}{b}\right),$$

which results in the desired expressions for q_1 and q_2.

It remains to shows that

$$F(r) := \frac{ea}{a-b}\Big(\frac{ar}{b}\Big)^{\frac{a}{b-a}} \ln\Big(\frac{ar}{b}\Big)$$

lies in $[-1,1]$ for $r \in (0, r_5]$. First,

$$F'(r) = -\frac{ea}{b-a}\Big(\frac{ar}{b}\Big)^{\frac{a}{b-a}} \left[1 + \frac{a}{b-a}\ln\Big(\frac{ar}{b}\Big)\right]\frac{1}{r},$$

and so $F'(r) = 0$ for $r > 0$ only when

$$1 + \frac{a}{b-a}\ln\Big(\frac{ar}{b}\Big) = 0,$$

or $r = (b/a)e^{1-b/a} := r_1$. Since F is increasing on $(0, r_1)$, $\lim_{r\to 0^+} F(r) = 0$, and $F(r_1) = 1$, it is clear that $F(r) \in (0, 1]$ for $r \in (0, r_1]$. Then, since F is decreasing on (r_1, ∞) and

$$F(r_5) = \frac{ea}{b}\big[eW_0(1/e)\big]^{-a/b} \ln\big[eW_0(1/e)\big] = -\frac{a}{b}\big[eW_0(1/e)\big]^{1-a/b} = -1/r_5$$

for $r_5 > 1$, we conclude that $-1 \leq F(r_5) \leq 1$ for all $r \in (0, r_5]$. ■

That the value r_1 in the proof of Proposition 7.4 is less than r_5 is apparent from the observation that $F(r) > 0$ for $r \in (0, r_1]$ and $F(r_5) < 0$, but in §9 (before the statement of Lemma 9.2) it is shown that in fact $r_1 < 1$.

Proposition 7.5. *The intersection points of* $\mathcal{S}_1$ *and* $\mathcal{S}_4$ *are*

$$o_1 = \rho_r \exp\left[i \arccos\Big(\frac{a}{\rho_r}\ln\frac{e\rho_r}{a}\Big)\right]$$

and

$$o_2 = \rho_r \exp\left[-i \arccos\Big(\frac{a}{\rho_r}\ln\frac{e\rho_r}{a}\Big)\right]$$

for all $r \in (0, r_5]$. *Moreover* $o_1, o_2 \in \mathcal{C}_r$.

Proof. We have $z \in \mathcal{S}_1 \cap \mathcal{S}_4$ if and only if

$$\Big|\varphi\Big(\frac{z}{a}\Big)\Big| = r\Big|\varphi\Big(\frac{z}{b}\Big)\Big| = 1,$$

so that

$$\left|\varphi\left(\frac{z}{a}\right)\right|^a = r^b \left|\varphi\left(\frac{z}{b}\right)\right|^b$$

holds and we again obtain (7.6). Thus $|z| = \rho_r$, and all intersection points of $\mathcal{S}_1$ and $\mathcal{S}_4$ lie on $\mathcal{C}_r$.

We next find t such that $z = \rho_r e^{it}$ satisfies $|\varphi(z/a)| = 1$. Obtaining

$$\left|e^{\rho_r e^{it}}\right| = \left(\frac{e\rho_r}{a}\right)^a,$$

it follows that $\cos t = G(r)$ for

$$G(r) := \frac{a}{\rho_r} \ln\left(\frac{e\rho_r}{a}\right) = \frac{eb}{a-b}\left(\frac{ar}{b}\right)^{\frac{b}{b-a}} \ln\left(\frac{ar}{b}\right).$$

Provided that $G(r) \in [-1, 1]$, there are two solutions for t,

$$t_1 = \arccos\left(\frac{a}{\rho_r} \ln \frac{e\rho_r}{a}\right) \quad \text{and} \quad t_2 = -\arccos\left(\frac{a}{\rho_r} \ln \frac{e\rho_r}{a}\right),$$

which results in the desired expressions for o_1 and o_2.

It remains to shows that $G(r) \in [-1, 1]$ indeed holds for $r \in (0, r_5]$. We have

$$G'(r) = -\frac{ea}{b-a}\left(\frac{ar}{b}\right)^{\frac{a}{b-a}} \left[1 + \frac{b}{b-a} \ln\left(\frac{ar}{b}\right)\right],$$

and so $G'(r) = 0$ for $r > 0$ only when

$$1 + \frac{b}{b-a} \ln\left(\frac{ar}{b}\right) = 0,$$

or $r = (b/a)e^{a/b-1} := r_3$. Since G is increasing on $(0, r_3)$, $\lim_{r\to 0^+} G(r) = 0$, and $G(r_3) = 1$, it is clear that $G(r) \in (0, 1]$ for $r \in (0, r_3]$. In addition, since G is decreasing on (r_3, ∞) and

$$G(r_5) = \frac{\ln[eW_0(1/e)]}{W_0(1/e)} = \frac{1 + \ln(1/e) - W_0(1/e)}{W_0(1/e)} = -1$$

we conclude both that $r_5 > r_3$ and $-1 \le G(r_5) \le 1$ for all $r \in (0, r_5]$. ■

Section 8: A Qualitative Overview

We undertake here a survey of how the zero attractor $\mathcal{A}$ of (6.2) evolves as $r \to \infty$ in the case when $C = 1$, $a = 1$, $b = 2$, and $\theta = 1$. There are five critical values, $\kappa_1 < \cdots < \kappa_5$, when the characterization of $\mathcal{A}$ as a union of regional boundaries passes to a new homotopy class as r passes from one open interval (κ_{i-1}, κ_i) to the next open interval (κ_i, κ_{i+1}), starting with $(0, \kappa_1)$ and ending with (κ_5, ∞). (Whether κ_i itself should be included in (κ_{i-1}, κ_i) or (κ_i, κ_{i+1}) depends on the value of i.) In the next section we will see that, in

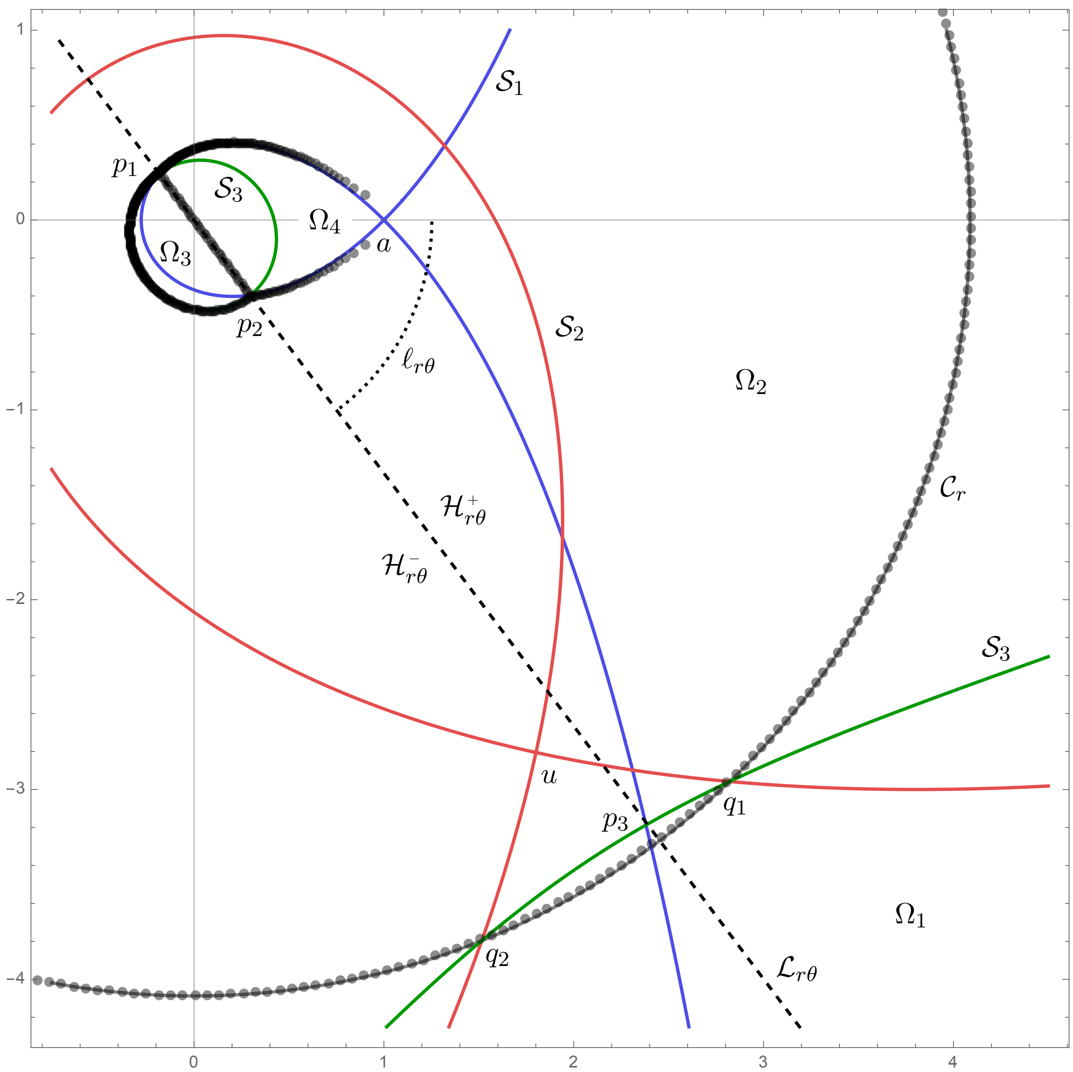

FIGURE 10. An $r \in (0, \kappa_1)$ case, illustrated by the zeros of $P_{400}(z)$ for $(a, b, \theta) = (1, 2, 1)$ and $r = 0.60$. Here $u = (b/r)e^{-i}$ and $\ell_{r\theta} \approx -0.929$.

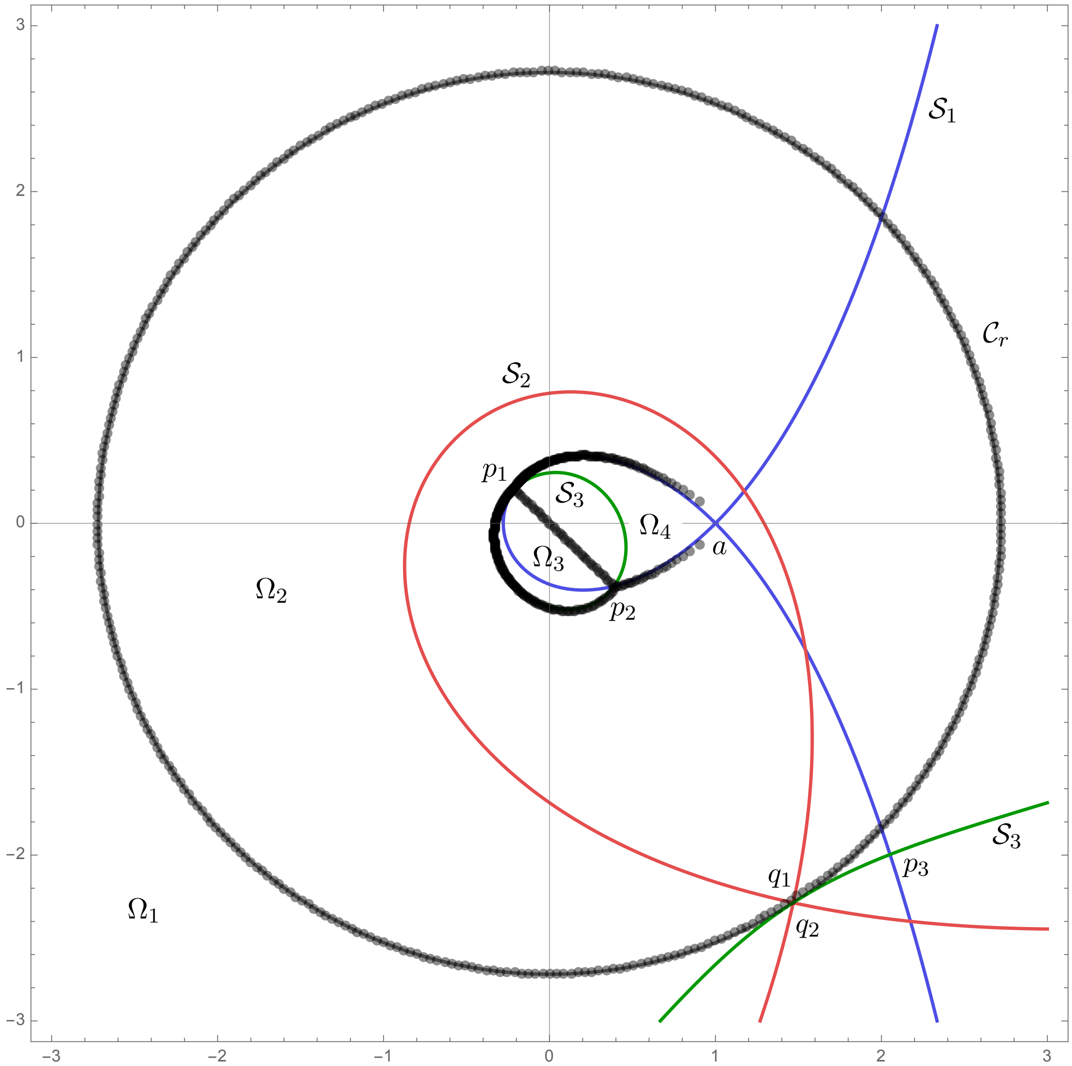

FIGURE 11. The $r = \kappa_1$ case illustrated by the zeros of $P_{400}(z)$, with $\kappa_1 = 2/e$ here.

the general case, most of the κ_i values are functions of at least one of the parameters a, b, θ.

In Figure 10 is shown the $r = 0.6$ case, featuring the points p_1, p_2, p_3 in $\mathcal{S}_1 \cap \mathcal{S}_3$, and q_1, q_2 in $\mathcal{S}_2 \cap \mathcal{S}_3$. (That $p_1, p_2, p_3 \in \mathcal{L}_{r\theta}$ and $q_1, q_2 \in \mathcal{C}_r$ follow from Propositions 7.1 and 7.4, respectively.) This is a fairly representative value in the interval $(0, \kappa_1)$. Indeed, for small $r \in (0, \kappa_1)$ the appearance of $\mathcal{A}$ changes little as $r \to 0^+$ save that the radius of $\mathcal{C}_r$ grows without bound. The union of the boundaries of the regions Ω_1, Ω_2, Ω_3, and Ω_4 in Figure 10 precisely equals $\mathcal{A}$. However, as $r \to \kappa_1^-$ the circle $\mathcal{C}_r$ shrinks until it includes the point

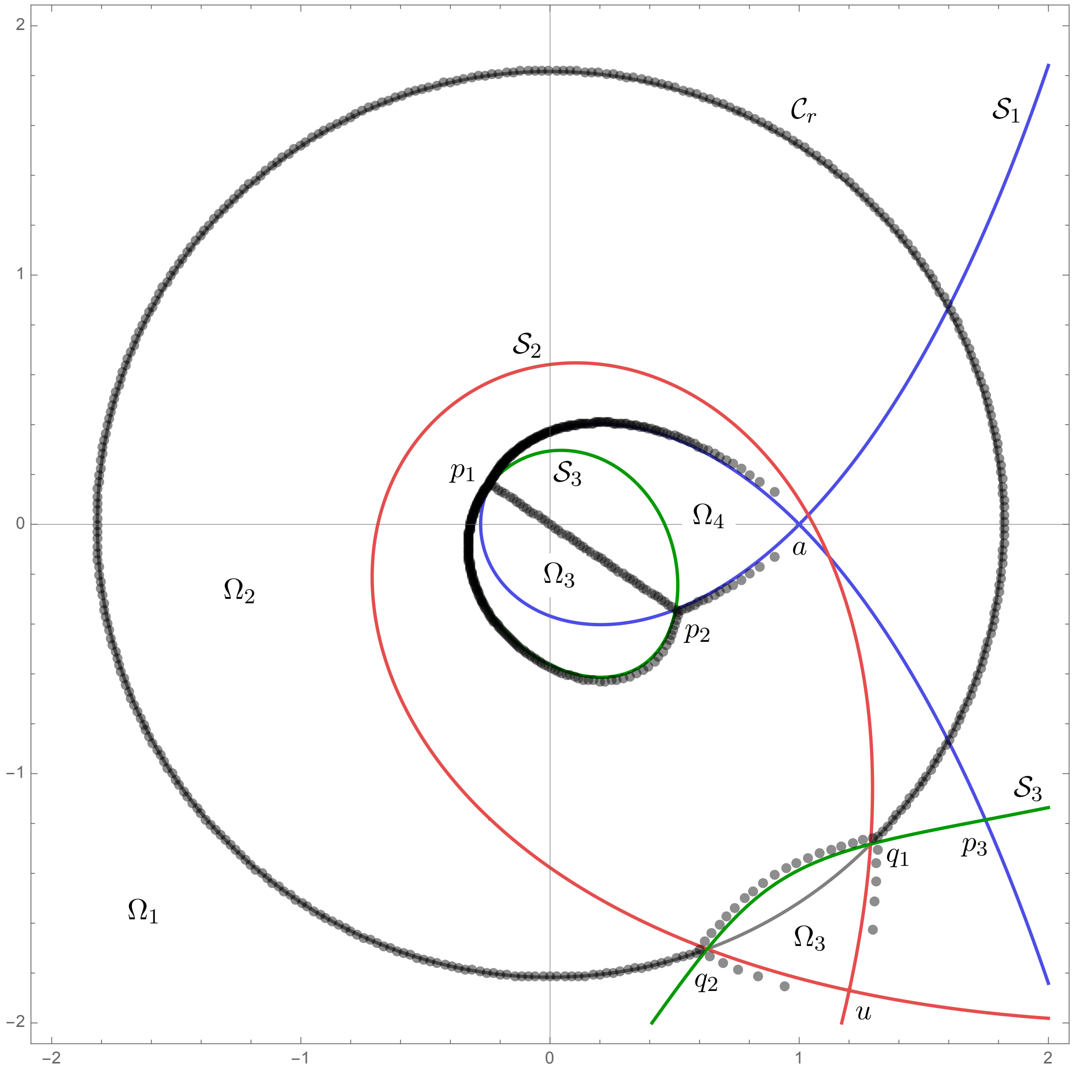

FIGURE 12. An $r \in (\kappa_1, \kappa_2)$ case illustrated with zeros of $P_{400}(z)$ for $r = 0.90$.

$u = (b/r)e^{-i}$ on $\mathcal{S}_2$. In the present setting in which $(a, b, \theta) = (1, 2, 1)$ we find this occurs when $r = 2/e \approx 0.736$, shown in Figure 11. Once r surpasses $2/e$ a new component of Ω_3 arises, shown in Figure 12, whose boundary contributes to $\mathcal{A}$. (The reason for regarding the new region as a component of Ω_3 will be made manifest when the two domains merge for higher values of r.) Thus we conclude that $\kappa_1 = 2/e$ is the first critical value.

Observation 8.1. *For $r \in (0, \kappa_1]$ with $\kappa_1 = 2/e$, $\mathcal{A}$ is disconnected and is the union of the boundaries of the connected regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \overline{\mathcal{D}}_r$$

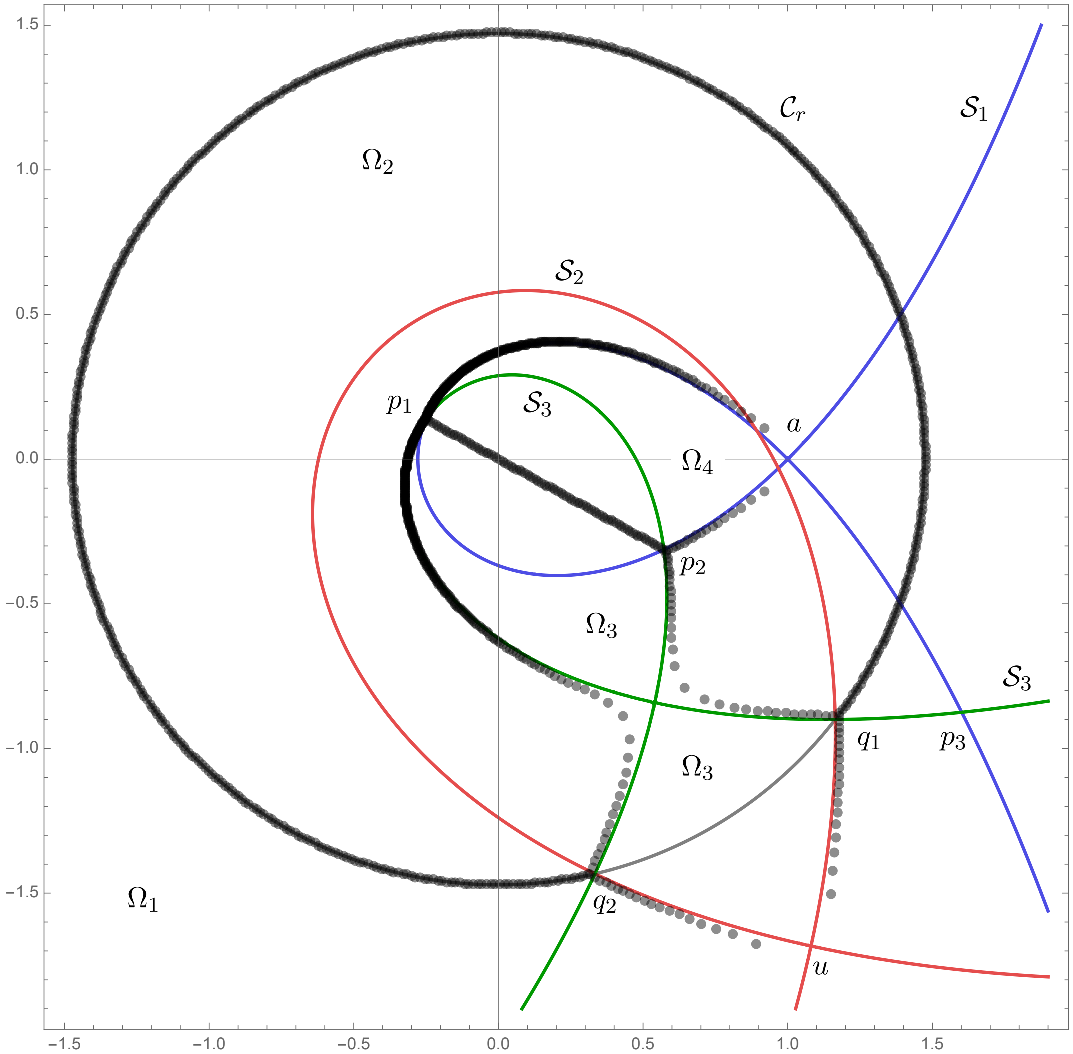

FIGURE 13. The $r = \kappa_2$ case illustrated by the zeros of $P_{400}(z)$, with $\kappa_2 = 1$ here.

$$\Omega_2: \quad \mathcal{D}_r \setminus \left[\left(\overline{\mathcal{S}}_1^< \cup \overline{\mathcal{S}}_3^<\right) \cap \overline{\mathbb{D}}_a\right]$$

$$\Omega_3: \quad \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a$$

$$\Omega_4: \quad \mathcal{S}_1^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a.$$

For $r \in (\kappa_1, \kappa_2)$ the zero attractor $\mathcal{A}$ has the appearance suggested by the zeros of $P_{400}(z)$ in Figure 12. In particular $\mathcal{A}$ consists of the boundaries of the regions Ω_1, Ω_2, Ω_3, and Ω_4. As $r \to \kappa_2^-$ the two components of the curve $\mathcal{S}_3$ draw toward one another, until finally when $r = 1$ they meet at the point $ae^{-i\theta}$ as shown in Figure 13, and $\mathcal{A}$ becomes a

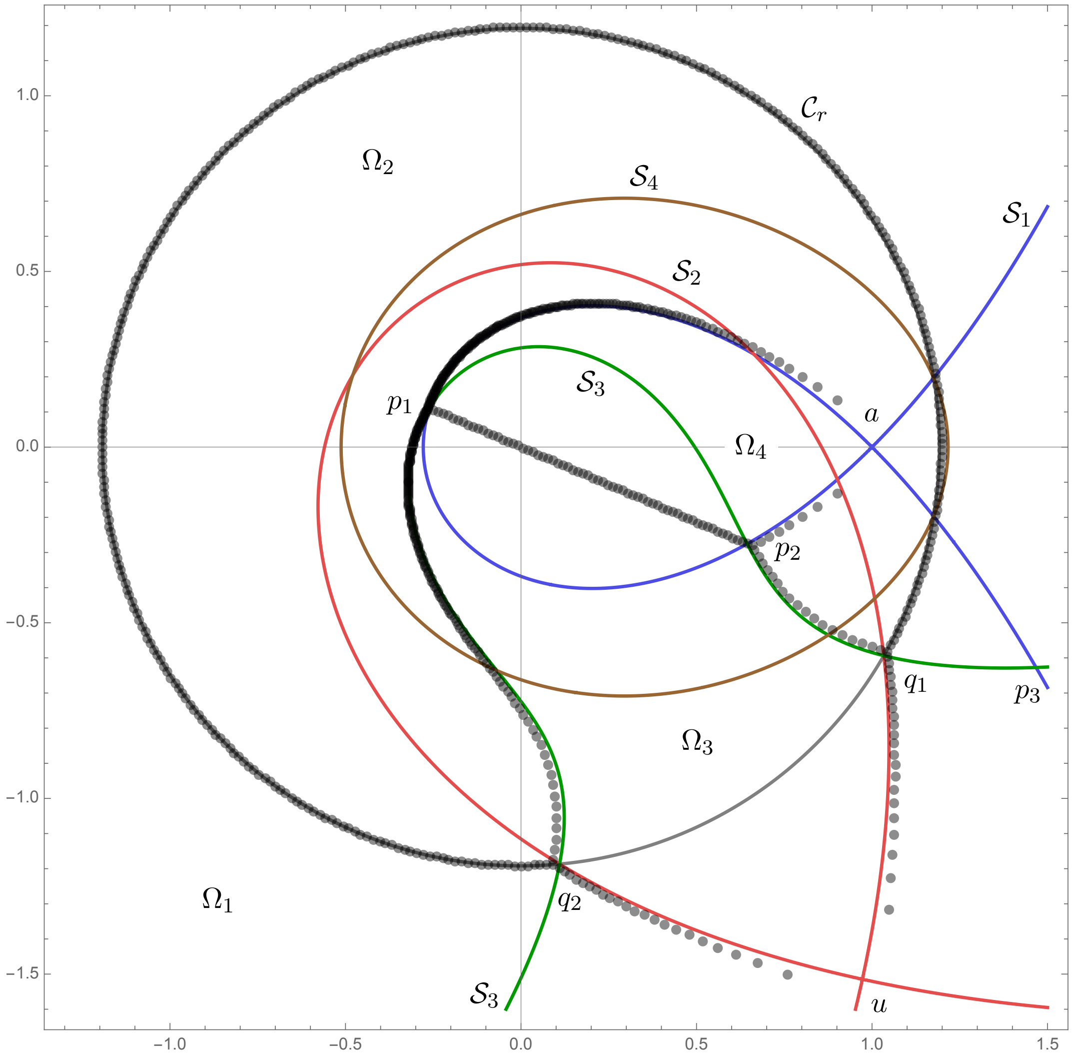

FIGURE 14. An $r \in (\kappa_2, \kappa_3)$ case illustrated with zeros of $P_{400}(z)$ for $r = 1.11$.

connected set. Once r exceeds 1 the two components of Ω_3 merge as in Figure 14, so that while $\mathcal{A}$ remains connected, it is in a decidedly different homotopy class. Thus $\kappa_2 = 1$ is the second critical value. Because $|u| = b/r$, and the bounded and unbounded components of $\mathcal{S}_3$ lie inside and outside $\mathbb{D}_a$, we have the following.

Observation 8.2. *For $r \in (\kappa_1, \kappa_2)$ with $\kappa_2 = 1$, $\mathcal{A}$ is disconnected and is the union of the boundaries of the regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \big[\overline{\mathcal{D}}_r \cup \big(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\big)\big]$$

$$\Omega_2 : \quad \mathcal{D}_r \setminus \big[\overline{\mathcal{S}}_3^< \cup \big(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\big)\big]$$

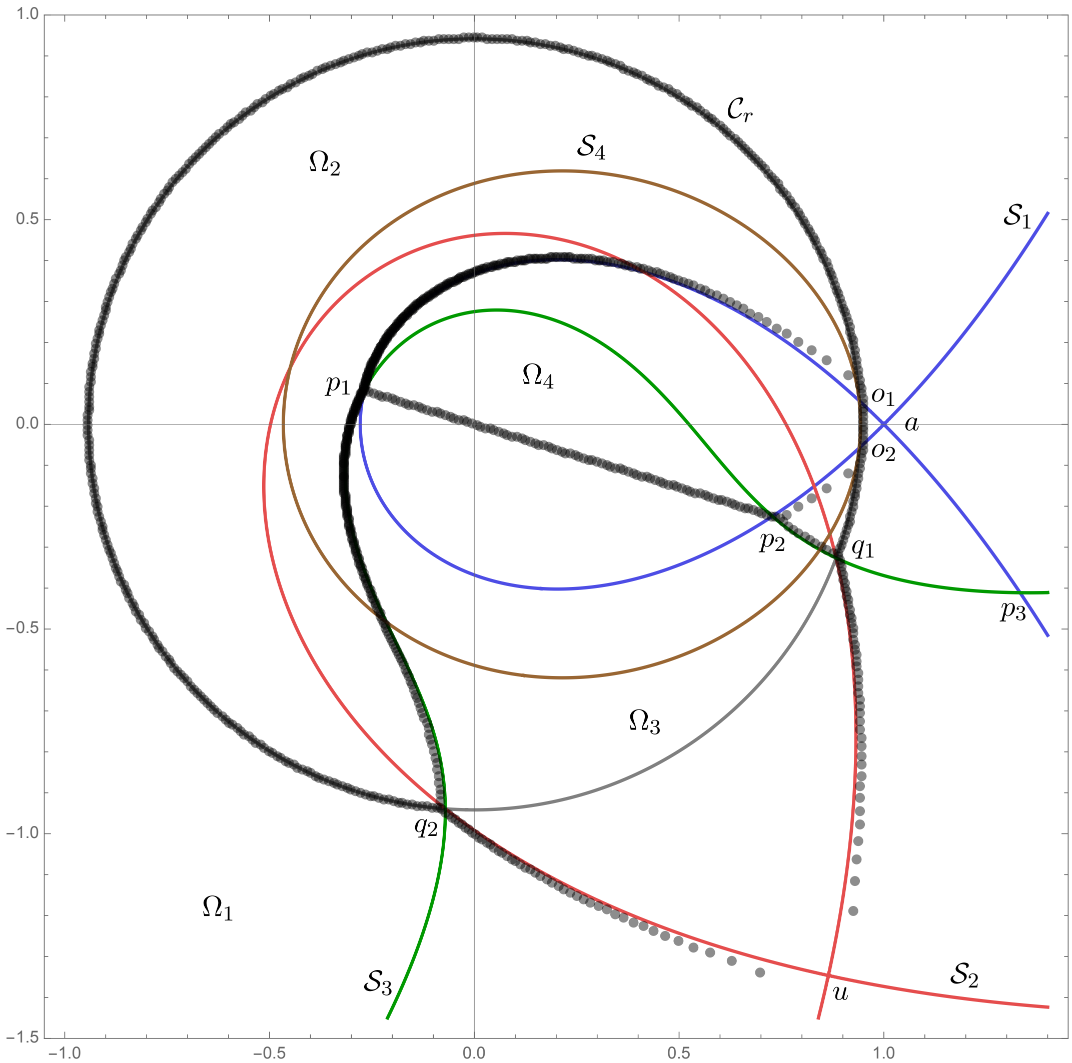

FIGURE 15. An $r \in (\kappa_3, \kappa_4)$ case illustrated with zeros of $P_{400}(z)$ for $r = 1.25$.

$$\Omega_3 : \quad \left[\mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a\right] \amalg \left[\mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathbb{A}_{a,b/r}\right]$$

$$\Omega_4 : \quad \mathcal{S}_1^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a,$$

where only Ω_3 *is disconnected*

For $r \in (\kappa_2, \kappa_3)$ we find $\mathcal{A}$ has a form that is largely traced by the zeros of $P_{400}(z)$ in Figure 14, where $r = 1.11$. The critical value κ_3 is achieved when r is such that the radius of $\mathcal{C}_r$ equals a. Then for $r \geq \kappa_3$ the region Ω_2 inside $\mathcal{C}_r$ becomes disconnected, as in Figure 15. Just as interesting, however, is that the curve $\mathcal{S}_4$ begins contributing to $\mathcal{A}$ for the first time. In Figure 15 the contribution lies between points o_1 and o_2 (the points where $\mathcal{C}_r$

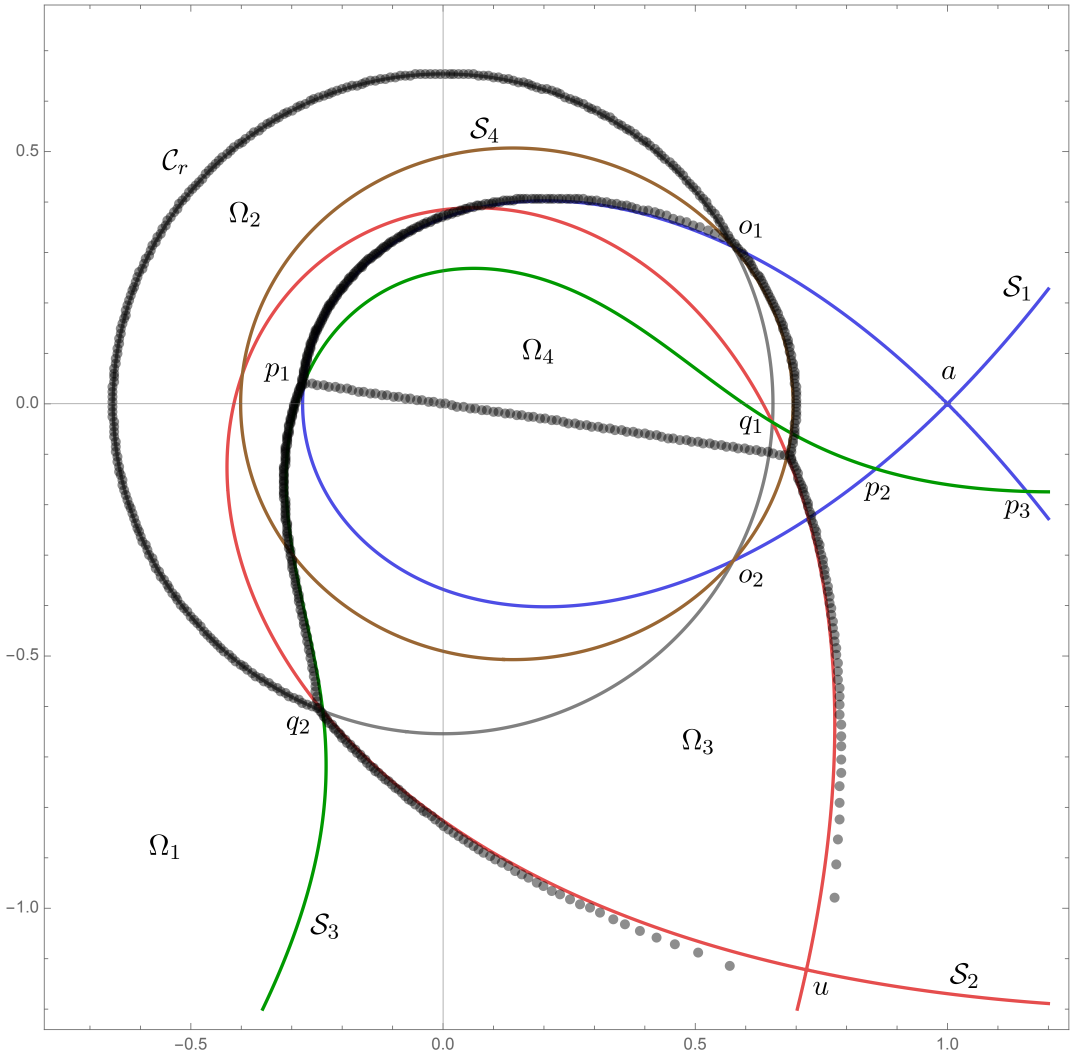

FIGURE 16. An $r \in (\kappa_4, \kappa_5)$ case illustrated with zeros of $P_{400}(z)$ for $r = 1.50$.

intersects $\mathcal{S}_1$), and while this is by no means visually obvious, it is borne out by a careful asymptotic analysis later on in §12. The radius of $\mathcal{C}_r$ when $a = 1$ and $b = 2$ is $4/er^2$, and so $\kappa_3 = 2/\sqrt{e} \approx 1.213$ in the present setting.

Observation 8.3. *For $r \in [\kappa_2, \kappa_3)$ with $\kappa_3 = 2/\sqrt{e}$, $\mathcal{A}$ is connected and is the union of the boundaries of the connected regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \big[\,\overline{\mathcal{D}}_r \cup \big(\overline{\mathcal{S}}_2^{<} \cap \overline{\mathbb{D}}_{b/r}\big)\big]$$

$$\Omega_2 : \quad \mathcal{D}_r \setminus \big[\,\overline{\mathcal{S}}_3^{<} \cup \big(\overline{\mathcal{S}}_1^{<} \cap \overline{\mathbb{D}}_a\big)\big]$$

$$\Omega_3 : \quad \mathcal{S}_2^{<} \cap \mathcal{S}_3^{<} \cap \mathcal{H}_{r\theta}^{-} \cap \mathbb{D}_{b/r}$$

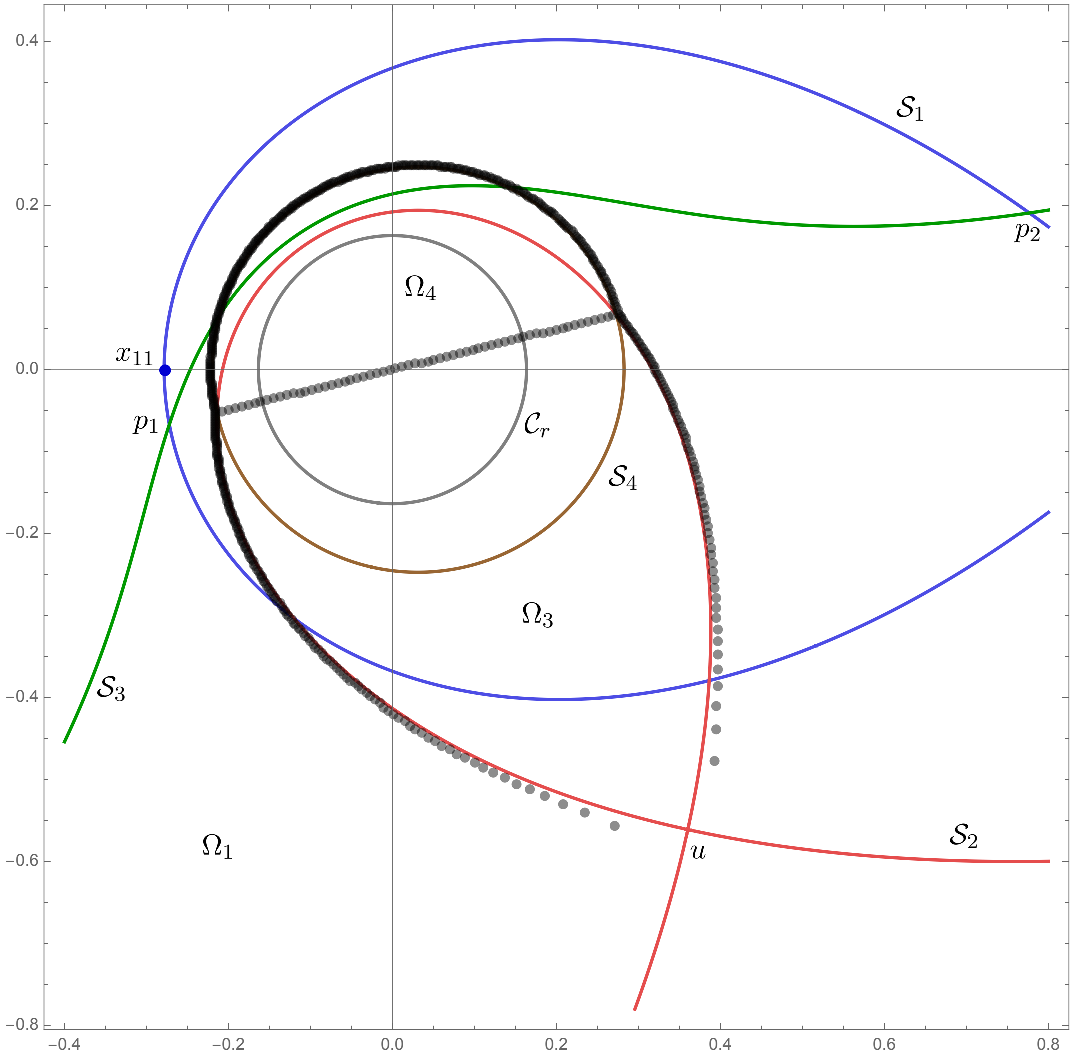

FIGURE 17. An $r \in (\kappa_5, \infty)$ case illustrated with zeros of $P_{400}(z)$ for $r = 3$.

$$\Omega_4: \quad \mathcal{S}_1^{<} \cap \mathcal{H}_{r\theta}^{+} \cap \mathbb{D}_a.$$

For $r \in (\kappa_3, \kappa_4)$ we find $\mathcal{A}$ has a form that is largely traced by the zeros of $P_{400}(z)$ in Figure 15, where $r = 1.25$. The region with boundary that traces $\mathcal{S}_1$ from o_2 to p_2, $\mathcal{S}_3$ from p_2 to q_1, and $\mathcal{C}_r$ from q_1 to o_2 is a component of Ω_2. The critical value κ_4 is reached when r is such that the radius of $\mathcal{C}_r$ equals $|p_2|$. Then the smaller component of Ω_2 vanishes, and Ω_2 becomes connected once again. For the statement of the next observation we emphasize that the curve $\mathcal{S}_4$ has two components for any $r > 1$, with the region $\mathcal{S}_4^{<}$ having a bounded and unbounded component. The unbounded component, however, always lies in the half-plane $b + \mathbb{H}$, a fact established by Corollary 6.5.

Observation 8.4. *For $r \in [\kappa_3, \kappa_4)$ with κ_4 being such that $\rho_{\kappa_4} = |p_2|$, $\mathcal{A}$ is the union of the boundaries of the regions*

$$\begin{aligned} \Omega_1 :&\quad \mathbb{C} \setminus \left[\overline{\mathcal{D}}_r \cup \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r} \right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b \right) \right] \\ \Omega_2 :&\quad \mathcal{D}_r \setminus \left[\overline{\mathcal{S}}_3^< \cup \overline{\mathcal{S}}_1^< \right] \\ \Omega_3 :&\quad \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_{b/r} \\ \Omega_4 :&\quad \mathcal{S}_1^< \cap \mathcal{S}_4^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a, \end{aligned}$$

where only Ω_2 is disconnected.

Region Ω_2 vanishes completely at the same time both the circle $\mathcal{C}_r$ and the curve $\mathcal{S}_1$ cease to contribute to $\mathcal{A}$, and so this is when the final critical value κ_5 has been attained. Since $\mathcal{C}_r$ is a part of the zero attractor so long as some piece of it lies outside $\overline{\mathcal{S}}_1^<$, we find that κ_5 is that value of r for which the radius of $\mathcal{C}_r$ equals the modulus of the point where $\mathcal{S}_1$ intersects $(-\infty, 0)$.

Observation 8.5. *For $r \in [\kappa_4, \kappa_5)$ with κ_5 as described above, $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\begin{aligned} \Omega_1 :&\quad \mathbb{C} \setminus \left[\overline{\mathcal{D}}_r \cup \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r} \right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b \right) \right] \\ \Omega_2 :&\quad \mathcal{D}_r \setminus \overline{\mathcal{S}}_1^< \\ \Omega_3 :&\quad \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_{b/r} \\ \Omega_4 :&\quad \mathcal{S}_1^< \cap \mathcal{S}_4^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a. \end{aligned}$$

For $r \in (\kappa_5, \infty)$ the zero attractor is relatively simple, as it no longer tracks along $\mathcal{C}_r$, $\mathcal{S}_1$, or $\mathcal{S}_3$. The $r = 3$ case in Figure 17 looks much the same as the $r = 10$ case or higher. With Ω_2 gone, only three regions remain.

Observation 8.6. *For $r \in [\kappa_5, \infty)$, $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\begin{aligned} \Omega_1 : &\quad \mathbb{C} \setminus \left[\left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r} \right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b \right) \right] \\ \Omega_3 : &\quad \mathcal{S}_2^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_{b/r} \\ \Omega_4 : &\quad \mathcal{S}_4^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a. \end{aligned}$$

Section 9: Critical Values of $r > 0$

Here we continue to lay the groundwork that will enable us to ascertain the zero attractor $\mathcal{A}$ of the sequence

$$P_n(z) = s_{an}(nz) + Cs_{bn}(re^{i\theta}nz)$$

for positive integers $a < b$, nonzero real C, positive real r, and real θ. The task is lengthy, for as the previous section showed, the topological nature of $\mathcal{A}$ depends on r; and as we'll see in the present section, even for fixed r it may be that the nature of $\mathcal{A}$ depends on θ. As stated at the beginning of §6 all $r = 1$ cases have already been treated. However, the developments of this section will be applicable to all $r = 1$ cases for which $\cos\theta \neq 1$. Since the $r = 1$, $\cos\theta = 1$ scenarios are treated in §4 for $C \neq -1$ and [3] for $C = -1$, we will henceforth assume that $r = 1$ and $\cos\theta = 1$ are never simultaneously the case. Working under this assumption, it will become apparent in the course of the asymptotic analyses in sections 11 and 12 that the value of C has no impact on the nature of $\mathcal{A}$. In any event the curves $\mathcal{S}_k$, $\mathcal{C}_r$, and $\mathcal{L}_{r\theta}$ defined early in §6, and the regions Ω_k introduced in §8, do not depend on C, and since the present section is concerned only with certain properties of these curves and regions, we will neglect making any reference to C until future sections.

It is convenient to think of r as being "time," and speak of the zero attractor $\mathcal{A}$ of $(P_n(z))$ as an object that evolves as time r increases and the other parameters a, b, and θ are held constant. In the previous section we let $a = 1$, $b = 2$, and $\theta = 1$ in particular, and presented a series of illustrations of $\mathcal{A}$ in order of increasing r. Five critical values of r were identified, denoted $\kappa_1, \ldots, \kappa_5$, which correspond to times when the homotopy class of the zero attractor changes. We will find in this section's general setting that no other critical values exist aside from the five already discovered, and indeed in some cases there are fewer than five. Most of the critical values depend in some way on at least some of the other parameters a, b, and θ. Also the four regions $\Omega_1, \ldots, \Omega_4$ featured in §8 figure prominently in the general setting, with no additional regions being necessary.

We now give decidedly topological definitions for the five critical values. A large part of the work done in the present section will be devoted to determining, where possible, explicit algebraic expressions for these critical values of r in terms of the other parameters. Any candidate for a critical value κ_j, or a bound on same, we will denote provisionally by r_j.

Definition 9.1. *Define the regions*

$$\Omega_2 = \mathcal{D}_r \setminus \left[\left(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\right) \cup \left(\overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r}\right)\right]$$

and

$$\Omega_3 = \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \left[(\mathcal{H}_{r\theta}^- \cap \mathbb{D}_a) \amalg \overline{\mathbb{A}}_{a,b/r}\right],$$

where $\overline{\mathbb{A}}_{a,b/r} = \varnothing$ if $b/r < a$. Taking $\epsilon > 0$ to be sufficiently small, we specify the following critical values of $r > 0$.

(1) *κ_1 is the least r such that Ω_3 is connected for $r \in (0, \kappa_1]$ and disconnected for $r \in (\kappa_1, \kappa_1 + \epsilon)$.*

(2) *κ_2 is the least r such that Ω_3 is disconnected for $r \in (\kappa_2 - \epsilon, \kappa_2]$ and connected for $r \in (\kappa_2, \infty)$.*

(3) *κ_3 is the least r such that Ω_2 is connected for $r \in (0, \kappa_3)$ and disconnected for $r = \kappa_3$.*

(4) *κ_4 is the least r such that Ω_2 is disconnected for $r \in (\kappa_4 - \epsilon, \kappa_4)$ and connected for $r = \kappa_4$.*

(5) *κ_5 is the least r such that $\Omega_2 = \varnothing$ for $r = \kappa_5$.*

Though not likely obvious at a glance, the regions Ω_2 and Ω_3 specified in Definition 9.1 correspond to the regions of the same name in the previous section's figures. (The complicated expressions given above for Ω_2 and Ω_3 stem from the need to accommodate all possible parameter values, whereas in §8 the two regions were able to assume relatively simple expressions owing to the specialized path within the parameter space to which that section restricted itself.) We also define the subregions

$$\Omega_3' = \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a \tag{9.1}$$

and

$$\Omega_3'' = \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \overline{\mathbb{A}}_{a,b/r}, \tag{9.2}$$

so that $\Omega_3 = \Omega_3' \amalg \Omega_3''$.

Before beginning our analyses we give here a broad overview of what they will discover. First, the regions Ω_1 and Ω_4 of §8 are never mentioned in this section because they never do anything interesting: they always exist and are always connected. As for Ω_2, it turns out that it is always connected if the angle θ falls within a sector I in $\mathbb{H}$ that is symmetrical about the real axis and depends only on a and b; otherwise it does become disconnected, as in Figure 15, for some bounded interval of r values. That Ω_2 is connected for all r if $\theta \in I$ means that the critical values κ_3 and κ_4 do not exist in this case. Moreover, for sufficiently large r the region Ω_2 vanishes altogether, as in Figure 17, which never occurs with any of the other regions. Finally, Ω_3 is always disconnected on some bounded r-interval, no matter the values of the other parameters. Tables 1 and Table 2 give the number of components constituting Ω_2 and Ω_3 when $\theta \notin I$ and $\theta \in I$, respectively, for r in the different open subintervals of $(0, \infty)$ determined by the critical values.

There may be points in the parameter space where the homotopy class of the zero attractor may change additional times as r increases in the interval (κ_4, κ_5). Fortunately this potentiality can be ruled out whenever $\cos\theta \leq 0$, and it will in any case be shown that the inequality $\kappa_j \leq \kappa_{j+1}$ always holds whenever κ_j and κ_{j+1} both exist. We now commence with our analyses.

For the statement of the first lemma consider the function

$$f(x) = \frac{x}{a} e^{1-x/a}.$$

Table 1. The number of components of Ω_2 and Ω_3 as r varies for $\theta \notin I$, where I depends only on a and b

$\theta \notin I$		
r-interval	Ω_2	Ω_3
$(0, \kappa_1)$	1	1
(κ_1, κ_2)	1	2
(κ_2, κ_3)	1	1
(κ_3, κ_4)	2	1
(κ_4, κ_5)	1	1
(κ_5, ∞)	$\varnothing$	1

TABLE 2. The number of components of Ω_2 and Ω_3 as r varies for $\theta \in I$. In this case κ_3 and κ_4 do not exist.

$\theta \in I$		
r-interval	Ω_2	Ω_3
$(0, \kappa_1)$	1	1
(κ_1, κ_2)	1	2
(κ_2, κ_5)	1	1
(κ_5, ∞)	$\varnothing$	1

Since $f(a) = 1$ and

$$f'(x) = \frac{1}{a}\Big(1 - \frac{x}{a}\Big)e^{1-x/a} < 0$$

for all $x > a$, it follows that $f(x) < 1$ on (a, ∞), and hence in particular

$$r_1 := \frac{b}{a}e^{1-b/a} < 1$$

for any $a, b \in \mathbb{Z}$ such that $1 \leq a < b$.

Lemma 9.2. *Fix $\theta \in \mathbb{R}$ and $a, b \in \mathbb{Z}$ with $1 \leq a < b$. Then the region Ω_3 is connected if and only if $r \in (0, r_1] \cup (1, \infty)$.*

Proof. Suppose $0 < r \leq r_1$. For x_3 as defined in Proposition 6.1 we first show that $x_3 \geq b/r$. From $r \leq r_1$ comes $-(b/a)e^{-b/a} \leq -r/e$, with both values in $[-1/e, 0)$. Recalling from page 39 that $W_{-1} : [-1/e, 0) \to (-\infty, -1]$ is a decreasing function, we next obtain

$$-\frac{b}{a} = W_{-1}\bigg(-\frac{b}{a}e^{-b/a}\bigg) \geq W_{-1}\Big(-\frac{r}{e}\Big),$$

whereupon multiplying by $-a/r$ gives

$$\frac{b}{r} \leq -\frac{a}{r}W_{-1}\Big(-\frac{r}{e}\Big) = x_3. \tag{9.3}$$

Now, since $r < 1$, by Corollary 6.4 the region $\mathcal{S}_3^{<}$ is disconnected with bounded component in the disc $\mathbb{D}_a$ and unbounded component in the half-plane $e^{-i\theta}(x_3+\mathbb{H})$, so that $\mathcal{S}_3^{<} \cap \overline{\mathbb{A}}_{a,b/r} = \varnothing$ here, and thus

$$\Omega_3 = \mathcal{S}_2^{<} \cap \mathcal{S}_3^{<} \cap \mathcal{H}_{r\theta}^{-} \cap \mathbb{D}_a. \tag{9.4}$$

Corollary 6.3(4) makes clear that $\mathcal{S}_2^< \cap \mathbb{D}_{b/r}$ is a convex set, while Proposition 6.1(7) implies $\mathcal{S}_3^< \cap \mathbb{D}_a$ is convex. The unbounded components of $\mathcal{S}_2^<$ and $\mathcal{S}_3^<$ both lie outside $\overline{\mathbb{D}}_{b/r}$, and since $\mathbb{D}_a \subseteq \mathbb{D}_{b/r}$ here, so that $\mathbb{D}_a = \mathbb{D}_a \cap \mathbb{D}_{b/r}$, (9.4) may be recast as an intersection of convex sets:

$$\Omega_3 = (\mathcal{S}_2^< \cap \mathbb{D}_{b/r}) \cap (\mathcal{S}_3^< \cap \mathbb{D}_a) \cap \mathcal{H}_{r\theta}^-.$$

Hence Ω_3 is convex, and therefore is connected.

Next suppose $r > 1$. By Corollary 6.4 the boundary of the region $\mathcal{S}_3^<$ is a simple continuous curve that partitions the complex plane, while $\mathcal{S}_2^< \cap \mathbb{D}_{b/r}$ is convex as before. In fact, with $v(x)$ as given by (6.9),

$$\mathcal{S}_3^< = \bigcup_{x > x_1} e^{-i\theta}\big(x - iv(x), x + iv(x)\big);$$

and with

$$w(x) = \sqrt{\frac{b^2}{r^2} e^{2rx/b-2} - x^2}$$

and $x_1' = -(b/r)W_0(1/e)$,

$$\mathcal{S}_2^< = \bigcup_{x > x_1'} e^{-i\theta}\big(x - iw(x), x + iw(x)\big).$$

Recalling that $be^{-i\theta}/r$ is the self-intersection point of $\mathcal{S}_2$, the above characterizations of $\mathcal{S}_3^<$ and $\mathcal{S}_2^<$ make clear that $\mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \overline{\mathbb{D}}_{b/r}$, which is $\mathcal{S}_2^< \cap \mathcal{S}_3^< \cap (\mathbb{D}_a \amalg \overline{\mathbb{A}}_{a,b/r})$, is connected, and therefore so too is Ω_3.

For the converse, suppose that $r_1 < r < 1$, so $-r/e < -(b/a)e^{-b/a}$, and by the steps leading to (9.3)—only with all inequalities reversed—we obtain $x_3 < b/r$. Corollary 6.4 implies $x_3 > a/r > a > x_2$, and the two components of $\mathcal{S}_3^<$ lie in $e^{-i\theta}(x_3 + \mathbb{H})$ and $\mathbb{D}_a$. It follows that $\mathcal{S}_3^< \cap \partial\mathbb{D}_a = \varnothing$, and thus $\Omega_3 \cap \partial\mathbb{D}_a = \varnothing$. With Ω_3', Ω_3'' as defined by (9.1) and (9.2), we find that $\Omega_3' \neq \varnothing$ since it contains points in $\mathcal{H}_{r\theta}^-$ that are sufficiently close to the origin, and also $\Omega_3'' \neq \varnothing$ since $e^{-i\theta}(x_3, b/r) \subseteq \Omega_3''$. Therefore Ω_3 is disconnected.

If $r = 1$, then $x_3 = x_2 = a$, so $\mathcal{S}_3^<$ has components in $e^{-i\theta}(a + \mathbb{H})$ and $\mathbb{D}_a$ by Corollary 6.4. Once again $\mathcal{S}_3^< \cap \partial\mathbb{D}_a = \varnothing$, and we conclude that Ω_3 is disconnected. ■

The next proposition now follows directly from Lemma 9.2 and the remarks made immediately before it. In particular the value r_1 is found to be κ_1, and the uniqueness of the critical values κ_1 and κ_2 is seen to be assured even if the word "least" were to be omitted from their descriptions in Definition 9.1.

Proposition 9.3. *For $\theta \in \mathbb{R}$ and $a, b \in \mathbb{Z}$ with $1 \leq a < b$,*

$$\kappa_1 = \frac{b}{a} e^{1-b/a} < 1 \quad \text{and} \quad \kappa_2 \equiv 1.$$

To determine the next critical value κ_3 we again start with a lemma, this time addressing the connectedness property of the region Ω_2 given in Definition 9.1. Before stating the lemma we first establish a needed fact. For fixed $a \geq 1$ let

$$f(x) = \frac{x}{a} e^{a/x-1}.$$

Since $f(a) = 1$ and

$$f'(x) = \left(\frac{1}{a} - \frac{1}{x}\right) e^{a/x-1} > 0$$

for all $x > a$, we have

$$r_3 := \frac{b}{a} e^{a/b-1} > 1$$

whenever $1 \leq a < b$. After the next lemma we will discover in short order that $\kappa_3 = r_3$.

Lemma 9.4. *For $\theta \in [0, \pi]$ let*

$$\Omega_2' = \Omega_2 \cap \{z : 0 < \arg z < 2\pi - \theta\} \quad \text{and} \quad \Omega_2'' = \Omega_2 \cap \{z : -\theta \leq \arg z \leq 0\},$$

so that $\Omega_2 = \Omega_2' \amalg \Omega_2''$. Defining

$$I = \big[- \arccos(1/r_3), \arccos(1/r_3) \big],$$

the following hold.

(1) *Ω_2 is connected if $0 < r < r_3$.*

(2) *Ω_2 is connected if $r \geq r_3$ and $\theta \in I$.*

(3) *Ω_2' and Ω_2'' are connected if $r \geq r_3$ and $\theta \notin I$.*

If $\theta \in [-\pi, 0]$ *the same conclusions hold for regions*

$$\tilde{\Omega}_2' = \Omega_2 \cap \{z : -\theta < \arg z < 2\pi\} \quad \textit{and} \quad \tilde{\Omega}_2'' = \Omega_2 \cap \{z : 0 \le \arg z \le -\theta\}.$$

Before proceeding with the proof the significance of the interval I of θ values bears some mention. It is easy to verify that r_3 is the unique value of r for which $\rho_r = a$, and since ρ_r decreases as r increases, we find that the teardrop-shaped portion of $\overline{\mathcal{S}}_1^<$ lies within the disc $\mathcal{D}_r$ for all $0 < r < r_3$. Figures 10 through 14 of the previous section illustrate this fact, which is wholly independent of the values of the other parameters a, b, and θ. By definition the region Ω_2 is what's left of $\mathcal{D}_r$ after removing both the teardrop-shaped portion of $\overline{\mathcal{S}}_1^<$ as well as the portion of $\overline{\mathcal{S}}_3^<$ that intersects $\mathcal{D}_r$, and part (1) of the lemma states that Ω_2 is connected in the case when $\rho_r > a$ (i.e. when $0 < r < r_3$). In the proof of part (1) we undertake some pains to "firm up" this assertion which might otherwise be passed off as being clear owing to the simple and well-studied nature of the curves involved.

Now, once r attains the value r_3 or exceeds it by a nominal amount (so that some part of the teardrop-shaped portion of $\overline{\mathcal{S}}_1^<$ falls outside $\mathcal{D}_r$) one of two things must occur: either Ω_2 becomes disconnected as in Figure 15, or Ω_2 remains connected as in Figure 18. Which scenario occurs depends on whether the point a (the tip of the teardrop) lies within $\overline{\mathcal{S}}_3^<$ when $\rho_r = a$ (i.e. when $r = r_3$). Given that both figures have $a = 1$, $b = 2$, and $r = 1.25$, it is clear that the parameter that makes the difference is θ. The endpoints of I are in fact the θ values for which the point a on the curve $\mathcal{S}_1$ lies on $\mathcal{S}_3$ when $\rho_r = a$ (i.e. when $r = r_3$). Thus Figure 15 illustrates a $\theta \notin I$ case for r slightly larger than $r_3 \approx 1.213$, with Ω_2' comprising all of Ω_2 save the portion with vertices p_2, q_1 and o_2, which is Ω_2''. If θ is changed from its §8 value of 1 to something smaller such as 0.4, however, we obtain the $\theta \in I$ case shown in Figure 18, where $I \approx [-0.6, 0.6]$ is also depicted. These situations are addressed in parts (2) and (3) of the lemma, and in the corresponding parts of the proof we will appeal much more to the known nature of the relevant curves to make the case for the connectedness of Ω_2 or its subregions Ω_2' and Ω_2'' so as to avoid becoming mired in topological technicalities.

Proof. We assume throughout that $\theta \in [0, \pi]$, as the proof for $\theta \in [-\pi, 0]$ is similar. A simple appeal to periodicity then extends the proof to all real θ.

Proof of (1). Suppose $r \in (0, \kappa_1]$. Then $r < 1$, so that $x_3 \geq b/r$ by the proof of Lemma 9.2 up to (9.3), and thus the point $x_{33} = x_3 e^{-i\theta}$ in Corollary 6.4 is such that $|x_{33}| \geq b/r > a$. Next, referencing parts (2) and (4) of Corollary 6.4, we find that the bounded and unbounded components of $\mathcal{S}_3^<$ lie inside $\mathbb{D}_a$ and outside $\overline{\mathbb{D}}_{b/r}$, respectively, so that $\overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r} = \overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_a$, and hence

$$\Omega_2 = \mathcal{D}_r \setminus \left[\left(\overline{\mathcal{S}}_1^< \cup \overline{\mathcal{S}}_3^<\right) \cap \overline{\mathbb{D}}_a\right]. \tag{9.5}$$

Since $\rho_{r_3} = a$ and ρ_R decreases as R increases, we have $\rho_R > a$ for all $R \in (0, r_3)$, and hence $\rho_r > a$ on account of the fact that $\kappa_1 < 1 < r_3$ by Proposition 9.3 and the remarks after it. This observation, together with (9.5), makes clear that the outer boundary of Ω_2 is simply the circle $\mathcal{C}_r$, while the inner boundary must lie in $\overline{\mathbb{D}}_a$ and is in fact the boundary of the domain $(\mathcal{S}_1^< \cup \mathcal{S}_3^<) \cap \mathbb{D}_a$. Therefore Ω_2 is homeomorphic to an annulus and hence connected.

For $r \in (\kappa_1, 1)$ we of course still have $\rho_r > a$ and $|x_{33}| > a$, so that, as before, the inner boundary of Ω_2 is a simple closed curve in $\overline{\mathbb{D}}_a$. The outer boundary, with no further analysis, is either $\mathcal{C}_r$ again, or else (as happens to be the case) is the portion of $\mathcal{C}_r$ lying outside $\mathcal{S}_3^<$ together with the portion of the unbounded component of $\mathcal{S}_3$ lying in $\mathcal{D}_r$. This results in a "dented" circle as in Figure 12, with the dent lying outside $\overline{\mathbb{D}}_a$ by Corollary 6.4(2). Again Ω_2 is homeomorphic to an annulus and so connected.

In the $r = 1$ case Corollary 6.4 informs us that $\mathcal{S}_3$ becomes connected, and in fact is the well-known curve $\mathcal{S}_1$ rotated by $-\theta$ about the origin (i.e. $\mathcal{S}_3 = e^{-i\theta}\mathcal{S}_1$) as in Figure 13; and since $\rho_1 > a$, removing $\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ and $\overline{\mathcal{S}}_3^<$ from $\mathcal{D}_1$ results in a set that is connected. Therefore Ω_2 is connected.

Suppose $r \in (1, r_3)$, so $\rho_r > a$ still holds, and thus the connectedness of $\mathcal{D}_r \setminus (\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a)$ is assured. By Corollary 6.4 the curve $\mathcal{S}_3$ is simple as well as connected, with $\mathcal{S}_3^<$ an unbounded connected and simply connected domain that is symmetrical about the line $t \mapsto te^{-i\theta}$ and contains the origin, as in Figure 14. These observations, together with Proposition 7.4, make clear that $\mathcal{S}_3$ intersects the circle $\mathcal{C}_r$ at least at the points q_1 and

q_2, with $q_1 \neq q_2$ in particular. Indeed, carrying out calculations like those in the proof of Proposition 7.4, we find the system of equations $|\varphi(re^{i\theta}z/a)| = r$, $|z| = \rho_r$ can have no more than two solutions for z, so $\mathcal{S}_3$ must pass once into $\mathcal{D}_r$ at q_1 and pass once out at q_2, with the implication that removal of $\overline{\mathcal{S}}_3^<$ from $\mathcal{D}_r$ will not disconnect the disc. Indeed $\mathcal{D}_r \setminus \overline{\mathcal{S}}_3^<$ is also simply connected since $\mathcal{D}_r \subseteq \overline{\mathbb{D}}_{b/r}$ on account of the equation $\rho_r = b/r$ having unique solution $r = \kappa_1 < 1$, with $\rho_r < b/r$ for $r > \kappa_1$. Next, $\mathcal{S}_1 \cap \mathcal{S}_3 \cap \overline{\mathbb{D}}_a$ consists of the two distinct points p_1 and p_2 by Proposition 7.1 and Lemma 7.2, so it may be reasonably surmised—and will soon be shown—that the simple curve $\mathcal{S}_3$ passes once into the domain $\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ (convex by Corollary 6.2) at p_1, and then passes once out at p_2. If this is the case, then removal of $\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ from the connected and simply connected region $\mathcal{D}_r \setminus \overline{\mathcal{S}}_3^<$ results in a connected set that is, by definition, Ω_2.

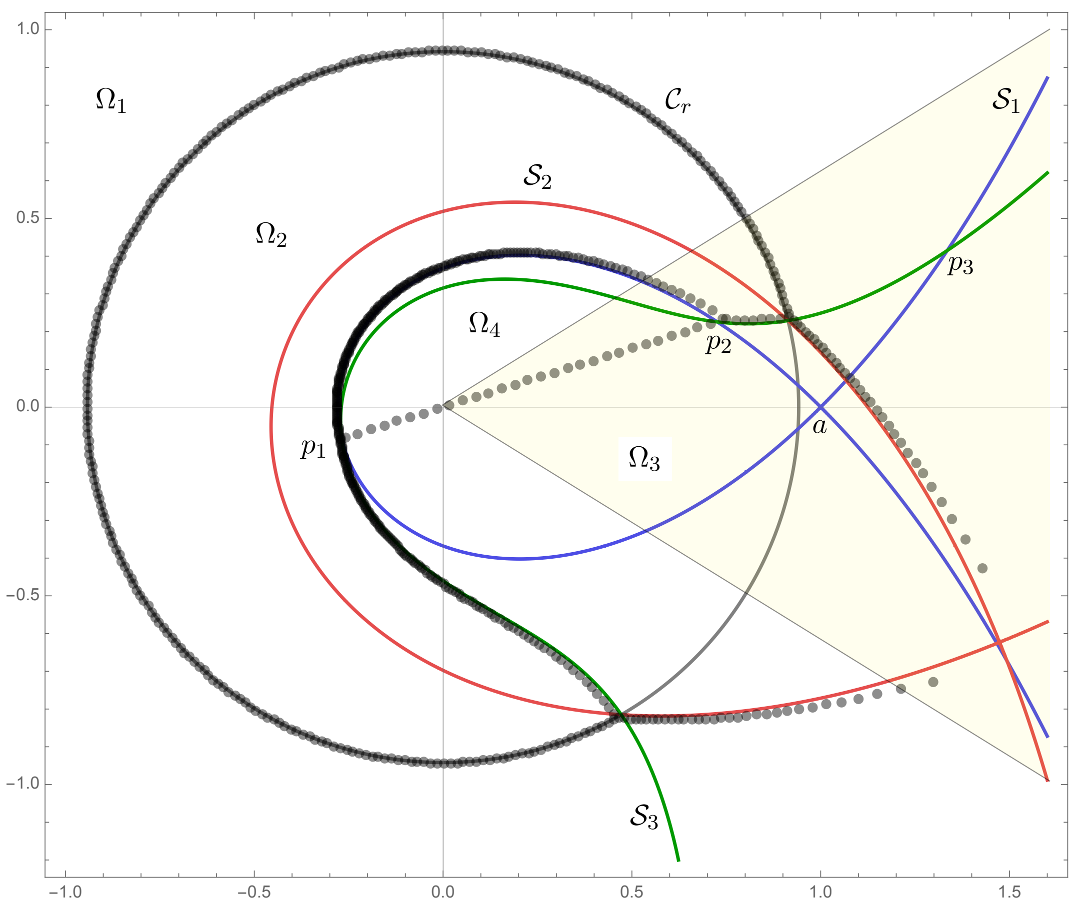

FIGURE 18. An $r \in (r_3, \infty)$, $\theta \in I$ case illustration with zeros of $P_{340}(z)$ and $(a, b, r, \theta) = (1, 2, 1.25, 0.4)$. Sector $I \approx [-0.6, 0.6]$ lies between the two rays in $\mathbb{H}$.

It remains to verify our "reasonable surmisal"; that is, we must show that $\mathcal{S}_3$ passes properly into and out of the interior of $\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ via the points p_1 and p_2, rather than "bounce" off the boundary of the region at these points and thereby cause a pocket of Ω_2 to be isolated from the rest. Working in $\mathbb{R}^2$, the leftmost point on the curve $\mathcal{S}$ of Proposition 6.1 that lies on the x-axis is $(x_1, 0)$. Recalling that $\mathcal{S}_3$ is the curve $\mathcal{S}$ of Proposition 6.1 rotated about the origin by $-\theta$, we wish to show that $(x_1, 0)$ is in fact the leftmost point on all of $\mathcal{S}$, and there are no other "local" leftmost points, which in light of the smoothness of $\mathcal{S}$ for $r > 1$ must necessarily be points where the curve has a vertical tangent line. If $\hat{\mathcal{S}}$ is $\mathcal{S}$ rotated by $-\pi/2$, then our objective is met if we show that $(0, -x_1)$ is the unique global maximum point on $\hat{\mathcal{S}}$, and there are no other local maxima (necessarily points where the tangent line is horizontal). The equation for $\hat{\mathcal{S}}$ in $\mathbb{C}$ is $|\varphi(irz/a)| = r$, which in $\mathbb{R}^2$ may be rendered as

$$\sqrt{x^2 + y^2} = ae^{-ry/a-1} \tag{9.6}$$

via manipulations like those in (6.8). Squaring both sides of (9.6) and differentiating with respect to x readily yields

$$y' = -\frac{x}{y + are^{-2ry/a-2}},$$

and thus $y' = 0$ if and only if $x = 0$. Setting $x = 0$ in (9.6) gives $|y|e^{ry/a}/a = 1/e$ after a bit of algebra, and hence

$$\frac{ry}{a}e^{ry/a} = \pm\frac{r}{e}.$$

Recalling that $E(x) = xe^x$ has range $[-1/e, \infty)$, the equation $E(ry/a) = -r/e$ has no solution for $r > 1$. The equation $E(ry/a) = r/e$, on the other hand, has solution $y = (a/r)W_0(r/e) = -x_1$. The point $(0, -x_1)$ on $\hat{\mathcal{S}}$ must be a unique (global) maximum, for Proposition 6.1 makes clear that $\hat{\mathcal{S}}$ is bounded above and unbounded below. (Of course $(0, -x_1)$ on $\hat{\mathcal{S}}$ corresponds to $(x_1, 0)$ on $\mathcal{S}$.) This implies a shape for $\mathcal{S}_3$ such that the curve must intersect $\mathcal{S}_1$ transversally at p_1 and p_2, thereby precluding any possibility of $\mathcal{S}_3$ being disjoint from the interior of $\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ for $r \in (1, r_3)$.

Proof of (2). Fix $r \geq r_3$ and $\theta \in I$, so in particular $0 \leq \theta \leq \arccos(1/r_3)$ since $\theta \in [0, \pi]$. Suppose $z \in \mathcal{D}_r$ is such that $-\theta \leq \arg z \leq 0$, so $z = se^{it}$ for some $s > 0$ and $t \in [-\theta, 0]$.

Let $f(x) = \ln(ex/a)$ and $g(x) = x/a$. For $E(x) = xe^x$, we find $f(x) = g(x)$ implies $E(-x/a) = -1/e$, and because $E(x)$ has a global minimum at $E(-1) = -1/e$, it follows that $-x/a = -1$ and thus $x = a$ is the only solution. Now, since $f(a/2) < g(a/2)$ and $f(2a) < g(2a)$, the intermediate value theorem makes clear that $f(x) < g(x)$ for all $x \in (0,a) \cup (a,\infty)$, so that

$$\ln\left(\frac{es}{a}\right) \leq \frac{s}{a} = \frac{sr_3}{b}e^{1-a/b} \leq \frac{sr}{b}e^{1-a/b},$$

and hence

$$\frac{a}{sr}\ln\left(\frac{es}{a}\right) \leq \frac{1}{r_3}. \tag{9.7}$$

On the other hand $t + \theta \in [0, \theta]$ implies

$$\cos(t+\theta) \geq \cos\theta \geq \cos\left(\arccos\frac{1}{r_3}\right) = \frac{1}{r_3}.$$

Taken together our findings indicate that

$$\cos(t+\theta) \geq \frac{a}{sr}\ln\left(\frac{es}{a}\right),$$

which leads to

$$\left|e^{sre^{i(\theta+t)}}\right| \geq \left(\frac{es}{a}\right)^a,$$

and finally

$$\left|\varphi\left(\frac{re^{i\theta}z}{a}\right)\right| \leq r.$$

That is, $z \in \overline{\mathcal{S}}_3^<$.

In the proof of part (1) we noted that $\rho_r < b/r$ for $r > \kappa_1$. Here $r \geq r_3$ by hypothesis, and since $r_3 > \kappa_1$ it follows that $\mathcal{D}_r \subseteq \mathbb{D}_{b/r}$ and hence $z \in \mathbb{D}_{b/r}$. Now we have $z \in \overline{\mathcal{S}}_3^< \cap \mathbb{D}_{b/r}$, implying $z \notin \Omega_2$ and more to the point $z \notin \Omega_2''$. Therefore $\Omega_2'' = \varnothing$, and so $\Omega_2 = \Omega_2'$. That Ω_2' is connected we take to be clear from the nature of the curves $\mathcal{S}_1$ and $\mathcal{S}_3$, noting as in the proof of part (1) that $\mathcal{S}_3$ intersects the teardrop-shaped portion of $\mathcal{S}_1$ at precisely two points.

Proof of (3). Fix $r \geq r_3$ and $\theta \notin I$. That Ω_2' is connected follows from the same observations made in the proof of part (2). As for Ω_2'', its connectedness is evident owing to its boundary being a simple closed curve that is the piecewise-smooth joining of portions of $\mathcal{S}_1$, $\mathcal{S}_3$, and the circle $\mathcal{C}_r$, with vertices being one of the points q_j, one of the points o_j, and point p_2 (as in Figure 15).

If $r = r_3$ in particular, so that $a \in \mathcal{C}_r$, Propositions 7.4 and 7.5 show the q_j and o_j points lie on $\mathcal{C}_r$ (so $|q_j| = |o_j| = a$), while $|p_2| < a$ by Lemma 7.2, and thus $\Omega_2'' \neq \varnothing$. Moreover Ω_2'' must be connected, for otherwise there must be at least one additional point of intersection amongst $\mathcal{C}_r$, $\mathcal{S}_1$, and $\mathcal{S}_3$ beyond those which were found to exist in §7. ■

The first two parts of Lemma 9.4 imply that κ_3 and κ_4 do not exist for any $\theta \in I$. For $\theta \notin I$ the next proposition informs us that κ_3 is none other than r_3.

Proposition 9.5. *If $\theta \notin I$, then*

$$\kappa_3 = \frac{b}{a} e^{a/b-1} > 1.$$

Proof. Fix $\theta \notin I$, assuming $\theta \in [0, \pi]$ for definiteness. In light of Lemma 9.4(1) it will suffice to show that Ω_2 is disconnected when $r = r_3$.

Since $\cos\theta \neq 1/r_3$ and $\rho_{r_3} = a$, Lemma 7.2 implies $|p_2| < \rho_{r_3}$, and so the situation is nearly as in Figure 15 except the circle $\mathcal{C}_{r_3}$ contains the point a. The boundary of Ω_2'' must therefore be the simple closed curve consisting of the piece of $\mathcal{S}_1$ in $\mathcal{H}^+_{r_3\theta}$ from a to p_2, the piece of $\mathcal{S}_3$ in $\mathcal{H}^-_{r_3\theta}$ from p_2 to q_1, and the piece of $\mathcal{C}_{r_3}$ in the lower half-plane from q_1 to a. The points p_2, q_1, and a are readily found to be distinct in the present setting by computing their coordinates using the formulas furnished in §7, and therefore $\Omega_2'' \neq \varnothing$. That Ω_2' is nonempty can be verified by showing that it contains $ai/2$. Since

$$\left|\varphi\left(\frac{ai/2}{a}\right)\right| = \frac{1}{2}\left|e^{1-i/2}\right| = \frac{e}{2} > 1,$$

we have $ai/2 \in \mathcal{S}_1^>$; and since, for $r = r_3$,

$$\theta \in [0,\pi] \;\;\Rightarrow\;\; -\frac{r}{2}\cos(\theta+\pi/2) \geq 0 \;\;\Rightarrow\;\; e^{-(r/2)\cos(\theta+\pi/2)} \geq 1,$$

so that

$$\left|\varphi\left(\frac{re^{i\theta}(ai/2)}{a}\right)\right| = \frac{er}{2}\left|e^{-rie^{i\theta}/2}\right| = \frac{er}{2}e^{-(r/2)\cos(\theta+\pi/2)} > r,$$

we also have $ai/2 \in \mathcal{S}_3^>$. Clearly $ai/2 \in \mathcal{D}_{r_3} = \mathbb{D}_a$, and so $ai/2 \in \Omega_2'$.

On the other hand, $[0,\infty) \subseteq \overline{\mathcal{S}}_1^<$ by Corollary 6.2(3) and $e^{-i\theta}[0,\infty) \subseteq \overline{\mathcal{S}}_3^<$ by Corollary 6.4(5), so that $[0,\infty) \cap \Omega_2 = \varnothing$ since $\mathcal{D}_r = \mathbb{D}_a$, and $e^{-i\theta}[0,\infty) \cap \Omega_2 = \varnothing$ also. Because the nonempty regions Ω_2' and Ω_2'' are each subsets of one of the two sectors formed by the rays $[0,\infty)$ and $e^{-i\theta}[0,\infty)$, we conclude that Ω_2 is disconnected. Therefore $\kappa_3 = r_3$, and it was already shown that $r_3 > 1$. A symmetrical argument leads to the same conclusion if $\theta \in [-\pi, 0]$, and so the conclusion holds for any $\theta \notin I$. ■

Lemma 9.6. *Define r_4 to be the smallest value of r for which $\rho_r = |p_2|$. Then r_4 exists for any θ, with $r_4 > \kappa_3$ whenever $\theta \notin I$. Moreover r_4 is the only solution to $\rho_r = |p_2|$ if* $\cos\theta \leq 0$.

Proof. First, $\theta \notin I$ implies $\cos\theta \neq 1/\kappa_3$, so for $r = \kappa_3$ we have $|p_2| < a$ by Lemma 7.2 while $\rho_r = a$. Since ρ_r decreases as r increases, and $|p_2| \leq a$ in any case, it's clear that $r_4 > \kappa_3$ if r_4 exists.

As for existence, $\rho_r = |p_2|$ is equivalent for any θ to

$$\frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} = -\frac{a}{\cos\ell_{r\theta}} W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right)$$

by (6.4) and Lemma 7.2, which with the property $W_0(x) = xe^{-W_0(x)}$ becomes

$$\frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} = -\frac{a}{\cos\ell_{r\theta}}\left(-\frac{\cos\ell_{r\theta}}{e} e^{-W_0(-e^{-1}\cos\ell_{r\theta})}\right),$$

and hence

$$W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right) = \frac{b}{b-a}\ln\left(\frac{ar}{b}\right). \tag{9.8}$$

The equation (9.8) can be shown to always have at least one solution for r in terms of the parameters a, b, and θ. Define

$$f_\theta(r) = W_0\left(-\frac{\cos\ell_{r\theta}}{e}\right) \quad \text{and} \quad \lambda(r) = \frac{b}{b-a}\ln\left(\frac{ar}{b}\right). \tag{9.9}$$

Noting that

$$0 < r < 1 \;\Rightarrow\; 2r < r^2 + 1 \;\Rightarrow\; r^2 - 2r\cos\theta + 1 > 0,$$

use of the identity (6.6) yields

$$\lim_{r\to 0^+} \cos \ell_{r\theta} = \lim_{r\to 0^+} \frac{r|\sin\theta|}{\sqrt{r^2 - 2r\cos\theta + 1}} = 0,$$

and so $f_\theta(r) \to W_0(0) = 0$ as $r \to 0^+$. Since $\lambda(r) \to -\infty$ as $r \to 0^+$ there exists some $r' > 0$ such that $\lambda(r') < f_\theta(r')$. On the other hand $f_\theta(r) \to W_0(-|\sin\theta|/e) \in [-1, 0]$ and $\lambda(r) \to \infty$ as $r \to \infty$, and so $\lambda(r'') > f_\theta(r'')$ for some $r'' > 0$. The intermediate value theorem now implies there exists some $r > 0$ such that $f_\theta(r) = \lambda(r)$, and therefore (9.8) has at least one solution for *any* $\theta \in \mathbb{R}$. The total number of solutions must be finite, and so we may further conclude that there is a smallest solution. This proves the existence of r_4.

Before commencing the uniqueness inquiry we note that, because the possibility that $\cos\theta = 1$ when $r = 1$ is excluded throughout this section, the inequality

$$r^2 - 2r\cos\theta + 1 > 0 \tag{9.10}$$

holds for all $r > 0$.

Suppose $\cos\theta \leq 0$. Since

$$W_0'(x) = \frac{1}{x + e^{W_0(x)}}$$

when $x > -1/e$, for $r > 0$ such that $\cos \ell_{r\theta} < 1$ we obtain

$$f_\theta'(r) = \frac{1}{\cos \ell_{r\theta} - e^{f_\theta(r)+1}} \cdot \frac{(1 - r\cos\theta)|\sin\theta|}{(r^2 - 2r\cos\theta + 1)^{3/2}} \tag{9.11}$$

after some simplification. In fact, $\cos \ell_{r\theta} < 1$ is assured whenever $\cos\theta \leq 0$, since

$$\begin{aligned}
\cos \ell_{r\theta} = 1 \quad &\Leftrightarrow \quad \frac{r^2 \sin^2\theta}{r^2 - 2r\cos\theta + 1} = 1 \\
&\Leftrightarrow \quad \cos^2\theta - \frac{2}{r}\cos\theta + \frac{1}{r^2} = 0 \\
&\Leftrightarrow \quad \left(\cos\theta - \frac{1}{r}\right)^2 = 0.
\end{aligned}$$

and yet $\cos\theta = 1/r$ is impossible. Moreover, since $\cos\ell_{r\theta} \geq 0$ is clear from (6.6), we find that

$$\frac{1}{r} \neq \cos\theta \;\Rightarrow\; -\frac{1}{e} < -\frac{\cos\ell_{r\theta}}{e} \leq 0 \;\Rightarrow\; -1 < f_\theta(r) \leq 0 \;\Rightarrow\; e^{f_\theta(r)+1} \in (1, e],$$

and so the denominator of the first fraction in (9.11) is always negative, whereas the denominator of the second fraction is always positive in light of (9.10). We conclude that $f'_\theta(r) < 0$ for all $r > 0$ whenever $\cos\theta \in (-1, 0]$, so that f_θ is decreasing on $(0, \infty)$ while λ is increasing, and the uniqueness of the solution to $f_\theta(r) = \lambda(r)$ is clear. When $\cos\theta = -1$ we of course have $\sin\theta = 0$, so $f_\theta(r) \equiv 0$ and again (9.8) must have a unique solution. ■

Difficulties arise when attempting to prove the uniqueness of the solution to (9.8) when $\cos\theta > 0$, as the factor $1 - r\cos\theta$ in (9.11) may then be negative. One mitigating factor is that the sector of θ values I, and all angles coterminal with the elements of I, comprise a subset of the θ values for which $\cos\theta > 0$, and so are of no concern to us. Nonetheless this leaves us with the symmetrical sectors $(-\pi/2, -\arccos(1/r_3))$ and $(\arccos(1/r_3), \pi/2)$ to worry about. At the end of this section we will push the uniqueness argument at least partway into these sectors, though a full resolution to the problem remains elusive. For this reason, and for the sake of simplicity, it will occasionally be assumed that $\cos\theta \leq 0$.

Proposition 9.7. *Suppose* $\cos\theta \leq 0$. *Then* κ_4 *is the unique solution to* $\rho_r = |p_2|$. *Moreover* $\kappa_4 > \kappa_3$ *always holds, with* $\kappa_4 = b/a$ *if* $\theta = (2k+1)\pi$ *for any* $k \in \mathbb{Z}$.

Proof. Assume for definiteness that $\theta \in [\pi/2, \pi]$. By Lemma 9.6 and its proof, $|p_2| < \rho_r$ for all $\kappa_3 < r < r_4$, and thus Ω_2 is disconnected by the same argument given in the proof of Proposition 9.5. When $r = r_4$, however, we have $\rho_r = |p_2|$, and in fact $p_2 = q_2 = o_2$ by Propositions 7.1, 7.4 and 7.5, and we conclude that $\Omega''_2 = \varnothing$. But since $\theta \in [\pi/2, \pi]$, when $r = r_4$ it can still be shown as in the proof of Proposition 9.5 that $ai/2 \in \Omega'_2$, and because Ω'_2 is connected by Lemma 9.4, it follows that $\Omega_2 = \Omega'_2 \amalg \Omega''_2$ is connected. Hence Ω_2 is disconnected for $r \in (\kappa_3, r_4)$ and connected for $r = r_4$, so that $\kappa_4 = r_4$ by Definition 9.1, and therefore κ_4 is the unique solution to $\rho_r = |p_2|$.

A similar argument can be made for the case $\theta \in [-\pi, -\pi/2]$, for all curves involved merely reflect about the real axis, and so the conclusion that κ_4 is the unique solution to $\rho_r = |p_2|$ holds for all θ such that $\cos\theta \leq 0$. That $\kappa_4 > \kappa_3$ is immediate from Lemma 9.6.

Next assume $\theta = (2k+1)\pi$. Then $|p_2| = a/e$ by Proposition 7.1, and so κ_4 is the unique solution to $\rho_r = a/e$. Solving this equation directly gives $\kappa_4 = b/a$. ■

Lemma 9.8. *For any fixed θ the value of r for which $\rho_r = |p_1|$ is the unique solution to*

$$W_0\left(\frac{\cos \ell_{r\theta}}{e}\right) = \frac{b}{b-a} \ln\left(\frac{ar}{b}\right). \tag{9.12}$$

Proof. Define $g_\theta(r) = W_0(e^{-1}\cos\ell_{r\theta})$ and let $\lambda(r)$ be as in (9.9). That $g_\theta(r) = \lambda(r)$, which is (9.12), is equivalent to $\rho_r = |p_1|$ is ascertained in the same manner that $\rho_r = |p_2|$ was found to be equivalent to (9.8). Also, nearly the identical argument that proved the existence of a solution to (9.8) for any $\theta \in \mathbb{R}$ will show the same for (9.12), only $g_\theta(r) \to W_0(|\sin\theta|/e)$ as $r \to \infty$.

The proof of the uniqueness of a solution to $g_\theta(r) = \lambda(r)$ will be carried out assuming $\theta \in [0, \pi]$. Then, since g_θ is unchanged when θ is replaced by $-\theta$, the proof will apply to all $\theta \in [-\pi, \pi]$. Finally, since g_θ is unchanged when θ is replaced by $\theta + 2\pi k$ for any $k \in \mathbb{Z}$, the proof will apply to all $\theta \in \mathbb{R}$.

The situation is especially simple if $\theta = 0$ or $\theta = \pi$, for then $\cos\ell_{r\theta} = 0$ and (9.12) becomes $\ln(ar/b) = 0$, which clearly has the unique solution $r = b/a$.

Assume $\theta \in (0, \pi)$, so that $\cos\ell_{r\theta} \in (0, 1]$. For $r \in (0, b/a]$ we have $\lambda(r) \leq 0$, whereas $e^{-1}\cos\ell_{r\theta} \in (0, e^{-1}]$ implies $g_\theta(r) > 0$. Thus $g_\theta(r) = \lambda(r)$ has no solution on $(0, b/a]$. Suppose there exist r', r'' such that $r'' > r' > b/a$, $g_\theta(r') = \lambda(r')$, and $g_\theta(r'') = \lambda(r'')$. Then by Rolle's Theorem there is some $r \in (r', r'')$ such that $(g_\theta - \lambda)'(r) = 0$, or equivalently

$$\frac{1}{e^{-1}\cos\ell_{r\theta} + e^{g_\theta(r)}} \cdot \frac{1}{2e}\left(\frac{r^2\sin^2\theta}{r^2 - 2r\cos\theta + 1}\right)^{-1/2} \frac{(2r\sin^2\theta)(1 - r\cos\theta)}{(r^2 - 2r\cos\theta + 1)^2} = \frac{b}{(b-a)r}.$$

Some algebra leads to

$$\frac{1}{\cos\ell_{r\theta} + e^{g_\theta(r)+1}} \cdot \frac{(1 - r\cos\theta)\sin\theta}{(r^2 - 2r\cos\theta + 1)^{3/2}} = \frac{b}{(b-a)r}, \tag{9.13}$$

which cannot be satisfied if $r\cos\theta \geq 1$ since the left-hand side would then be either zero or negative, contradicting the assumption that $1 \leq a < b$.

Assume $r\cos\theta < 1$. From (9.13) we have

$$\left[\cos\ell_{r\theta} + e^{g_\theta(r)+1}\right] \cdot \frac{(r^2 - 2r\cos\theta + 1)^{3/2}}{(1 - r\cos\theta)\sin\theta} = \left(1 - \frac{a}{b}\right)r,$$

which with (6.6) becomes

$$\frac{r(r^2 - 2r\cos\theta + 1)}{1 - r\cos\theta} + \frac{(r^2 - 2r\cos\theta + 1)^{3/2}e^{g_\theta(r)+1}}{(1 - \cos\theta)\sin\theta} = \left(1 - \frac{a}{b}\right)r. \tag{9.14}$$

Now, because $r > b/a > 1$, we find that

$$\frac{r(r^2 - 2r\cos\theta + 1)}{1 - r\cos\theta} \geq \frac{r(1 - 2r\cos\theta + 1)}{1 - r\cos\theta} = 2r,$$

and hence the left-hand side of (9.14) is greater than $2r$. However the right-hand side is less than r, so the equation cannot be satisfied if $r\cos\theta < 1$.

It is now clear that $(g_\theta - \lambda)'(r) = 0$ cannot be satisfied for any $r > 0$, so that the equation $g_\theta(r) = \lambda(r)$ must have at most one solution on $(b/a, \infty)$ whenever $\theta \in (0, \pi)$. Therefore any solution to (9.12) on $(0, \infty)$ must be unique for any $\theta \in [0, \pi]$. ■

Proposition 9.9. *Let m_1 be the point on $\mathcal{S}_1$ with minimum modulus and let $k \in \mathbb{Z}$. Then*

$$\kappa_5 \in \{r : |m_1| \leq \rho_r \leq |p_1|\} \tag{9.15}$$

for all $\theta \in \mathbb{R}$, with the inequalities being sharp. Moreover the following hold:

(1) $\kappa_5 > \kappa_4$ *if* $\cos\theta \leq 0$ *and* $\theta \neq (2k+1)\pi$.

(2) $\kappa_5 = \kappa_4 = b/a$ *if* $\theta = (2k+1)\pi$.

(3) $\kappa_5 > 1$ *for all* θ.

In any case

$$\kappa_5 \leq \frac{b}{a}\left[eW_0\left(\frac{1}{e}\right)\right]^{a/b-1} := r_5. \tag{9.16}$$

Proof. We first lay down a few needed facts. In the proof of Lemma 7.3 it was established that $m_1 = -aW_0(1/e)$, while the lemma itself showed r_5 is the unique value of r for which $\rho_r = |m_1|$. Also $|p_1| = a/e$ for $\theta = k\pi$ by Proposition 7.1, and since $E(t) = te^t$ is increasing on $(-1,\infty)$,

$$1/e < (1/e)e^{1/e} \;\;\Leftrightarrow\;\; E(W_0(1/e)) < E(1/e) \;\;\Leftrightarrow\;\; W_0(1/e) < 1/e,$$

so $|m_1| = aW_0(1/e) < a/e = |p_1|$ whenever $\theta = k\pi$.

For fixed θ suppose $r \geq r_5$. Then $\rho_r \leq |m_1|$, and so $z \in \mathcal{D}_r$ implies $|z| < |m_1|$. Now, since

$$e^{|z|/a} \geq e^{|\operatorname{Re}(z/a)|} = e^{|-\operatorname{Re}(z/a)|} \geq e^{-\operatorname{Re}(z/a)} = |e^{-z/a}|,$$

we find that

$$|z| < |m_1| \;\;\Rightarrow\;\; \frac{|z|}{a} < W_0(1/e) \;\;\Rightarrow\;\; \frac{|z|}{a}e^{|z|/a} < \frac{1}{e} \;\;\Rightarrow\;\; \frac{|z|}{a}|e^{-z/a}| < \frac{1}{e},$$

so $|\varphi(z/a)| < 1$ and hence $z \notin \Omega_2$. Since $\Omega_2 \subseteq \mathcal{D}_r$ by definition, it follows that $\Omega_2 = \varnothing$ for any $r \geq r_5$, thereby affirming (9.16) and making clear that no $r > r_5$ can be in the range of κ_5. Given ρ_r decreases as r increases, it follows that no r for which $\rho_r < |m_1|$ can be in the interval (9.15).

We now show that $\kappa_5 = r_5$ when $\theta = 0$, so that r_5 must be the least upper bound on the range of κ_5, and the first inequality in the interval (9.15) is sharp. For any $r > 0$ define $m_3 = -(a/r)W_0(r/e)$, and suppose $x < m_3$. Since $x < 0$ and $E(t) = te^t$ is increasing on $(0,\infty)$,

$$\begin{aligned} x < m_3 \;\;&\Rightarrow\;\; -\frac{r}{a}x > W_0\Big(\frac{r}{e}\Big) \;\;\Rightarrow\;\; E\Big(-\frac{r}{a}x\Big) > E\Big(W_0\Big(\frac{r}{e}\Big)\Big) \\ &\Rightarrow\;\; -\frac{rx}{a}e^{-rx/a} > \frac{r}{e} \;\;\Rightarrow\;\; -\frac{rx}{a}e^{1-rx/a} > r, \end{aligned}$$

so that $|\varphi(rx/a)| > r$ and we have $x \in \mathcal{S}_3^{>}$ when $\theta = 0$. Assuming also that $x < m_1$, the same chain of implications with r replaced by 1 shows that $|\varphi(x/a)| > 1$ and hence $x \in \mathcal{S}_1^{>}$. Now, it's readily shown that $r_3 > 1$ as defined in Lemma 9.4 is the unique r value for which $\rho_r = a$, with $\rho_r > a$ if $r < r_3$ and $\rho_r < a$ if $r > r_3$. Since $-a < m_1$ and $-a < m_3$ by Corollary 6.2, we find $-\rho_r \leq -a < x_m := \min\{m_1, m_3\}$ for $r \in (0, r_3]$, and hence $(-\rho_r, x_m) \neq \varnothing$ for $r \leq 1$ with $(-\rho_r, x_m) \subseteq \Omega_2$. That is, $\Omega_2 \neq \varnothing$ for $r \leq 1$.

The case when $\theta = 0$ and $r > 1$ we handle separately. Define $f(t) = W_0(t/e)$ and $g(t) = tW_0(1/e)$. Clearly $f(1) = g(1)$, and so suppose $f(t_0) = g(t_0)$ for some

$t_0 > 1$. By Rolle's Theorem there exists $1 < t < t_0$ such that $f'(t) = g'(t)$. Then since $eW_0(1/e) = e^{-W_0(1/e)}$,

$$f'(t) = g'(t) \;\; \Rightarrow \;\; \frac{1/e}{t/e + e^{W_0(t/e)}} = W_0(1/e) \;\; \Rightarrow \;\; \frac{1}{t/e + e^{W_0(t/e)}} = e^{-W_0(1/e)},$$

so that

$$\frac{t}{e} + e^{W_0(t/e)} = e^{W_0(1/e)},$$

and hence $W_0(t/e) < W_0(1/e)$. This is a contradiction, as W_0 is known to be strictly increasing on $(0, \infty)$, and so $f(t) \neq g(t)$ for all $t > 1$. Continuity considerations and the observation that $f(2) < g(2)$ then establish that $f(r) < g(r)$ for all $r > 1$. Now,

$$f(r) < g(r) \;\; \Rightarrow \;\; W_0\left(\frac{r}{e}\right) < rW_0(1/e) \;\; \Rightarrow \;\; -\frac{a}{r}W_0\left(\frac{r}{e}\right) > -aW_0\left(\frac{1}{e}\right),$$

and so $m_1 < m_3$ for all $r > 1$. Now, with Lemma 7.3 we find that $-\rho_r < m_1 = x_m$ for $1 < r < r_5$, and so once again $(-\rho_r, x_m) \neq \varnothing$ with $(-\rho_r, x_m) \subseteq \Omega_2$. That is $\Omega_2 \neq \varnothing$ for $1 < r < r_5$, and hence for all $r < r_5$. Recalling that $\Omega_2 = \varnothing$ for all $r \geq r_5$, we conclude $\kappa_5 = r_5$ when $\theta = 0$.

The full verification of (9.15) is still a few steps away, but the next step is achieved by establishing statement (2) in the proposition. In the proof of Proposition 9.7 we found that κ_4 is the smallest value of r for which $\Omega_2'' = \varnothing$ for any $\theta \notin I$, and so the symmetry of the regions Ω_2' and Ω_2'' about the real axis when $\theta = k\pi$ implies that κ_4 is the smallest r value for which $\Omega_2 = \varnothing$ when $\theta = \pi$. Thus $\kappa_5 = \kappa_4 = b/a$ for any $\theta = (2k+1)\pi$ by Definition 9.1 and Proposition 9.7, and since $|p_1| = |p_2|$ for any $\theta = k\pi$ by Lemma 7.2, when $\theta = \pi$ we find κ_5 to be the r value for which $\rho_r = |p_1|$.

Next we show that if r is such that $\rho_r > |p_1|$ then $\kappa_5 \neq r$ for any θ, thereby establishing not only that the range of κ_5 must lie in the interval (9.15), but also that the second inequality in the interval's definition is sharp. Lemma 9.8 establishes that there is a unique r value, say r', for which $\rho_r = |p_1|$. Supposing that $r \in (0, r')$, then $\rho_r > |p_1|$ and there exists some $t > 1$ such that $tp_1 \in \mathcal{D}_r$. Since $|\varphi(p_1/a)| = 1$ and $|\varphi(re^{i\theta}p_1/a)| = r$, with

Lemma 7.2(1) we obtain

$$\left|\varphi\!\left(\frac{tp_1}{a}\right)\right| = \frac{t|p_1|}{a}\left|e^{1-tp_1/a}\right| = t\left(\frac{e|p_1|}{a}\right)^{1-t}\left|\varphi\!\left(\frac{p_1}{a}\right)\right|^t = t\left(\frac{a/e}{|p_1|}\right)^{t-1} \geq t > 1,$$

and

$$\begin{aligned}\left|\varphi\!\left(\frac{re^{i\theta}tp_1}{a}\right)\right| &= \frac{rt|p_1|}{a}\left|e^{1-re^{i\theta}tp_1/a}\right| \\ &= t\!\left(\frac{r|p_1|}{a}\right)^{1-t}\left(\frac{r|p_1|}{a}\right)^t\left|e^{t-re^{i\theta}tp_1/a}\right|e^{1-t} \\ &= t\left(\frac{er|p_1|}{a}\right)^{1-t}\left[\frac{r|p_1|}{a}\left|e^{1-re^{i\theta}p_1/a}\right|\right]^t \\ &= t\left(\frac{er|p_1|}{a}\right)^{1-t}\left|\varphi\!\left(\frac{re^{i\theta p_1}}{a}\right)\right|^t \\ &= tr^t\!\left(\frac{er|p_1|}{a}\right)^{1-t} = rt\left(\frac{a/e}{|p_1|}\right)^{t-1} > r.\end{aligned}$$

Thus $tp_1 \in \Omega_2$, so that $\Omega_2 \neq \varnothing$ and we conclude that $r \neq \kappa_5$ for any θ. The proof of (9.15) is done.

It remains to verify the statements (1) and (3) in the proposition. Suppose θ is such that $\cos\theta \leq 0$ and $\theta \neq (2k+1)\pi$. Then κ_4 is the unique solution to (9.8) by Proposition 9.7, and we let r_5' be the unique solution to (9.12). By Lemmas 7.2, 9.6, and 9.8,

$$\rho_{r_5'} = |p_1| < |p_2| = \rho_{\kappa_4},$$

and thus $r_5' > \kappa_4$ since $r \mapsto \rho_r$ is a decreasing function. As we found earlier, κ_5 is an r value for which $\rho_r \leq |p_1|$, implying $\rho_{\kappa_5} \leq \rho_{r_5'}$, and hence $r_5' \leq \kappa_5$. Therefore $\kappa_4 < \kappa_5$, and statement (1) is proven.

Finally, for fixed θ suppose r is such that $\rho_r = |p_1|$. In the proof of Lemma 9.8 it was found that (9.12) has no solution on $(0, b/a]$, and so $r \geq b/a > 1$. Now, from (9.15) we have $\rho_{\kappa_5} \leq |p_1| = \rho_r$, whence $\kappa_5 \geq r$ obtains, and we conclude that $\kappa_5 > 1$ for any θ. Statement (3) is proven. ■

If r_5' is the unique solution to (9.12) for a given value of θ, Proposition 9.9 states that κ_5 lies in the interval $[r_5', r_5]$. In particular, as we have just seen, $\kappa_5 = r_5$ for $\theta = 0$ and

$\kappa_5 = r_5'$ for $\theta = \pi$. That is, when $\theta = 0$, κ_5 is that value of r for which the circle $\mathcal{C}_r$ contains the point m_1; and, when $\theta = \pi$, κ_5 is the r value for which $\mathcal{C}_r$ contains p_1.

We now summarize our findings for the critical values κ_j. For fixed $\theta \notin I$ we have

$$\kappa_1 = \frac{b}{a}e^{1-b/a}, \quad \kappa_2 \equiv 1, \quad \kappa_3 = \frac{b}{a}e^{a/b-1},$$

and if $\cos\theta \leq 0$ then $\kappa_4 > \kappa_3$ is the unique r value that satisfies (9.8), while $\kappa_5 \geq \kappa_4$ lies between the solution r_5' to (9.12) and $r_5 = (b/a)[eW_0(1/e)]^{a/b-1}$. If $\theta \in I$ the critical values κ_3 and κ_4 do not exist, and we simply note that $\kappa_5 > \kappa_2 = 1$.

As promised, we now extend the argument that equation (9.8) has a unique solution for r so as to include at least some θ values for which $\cos\theta > 0$.

First we establish a few more general properties of the function f_θ as defined by (9.9), at least for θ values for which $\cos\theta > 0$. If $r < \sec\theta$, then $r\cos\theta < 1$ and the second fraction in equation (9.11) is positive while the first fraction (by the same arguments that follow (9.11)) is negative, and thus $f_\theta'(r) < 0$. If $r > \sec\theta$, then both fractions are negative and $f_\theta'(r) > 0$ results. Therefore f_θ is decreasing on $(0, \sec\theta)$, increasing on $(\sec\theta, \infty)$, with $f_\theta(\sec\theta) = -1$ an easy matter to verify, and

$$\lim_{r\to\infty} f_\theta(r) = W_0\left(-\frac{|\sin\theta|}{e}\right) \leq 0. \tag{9.17}$$

Next, recalling (9.9), we readily find that $\lambda(b/a) = 0$. Set $\hat{r} = b/a$ and $\hat{\theta} = \operatorname{arcsec}\hat{r}$, so $\hat{\theta} \in (0, \pi/2)$ and

$$f_{\hat{\theta}}(\hat{r}) = f_{\hat{\theta}}(\sec\hat{\theta}) = -1 < 0 = \lambda(\hat{r}).$$

Suppose $\theta \in [\hat{\theta}, \pi/2)$, so $\sec\theta \geq \sec\hat{\theta} = \hat{r}$. Fix $r \geq \sec\theta$. Then $\lambda(r) \geq \lambda(\hat{r}) = 0$ while (9.17) and other findings of the previous paragraph make clear that $f_\theta(r) < 0$. Hence $f_\theta(r) \neq \lambda(r)$ for all $r \in [\sec\theta, \infty)$, and the existence part of the proof of Lemma 9.6 leads to the conclusion that $f_\theta(r) = \lambda(r)$ must hold for at least one r value in $(0, \sec\theta)$. That λ is increasing while f_θ is decreasing on $(0, \sec\theta)$ then finishes the proof of uniqueness for all $\theta \in [\hat{\theta}, \pi/2)$. Since $f_{-\theta} = f_\theta$ in general, the uniqueness of the solution to (9.8) is also assured for $\theta \in (-\pi/2, -\hat{\theta}]$.

So for example, if $a = 1$ and $b = 2$, then $\hat{\theta} = \operatorname{arcsec}(2) = \pi/3$, and (9.8) has a unique solution for all $\theta \notin (-\pi/3, \pi/3)$. This is a substantial improvement over $\theta \notin (-\pi/2, \pi/2)$, which we have been more properly characterizing as $\cos\theta \leq 0$.

While an improvement, our new result does not capture all θ values outside the sector

$$I = [-\arccos(1/\kappa_3), \arccos(1/\kappa_3)],$$

since

$$\cos\left(\operatorname{arcsec}\frac{b}{a}\right) = \frac{a}{b} < \frac{a}{b}e^{1-a/b} = \frac{1}{\kappa_3}$$

implies $\hat{\theta} = \operatorname{arcsec}(b/a) > \arccos(1/\kappa_3)$. We could sharpen the result by, for instance, letting $\hat{r}$ be such that $\lambda(\hat{r}) = W_0(-|\sin\theta|/e)$, setting $\hat{\theta} = \operatorname{arcsec}\hat{r}$, and taking $W_0(-|\sin\theta|/e)$ to be the upper bound on $f(r)$ for $r \geq \sec\theta$ and $\theta \in [\hat{\theta}, \pi/2)$. Capturing more θ values in this manner, or by some other means, will be left as a possible avenue for future research.

Section 10: Additional Tools

Here we establish some more results that will help in the course of proving upcoming theorems about the zero attractors of sequences of the form (6.2). The foremost result we've already seen, namely Theorem 2.5, which characterizes zero attractors as explicit point sets. However, the following asymptotic result deriving from [4, Proposition 2.1] will be useful in determining the limits mentioned in part (2) of Theorem 2.5.

Proposition 10.1. *There exists $\nu < 0$ such that, for each $n \in \mathbb{N}$,*

$$s_{bn}(re^{i\theta}nz) = \left[1 + \frac{\varphi^{bn+1}\left(\dfrac{re^{i\theta}nz}{bn+1}\right)}{\left(\dfrac{re^{i\theta}nz}{bn+1} - 1\right)\sqrt{2\pi(bn+1)}}[1 + O(n^{\nu})]\right] e^{re^{i\theta}nz} \tag{10.1}$$

for all $z \in \mathbb{C} \setminus e^{-i\theta}(b/r + \overline{\mathbb{H}})$, where $O(n^{\nu})$ holds uniformly on compact sets.

Proof. A direct adaptation of a result given in [4] yields (10.1) for all z such that $\operatorname{Re}(re^{i\theta}z) < b + 1/n$, with $\nu \in (-1/2, 0)$ fixed and $O(n^{\nu})$ holding uniformly on compact sets. Thus (10.1) holds for all n for any z such that $\operatorname{Re}(e^{i\theta}z) < b/r$, and since

$$\begin{aligned}\{z : \operatorname{Re}(e^{i\theta}z) < b/r\} &= \{e^{-i\theta}z : \operatorname{Re} z < b/r\} = \mathbb{C} \setminus \{e^{-i\theta}z : z \in b/r + \overline{\mathbb{H}}\} \\ &= \mathbb{C} \setminus \{z : e^{-i\theta}z \in b/r + \overline{\mathbb{H}}\} = \mathbb{C} \setminus \{z : z \in e^{-i\theta}(b/r + \overline{\mathbb{H}})\},\end{aligned}$$

the claimed region of validity is obtained. ■

The above asymptotic covers an open half-plane of $\mathbb{C}$, while the following asymptotic derived from [4, Proposition 2.2] covers all of $\mathbb{C}$ outside a closed disc. It can be seen that the two asymptotics combined cover all of $\mathbb{C}$ except for the single point $e^{-i\theta}b/r$.

Proposition 10.2. *There exists $\nu < 0$ such that, for each $n \in \mathbb{N}$,*

$$s_{bn}(re^{i\theta}nz) = \frac{\varphi^{bn+1}\left(\dfrac{re^{i\theta}nz}{bn+1}\right)}{\left(\dfrac{re^{i\theta}nz}{bn+1} - 1\right)\sqrt{2\pi(bn+1)}}[1 + O(n^{\nu})]e^{re^{i\theta}nz}$$

for all z such that $|z| > b/r + 1/n$, where $O(n^{\nu})$ holds uniformly on compact sets.

The next two asymptotics were already introduced in the beginning of §3, and they in fact can be readily adapted to give the two more general asymptotics above. In the interests of thoroughness we will give some of the steps whereby the final asymptotic, in Proposition 10.4, is derived from [4, Proposition 2.2].

Proposition 10.3. *There exists $\nu < 0$ such that, for each $n \in \mathbb{N}$,*

$$s_{an}(nz) = \left[1 + \frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1} - 1\right)\sqrt{2\pi(an+1)}}[1 + O(n^\nu)] \right] e^{nz}$$

for all $z \in \mathbb{C} \setminus (a + \overline{\mathbb{H}})$, where $O(n^\nu)$ holds uniformly on compact sets.

Proposition 10.4. *There exists $\nu < 0$ such that, for each $n \in \mathbb{N}$,*

$$s_{an}(nz) = \frac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)}{\left(\dfrac{nz}{an+1} - 1\right)\sqrt{2\pi(an+1)}}[1 + O(n^\nu)]e^{nz}$$

for all z such that $|z| > a + 1/n$, where $O(n^\nu)$ holds uniformly on compact sets.

Proof. By [4, Proposition 2.2], for any $1/3 < \alpha < 1/2$, we have

$$\frac{s_{n-1}(nw)}{e^{nw}} = \frac{(we^{1-w})^n}{\sqrt{2\pi n}(w-1)}\big(1 + O(n^{1-3\alpha})\big)$$

for any $|w| > 1$, with the term $O(n^{1-3\alpha})$ holding uniformly on compacta in $\mathbb{A}_{1,\infty}$. Setting $\nu = 1 - 3\alpha$, so that $-1/2 < \nu < 0$, for $w \in \mathbb{A}_{n,\infty}$ we obtain

$$s_{n-1}(w) = \frac{e^w\left(\dfrac{w}{n}e^{1-w/n}\right)^n}{(w/n - 1)\sqrt{2\pi n}}\big(1 + O(n^\nu)\big)$$

since $w/n \in \mathbb{A}_{1,\infty}$. Noting that a function of order $O((n+1)^\nu)$ as $n \to \infty$ is necessarily of order $O(n^\nu)$, it follows that

$$s_n(w) = \frac{e^w\left(\dfrac{w}{n+1}e^{1-w/(n+1)}\right)^{n+1}}{\left(\dfrac{w}{n+1} - 1\right)\sqrt{2\pi(n+1)}}\big(1 + O(n^\nu)\big)$$

for $|w| > n+1$, and hence

$$s_{an}(w) = \frac{e^w \left(\dfrac{w}{an+1} e^{1-w/(an+1)} \right)^{an+1}}{\left(\dfrac{w}{an+1} - 1 \right) \sqrt{2\pi(an+1)}} \big(1 + O(n^\nu)\big)$$

for $|w| > an+1$. This last equation then yields

$$s_{an}(nz) = \frac{e^{nz} \left(\dfrac{nz}{an+1} e^{1-nz/(an+1)} \right)^{an+1}}{\left(\dfrac{nz}{an+1} - 1 \right) \sqrt{2\pi(an+1)}} \big(1 + O(n^\nu)\big)$$

for $|z| > a + 1/n$. ■

In addition to the preceding asymptotics, the following proposition and several lemmas will also prove to be valuable aids.

Proposition 10.5. *If $z \in \mathcal{H}^-_{r\theta}$ then*

$$\operatorname{Re}[(1 - re^{i\theta})z] < 0,$$

and if $z \in \mathcal{H}^+_{r\theta}$ then

$$\operatorname{Re}[(1 - re^{i\theta})z] > 0.$$

Proof. Let $z \in \mathcal{H}^-_{r\theta} = -e^{(\ell_{r\theta}+\pi/2)i}\mathbb{H}$, so $z = -e^{(\ell_{r\theta}+\pi/2)i}\zeta$ for some ζ such that $\operatorname{Re}\zeta > 0$. Now, letting

$$\zeta_0 = \frac{\zeta}{\sqrt{r^2 - 2r\cos\theta + 1}} \quad \text{and} \quad \zeta_1 = (r\cos\theta - 1) + i(r\sin\theta),$$

with (6.5) we obtain

$$\begin{aligned}(1 - re^{i\theta})z &= (re^{i\theta} - 1)e^{(\ell_{r\theta}+\pi/2)i}\zeta \\ &= i(re^{i\theta} - 1)\exp\left[i\arctan\left(\frac{r\cos\theta - 1}{r\sin\theta}\right)\right]\zeta \\ &= i(re^{i\theta} - 1)\left(\frac{r\sin\theta}{\sqrt{r^2 - 2r\cos\theta + 1}} + i\frac{r\cos\theta - 1}{\sqrt{r^2 - 2r\cos\theta + 1}}\right)\zeta \\ &= -\big[(r\cos\theta - 1) + i(r\sin\theta)\big]\big[(r\cos\theta - 1) - i(r\sin\theta)\big]\zeta_0\end{aligned}$$

$$= -\zeta_1\overline{\zeta}_1\zeta_0 = -|\zeta_1|^2\zeta_0.$$

Thus $\mathrm{Re}[(1 - re^{i\theta})z] < 0$ since $\mathrm{Re}\,\zeta_0 > 0$. The proof that $\mathrm{Re}[(1 - re^{i\theta})z] > 0$ if $z \in \mathcal{H}^+_{r\theta}$ is similar. ■

Lemma 10.6. *For $1 \le a < b$, $r > 0$, and $\theta \in \mathbb{R}$, let*

$$M = \frac{|e^z|\left|\varphi\left(\dfrac{z}{a}\right)\right|^a}{|e^{re^{i\theta}z}|\left|\varphi\left(\dfrac{re^{i\theta}z}{b}\right)\right|^b}.$$

Then $z \in \mathbb{C} \setminus \overline{\mathcal{D}}_r$ implies $M < 1$, and $z \in \mathcal{D}_r \setminus \{0\}$ implies $M > 1$.

Proof. Since

$$z \in \mathbb{C} \setminus \overline{\mathcal{D}}_r \quad \Leftrightarrow \quad |z| > \frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}} \quad \Leftrightarrow \quad |z|^{a-b} < \frac{e^b a^a r^b}{e^a b^b},$$

we have

$$M = \frac{|e^z|\left|\varphi\left(\dfrac{z}{a}\right)\right|^a}{|e^{re^{i\theta}z}|\left|\varphi\left(\dfrac{re^{i\theta}z}{b}\right)\right|^b} = \frac{e^a b^b}{e^b a^a r^b}|z|^{a-b} < 1.$$

Similarly we obtain $M > 1$ if $z \in \mathcal{D}_r \setminus \{0\}$. ■

Lemma 10.7. *Suppose $1 \le a < b$ and $r < 1$. If $z \in \mathbb{D}_a$ is such that $|\varphi(re^{i\theta}z/a)| < r$, then $|\varphi(re^{i\theta}z/b)| < 1$.*

Proof. The result is trivial if $z = 0$, so fix $z \neq 0$ such that $|z| < a$ and $|\varphi(re^{i\theta}z/a)| < r$. We need two auxiliary inequalities. First, from $|z| < a$ and $b - a > 0$ we have

$$\frac{|z|^b}{|z|^a} < \frac{a^b}{a^a}. \tag{10.2}$$

Next, defining

$$g(x) = \frac{a}{x}e^{1-a/x},$$

we note that $g = 1/f$ for f defined before Lemma 9.4, and so $g < 1$ on (a, ∞), and hence $(a/b)e^{1-a/b} < 1$ in particular. From this we obtain our second needed inequality:

$$\frac{a^b}{b^b} e^{b-a} < 1. \tag{10.3}$$

Now, $|\varphi(re^{i\theta}z/a)| < r$ implies that $|e^{a-re^{i\theta}z}| < a^a/|z|^a$, and then with (10.2), the fact that $r < 1$, and (10.3) it follows that

$$|e^{b-re^{i\theta}z}| < \frac{a^a}{|z|^a} \cdot \frac{e^b}{e^a} = \frac{b^b}{r^b|z|^b} \cdot \frac{|z|^b a^a e^b r^b}{|z|^a b^b e^a} < \frac{b^b}{r^b|z|^b} \cdot \frac{a^b e^b}{b^b e^a} < \frac{b^b}{r^b|z|^b}.$$

Therefore

$$\frac{r^b|z|^b}{b^b}|e^{b-re^{i\theta}z}| < 1,$$

which immediately yields $|\varphi(re^{i\theta}z/b)| < 1$. ■

Lemma 10.8. *Suppose* $1 \le a < b$. *If* $z \in \mathbb{D}_a$ *is such that* $|\varphi(z/a)| < 1$, *then* $|\varphi(z/b)| < 1/r$ *for all* $r \in (0, \kappa_3]$.

Proof. The result is trivial if $z = 0$. Suppose $z \neq 0$ is such that $|z| < a$ and $|\varphi(z/a)| < 1$. From the latter inequality comes

$$\frac{|z|^a}{a^a}|e^{a-z}| < 1,$$

and hence

$$|e^{a-z}| < \left(\frac{a}{|z|}\right)^a < \left(\frac{a}{|z|}\right)^b$$

since $a/|z| > 1$ and $0 < a < b$. Thus $|z|^b|e^{-z}| < a^b e^{-a}$, implying

$$|\varphi(z/b)| = \frac{|z|}{b}|e^{1-z/b}| < \frac{a}{b}e^{1-a/b} = \frac{1}{\kappa_3},$$

which in turn implies $|\varphi(z/b)| < 1/r$ for $r \le \kappa_3$. ■

Lemma 10.9. *Suppose* $1 \le a < b$. *Then*

$$\mathcal{S}_1^< \cap \mathcal{D}_r \subseteq \mathcal{S}_4^< \quad \textit{and} \quad \mathcal{S}_3^< \cap \mathcal{D}_r \subseteq \mathcal{S}_2^< \tag{10.4}$$

for all $r > 0$ *and* $\theta \in \mathbb{R}$.

Proof. Suppose $z \in \mathcal{S}_1^< \cap \mathcal{D}_r$. If $z = 0$ then $z \in \mathcal{S}_4^<$ is immediate, so we assume $z \neq 0$. Then

$$\begin{aligned} |\varphi(z/a)| < 1 \ \Rightarrow \ \left|e^{1-z/a}\right| < \frac{a}{|z|} \ &\Rightarrow \ \left|e^{1-z/b}\right| < \frac{a^{a/b}}{|z|^{a/b}} \cdot e^{1-a/b} \\ &\Rightarrow \ |\varphi(z/b)| = \frac{|z|}{b}\left|e^{1-z/b}\right| < |z|^{1-a/b} \cdot \frac{a^{a/b}}{b} \cdot e^{1-a/b}, \end{aligned}$$

and so, since $|z| < \rho_r$,

$$|\varphi(z/b)| < \left[\frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}}\right]^{1-a/b} \left(\frac{a^{a/b}e^{1-a/b}}{b}\right) = \frac{1}{r},$$

and therefore $z \in \mathcal{S}_4^<$.

To prove the second containment in (10.4) it suffices to examine only the case $\theta = 0$. Setting $\theta = 0$, suppose that $z \in \mathcal{S}_3^< \cap \mathcal{D}_r$. Then

$$\begin{aligned} |\varphi(rz/a)| < r \ \Rightarrow \ \left|e^{1-rz/a}\right| < \frac{a}{|z|} \ &\Rightarrow \ \left|e^{1-rz/b}\right| < \frac{a^{a/b}}{|z|^{a/b}} \cdot e^{1-a/b} \\ &\Rightarrow \ |\varphi(rz/b)| = \frac{r|z|}{b}\left|e^{1-rz/b}\right| < |z|^{1-a/b} \cdot \frac{ra^{a/b}}{b} \cdot e^{1-a/b}, \end{aligned}$$

so that

$$|\varphi(rz/b)| < \left[\frac{a}{e}\left(\frac{b}{ar}\right)^{\frac{b}{b-a}}\right]^{1-a/b} \left(\frac{ra^{a/b}e^{1-a/b}}{b}\right) = 1,$$

and therefore $z \in \mathcal{S}_2^<$. ■

Section 11: The Zero Attractor for $0 < r \le \kappa_1$

Let $\mathcal{A}$ denote the zero attractor for the sequence of polynomials

$$P_n(z) = s_{an}(nz) + Cs_{bn}(re^{i\theta}nz).$$

Since the compact set $\mathcal{A}$ passes through different homotopy classes as $r \to \infty$, it is necessary to carry out separate analyses over a series of disjoint intervals of r values that are determined by the critical values given in Definition 9.1. The regions Ω_2 and Ω_3 given in that definition are involved in nearly all these analyses, though we will often alter their expressions in the upcoming theorems (depending on the interval in which r is assumed to lie) for purposes of using the propositions of the previous section. In the present section we concern ourselves strictly with $0 < r \le \kappa_1$ for all admissible a, b, θ.

Theorem 11.1. *If $0 < r \le \kappa_1$, then $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\begin{aligned}
\Omega_1 &: \quad \mathbb{C} \setminus \overline{\mathcal{D}}_r \\
\Omega_2 &: \quad \mathcal{D}_r \setminus \left[\left(\overline{\mathcal{S}}_1^< \cup \overline{\mathcal{S}}_3^<\right) \cap \overline{\mathbb{D}}_a\right] \\
\Omega_3 &: \quad \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a \\
\Omega_4 &: \quad \mathcal{S}_1^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a.
\end{aligned}$$

To facilitate the proof of the theorem we first establish lemmas that determine the asymptotics in each of the four regions.

Lemma 11.2. *If $0 < r \le \kappa_1$, then*

$$\lim_{n\to\infty} \frac{\ln |P_n(z)|}{n} = b \ln \left|\frac{rez}{b}\right|$$

uniformly on compact sets of $\Omega_1 = \mathbb{C} \setminus \overline{\mathcal{D}}_r$.

Proof. Let $K \subseteq \mathbb{C} \setminus \overline{\mathcal{D}}_r$ be compact. Since $\rho_r = a$ if and only if $r = \kappa_3$, the fact that $r \mapsto \rho_r$ is a decreasing function and $\kappa_1 < \kappa_3$ makes clear that $\mathbb{D}_a \subseteq \mathcal{D}_r$. Some algebra shows

$$\rho_r \geq \frac{b}{r} \quad \Leftrightarrow \quad \frac{e}{a}\left(\frac{ar}{b}\right)^{\frac{b}{b-a}} \leq \frac{r}{b} \quad \Leftrightarrow \quad r \leq \frac{b}{a}e^{1-b/a} = \kappa_1,$$

and so $\mathbb{D}_{b/r} \subseteq \mathcal{D}_r$. Hence K is a subset of $\mathbb{C} \setminus \overline{\mathbb{D}}_a$ and $\mathbb{C} \setminus \overline{\mathbb{D}}_{b/r}$, so Propositions 10.2 and 10.4 imply that

$$\lim_{n\to\infty}\left|\frac{s_{an}(nz)}{Cs_{bn}(re^{i\theta}nz)}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{\dfrac{\varphi^{an+1}\left(\dfrac{nz}{an+1}\right)[1+O(n^{\nu_1})]e^{nz}}{\left(\dfrac{nz}{an+1}-1\right)\sqrt{2\pi(an+1)}}}{\dfrac{\varphi^{bn+1}\left(\dfrac{re^{i\theta}nz}{bn+1}\right)[1+O(n^{\nu_2})]e^{re^{i\theta}nz}}{\left(\dfrac{re^{i\theta}nz}{bn+1}-1\right)\sqrt{2\pi(bn+1)}}}\right|^{1/n} \tag{11.1}$$

on K, where the order terms $O(n^{\nu_1})$ and $O(n^{\nu_2})$ hold uniformly on K for some $\nu_1, \nu_2 < 0$. This readily simplifies, giving

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{s_{an}(nz)}{Cs_{bn}(re^{i\theta}nz)}\right|^{1/n} &= \lim_{n\to\infty}\frac{\left|\varphi^a\left(\dfrac{nz}{an+1}\right)\right||e^z|}{\left|\varphi^b\left(\dfrac{re^{i\theta}nz}{bn+1}\right)\right||e^{re^{i\theta}z}|}\\ &= \frac{\left|\varphi\left(\dfrac{z}{a}\right)\right|^a|e^z|}{\left|\varphi\left(\dfrac{re^{i\theta}z}{b}\right)\right|^b|e^{re^{i\theta}z}|} = \frac{e^ab^b}{e^ba^ar^b}|z|^{a-b} < 1\end{aligned}$$

for each $z \in K$, with the last inequality following from Lemma 10.6. Since $f : K \to \mathbb{C}$ defined as

$$f(z) = \frac{e^ab^b}{e^ba^ar^b}|z|^{a-b}$$

is continuous on K, we have $\|f\|_K < 1$. Also it was shown in §3 that $|s_{an}(nz)|^{1/n}$ converges uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$, and a similar argument shows that

$$\lim_{n\to\infty}|Cs_{bn}(re^{i\theta}nz)|^{1/n} = \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right|^b|e^{re^{i\theta}z}| = \frac{e^br^b}{b^b}|z|^b$$

uniformly on compact subsets of $\mathbb{C} \setminus \{e^{-i\theta}b/r\}$. Hence the limit (11.1) must converge uniformly to $f(z)$ on K, so that

$$\lim_{n\to\infty} \left| \frac{s_{an}(nz)}{Cs_{bn}(re^{i\theta}nz)} + 1 \right|^{1/n} = 1$$

uniformly on K by Proposition 3.3, and therefore

$$\begin{aligned}
\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} |s_{an}(nz) + Cs_{bn}(re^{i\theta}nz)|^{1/n} \\
&= \lim_{n\to\infty} |Cs_{bn}(re^{i\theta}nz)|^{1/n} \left| \frac{s_{an}(nz)}{Cs_{bn}(re^{i\theta}nz)} + 1 \right|^{1/n} \\
&= \frac{e^b r^b}{b^b} |z|^b
\end{aligned}$$

uniformly on K. ■

That our limits hold uniformly on compact subsets of whatever region is under consideration can be established in much the same way as in the proof of the lemma above, and so in subsequent proofs we shall concern ourselves largely with the evaluation of pointwise limits.

Lemma 11.3. *In the region Ω_2 that is $\mathcal{D}_r \setminus \left[(\overline{\mathcal{S}}_1^< \cup \overline{\mathcal{S}}_3^<) \cap \overline{\mathbb{D}}_a \right]$,*

$$\lim_{n\to\infty} \frac{\ln |P_n(z)|}{n} = a \ln \left| \frac{ez}{a} \right|$$

uniformly on compact sets.

Proof. Suppose z is a point in Ω_2 that lies in the intersection of $\mathbb{C} \setminus e^{-i\theta}(b/r + \overline{\mathbb{H}})$ and $\mathbb{C} \setminus (a + \overline{\mathbb{H}})$. By Propositions 10.1 and 10.3,

$$\begin{aligned}
\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} \left| e^{nz} + e^{nz}\varphi^{an}\left(\frac{z}{a}\right) + Ce^{re^{i\theta}nz} + Ce^{re^{i\theta}nz}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right) \right|^{1/n} \\
&= \lim_{n\to\infty} |e^{re^{i\theta}z}| \left| e^{(1-re^{i\theta})nz} + e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) + C + C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right) \right|^{1/n}.
\end{aligned} \tag{11.2}$$

Since

$$e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) = (e^{-i\theta})^{an} \left[\frac{1}{r}\varphi\left(\frac{re^{i\theta}z}{a}\right) \right]^{an} \tag{11.3}$$

and $|\varphi(re^{i\theta}z/a)| > r$, the constant term in (11.2) may be neglected, and since $|\varphi(z/a)| > 1$ we obtain

$$\begin{aligned}
\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} |e^{re^{i\theta}z}| \left| e^{(1-re^{i\theta})nz} + e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) + C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n} \\
&= \lim_{n\to\infty} |e^{z}| \left| 1 + \varphi^{an}\left(\frac{z}{a}\right) + \frac{C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)}{e^{(1-re^{i\theta})nz}} \right|^{1/n} \qquad (11.4) \\
&= \lim_{n\to\infty} |e^{z}| \left|\varphi\left(\frac{z}{a}\right)\right|^{a} \left| 1 + \frac{C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)}{e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right)} \right|^{1/n}.
\end{aligned}$$

Now, by Lemma 10.6,

$$\left| \frac{C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)}{e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right)} \right|^{1/n} = \frac{|e^{re^{i\theta}z}| \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right|^{b}}{|e^{z}| \left|\varphi\left(\frac{z}{a}\right)\right|^{a}} < 1,$$

and so Proposition 3.3 implies that

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \lim_{n\to\infty} |e^{z}| \left|\varphi\left(\frac{z}{a}\right)\right|^{a} = \frac{e^{a}}{a^{a}}|z|^{a}. \qquad (11.5)$$

If z lies in the portion of Ω_2 that is in the intersection of $\mathbb{C} \setminus \overline{\mathbb{D}}_a$ and $\mathbb{C} \setminus e^{-i\theta}(b/r + \overline{\mathbb{H}})$, then it is not necessarily the case that $|\varphi(z/a)| > 1$, but then Propositions 10.1 and 10.4 may be used in order to obtain (11.4) without the 1 term to begin with, and the rest of the analysis is the same.

Finally, if z lies in the part of Ω_2 where $\mathbb{C} \setminus \overline{\mathbb{D}}_a$ and $\mathbb{C} \setminus \overline{\mathbb{D}}_{b/r}$ intersect, then neither $|\varphi(z/a)| > 1$ nor $|\varphi(re^{i\theta}z/a)| > r$ necessarily hold, but Propositions 10.2 and 10.4 may be used to find that

$$\lim_{n\to\infty} \left| \frac{Cs_{bn}(re^{i\theta}nz)}{s_{an}(nz)} \right|^{1/n} < 1,$$

much as the reciprocal expression was treated in the proof of Lemma 11.2. Then via Proposition 3.3 we arrive at

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \lim_{n\to\infty} |s_{an}(nz)|^{1/n} \left|\frac{Cs_{bn}(re^{i\theta}nz)}{s_{an}(nz)} + 1\right|^{1/n} = |s_{an}(nz)|^{1/n},$$

which again leads to (11.5). We have now covered all of Ω_2 save for the point b/r. ■

Lemma 11.4. *In the region* Ω_3 *that is* $\mathcal{S}_3^{<} \cap \mathcal{H}_{r\theta}^{-} \cap \mathbb{D}_a$*,*

$$\lim_{n\to\infty} \frac{\ln |P_n(z)|}{n} = \ln |e^{re^{i\theta}z}|$$

uniformly on compact sets.

Proof. Let z be a point in Ω_3. As in the proof of Lemma 11.3 we employ Propositions 10.1 and 10.3 to obtain (11.2). Now, $\mathrm{Re}[(1 - re^{i\theta})z] < 0$ by Proposition 10.5, and also $|\varphi(re^{i\theta}z/b)| < 1$ by Lemma 10.7. Then, in light of (11.2) and the fact that $|\varphi(re^{i\theta}z/a)| < r$, we find that the nonzero constant term C in (11.2) dominates, and hence

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \lim_{n\to\infty} |e^{re^{i\theta}z}||C|^{1/n} = |e^{re^{i\theta}z}|.$$

This immediately implies the desired result. ■

Lemma 11.5. *In the region* Ω_4 *that is* $\mathcal{S}_1^{<} \cap \mathcal{H}_{r\theta}^{+} \cap \mathbb{D}_a$*,*

$$\lim_{n\to\infty} \frac{\ln |P_n(z)|}{n} = \ln |e^{z}|$$

uniformly on compact sets.

Proof. Let z be a point in Ω_4. Again using (11.2),

$$\begin{aligned}\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} |e^{z}| \left|1 + \varphi^{an}\left(\frac{z}{a}\right) + Ce^{(re^{i\theta}-1)nz} + Ce^{(re^{i\theta}-1)nz}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n} \\ &= \lim_{n\to\infty} |e^{z}| \left|1 + \varphi^{an}\left(\frac{z}{a}\right) + Ce^{(re^{i\theta}-1)nz} + C\left[re^{i\theta}\varphi\left(\frac{z}{b}\right)\right]^{bn}\right|^{1/n}. \qquad (11.6)\end{aligned}$$

Now, $\mathrm{Re}[(re^{i\theta} - 1)z] < 0$ by Proposition 10.5 and $|\varphi(z/b)| < 1/r$ by Lemma 10.8. Moreover, $r \le \kappa_1$ implies $r < 1$ by Proposition 9.3, so that $|re^{i\theta}| < 1$. Since $|\varphi(z/a)| < 1$ as well, the constant term 1 in (11.6) dominates, and therefore

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |e^z|$$

as claimed. ■

Proof of Theorem 11.1. From the lemmas we find that

$$\lim_{n\to\infty} \frac{\ln|P_n(z)|}{n} = \begin{cases} b\ln|rez/b|\,, & z\in\Omega_1 \\ a\ln|ez/a|\,, & z\in\Omega_2 \\ \ln|e^{re^{i\theta}z}|, & z\in\Omega_3 \\ \ln|e^z|, & z\in\Omega_4 \end{cases}$$

and so by Theorem 2.5 the theorem is proven. ■

Section 12: The Zero Attractor for r in Other Intervals

We now consider the nature of $\mathcal{A}$ for r in other intervals beyond $(0, \kappa_1]$. As Proposition 9.3 established, $\kappa_2 \equiv 1$, and of course when $r = 1$ we are back to the unimodular case treated in §5. Thus in the course of proving the following theorem we may assume $r < 1$, though $r = 1$ should not present any difficulties.

Theorem 12.1. *If $\kappa_1 < r \leq 1$, with $\theta \neq 2\pi k$ for $k \in \mathbb{Z}$ if $r = 1$, then $\mathcal{A}$ is the union of the boundaries of the regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \big[\overline{\mathcal{D}}_r \cup \big(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\big)\big]$$

$$\Omega_2 : \quad \mathcal{D}_r \setminus \big[\overline{\mathcal{S}}_3^< \cup \big(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\big)\big]$$

$$\Omega_3 : \quad \big[\mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a\big] \amalg \big[\mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathbb{A}_{a,b/r}\big]$$

$$\Omega_4 : \quad \mathcal{S}_1^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a,$$

where only Ω_3 is disconnected.

Proof. The analysis of regions Ω_2 and Ω_4 is the same as in the proofs of Lemmas 11.3 and 11.5 in the previous section, and also the component

$$\Omega_3' = \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_a$$

of the present region Ω_3 is handled as in the proof of Lemma 11.4. It remains to consider Ω_1 and the component

$$\Omega_3'' = \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathbb{A}_{a,b/r}.$$

Any $z \in \Omega_1$ is such that either $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_{b/r}$ or $z \in \mathbb{C} \setminus e^{-i\theta}(b/r + \overline{\mathbb{H}})$. In the former case the analysis proceeds as in the proof of Lemma 11.2, while in the latter case Proposition 10.1 leads to

$$\lim_{n\to\infty} \big|Cs_{bn}(re^{i\theta}nz)\big|^{1/n} = \lim_{n\to\infty} \left|\left[C + \frac{C\varphi^{bn}(re^{i\theta}z/b)}{(re^{i\theta}z/b - 1)\sqrt{2\pi bn}}\right] e^{re^{i\theta}nz}\right|^{1/n};$$

and then since $|\varphi(re^{i\theta}z/b)| > 1$ we may neglect the constant term C to obtain

$$\lim_{n\to\infty} \left|Cs_{bn}(re^{i\theta}nz)\right|^{1/n} = \lim_{n\to\infty} \left|\frac{C\varphi^{bn}(re^{i\theta}z/b)}{(re^{i\theta}z/b-1)\sqrt{2\pi bn}} e^{re^{i\theta}nz}\right|^{1/n},$$

This leads to (11.1), whereafter the analysis is the same.

For $z \in \Omega_3''$, Propositions 10.1 and 10.4 imply

$$\begin{aligned}
\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} \left|e^{nz}\varphi^{an}\left(\frac{z}{a}\right) + Ce^{re^{i\theta}nz} + Ce^{re^{i\theta}nz}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n} \\
&= \lim_{n\to\infty} |e^{re^{i\theta}z}| \left|e^{(1-re^{i\theta})nz}\varphi^{an}\left(\frac{z}{a}\right) + C + C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n} \\
&= \lim_{n\to\infty} |e^{re^{i\theta}z}| \left|\left[\frac{1}{re^{i\theta}}\varphi\left(\frac{re^{i\theta}z}{a}\right)\right]^{an} + C + C\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n},
\end{aligned}$$

with the last equality following from (11.3). The constant term C dominates since $|\varphi(re^{i\theta}z/a)| < r$ and $|\varphi(re^{i\theta}z/b)| < 1$ by the definition of Ω_3'', and so

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |e^{re^{i\theta}z}|.$$

Hence

$$\lim_{n\to\infty} \frac{\ln|P_n(z)|}{n} = \begin{cases} b\ln|rez/b|, & z \in \Omega_1 \\ a\ln|ez/a|, & z \in \Omega_2 \\ \ln|e^{re^{i\theta}z}|, & z \in \Omega_3 \\ \ln|e^z|, & z \in \Omega_4 \end{cases}$$

■

For the statement of the following theorem recall the set I as defined in Lemma 9.4. It will also be convenient to define

$$\Omega_3^* = \mathcal{S}_2^< \cap \mathcal{S}_3^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{C} \setminus (a + \overline{\mathbb{H}})$$

and, for aesthetic reasons, $\Omega_3^{**} = \Omega_3''$.

Theorem 12.2. *If $\theta \notin I$ and $1 < r < \kappa_3$, then $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\begin{aligned}
\Omega_1 &: \quad \mathbb{C} \setminus \left[\,\overline{\mathcal{D}}_r \cup \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\right)\right] \\
\Omega_2 &: \quad \mathcal{D}_r \setminus \left[\,\overline{\mathcal{S}}_3^< \cup \left(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\right)\right] \\
\Omega_3 &: \quad \Omega_3^* \cup \Omega_3^{**} \\
\Omega_4 &: \quad \mathcal{S}_1^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a.
\end{aligned}$$

Proof. The analysis of Ω_1 is as in Theorem 12.1, while Ω_2 and Ω_4 are treated as in the previous section. The current Ω_3, however, is the result of a merging of the former regions Ω_3 and Ω_3', and so has properties of both. To fully cover Ω_3 may require two separate analyses: one using Propositions 10.1 and 10.3, and another using Propositions 10.1 and 10.4 provided that $\Omega_3^{**} \setminus \Omega_3^* \neq \varnothing$. However, for $z \in \Omega_3^*$ the analysis proceeds precisely as in Lemma 11.4, and for $z \in \Omega_3^{**}$ the analysis is the same as that carried out for Ω_3' in the proof of Theorem 12.1. Therefore

$$\lim_{n\to\infty} \frac{\ln|P_n(z)|}{n} = \begin{cases} b\ln|rez/b|\,, & z \in \Omega_1 \\ a\ln|ez/a|\,, & z \in \Omega_2 \\ \ln|e^{re^{i\theta}z}|, & z \in \Omega_3 \\ \ln|e^{z}|, & z \in \Omega_4 \end{cases}$$

■

When $r \geq \kappa_3$, the radius of $\mathcal{C}_r$ is less than a, and thus the portion of the set $\mathcal{S}_1^<$ that lies outside this circle is no longer a part of Ω_2. This simplifies the expression for Ω_2 in the next theorem. A complicating feature is that a portion of $\mathcal{S}_4$ now forms part of the boundary of Ω_1 and Ω_4.

Theorem 12.3. *If $\theta \notin I$ and $\kappa_3 \le r < \kappa_5$, then $\mathcal{A}$ is the union of the boundaries of the regions*

$$\Omega_1: \quad \mathbb{C} \setminus \left[\overline{\mathcal{D}}_r \cup \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b\right)\right]$$

$$\Omega_2: \quad \mathcal{D}_r \setminus \left(\overline{\mathcal{S}}_3^< \cup \overline{\mathcal{S}}_1^<\right)$$

$$\Omega_3: \quad \Omega_3^* \cup \Omega_3^{**}$$

$$\Omega_4: \quad \mathcal{S}_1^< \cap \mathcal{S}_4^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a,$$

where Ω_2 is disconnected for $\kappa_3 \le r < \kappa_4$ and connected for $\kappa_4 \le r < \kappa_5$ provided that $\cos\theta \le 0$.

Proof. As in the proof of Theorem 12.1, any $z \in \Omega_1$ lies either in $\mathbb{C}\setminus\overline{\mathbb{D}}_{b/r}$ or $\mathbb{C}\setminus e^{-i\theta}(b/r+\overline{\mathbb{H}})$, and in the former case the analysis is the same as that done in the previous section. If $z \in \mathbb{C} \setminus e^{-i\theta}(b/r + \overline{\mathbb{H}})$, then Propositions 10.1 and 10.3 imply that

$$\begin{aligned}
\lim_{n\to\infty} |P_n(z)|^{1/n} &= \lim_{n\to\infty} \left| e^{nz}\left[1+\varphi^{an}\left(\frac{z}{a}\right)\right] + Ce^{re^{i\theta}nz}\left[1+\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right]\right|^{1/n} \\
&= \lim_{n\to\infty} \left| e^{nz}\left[1+\varphi^{an}\left(\frac{z}{a}\right)\right] + Ce^{re^{i\theta}nz}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n} \\
&= \lim_{n\to\infty} |e^{z}| \left| 1+\varphi^{an}\left(\frac{z}{a}\right) + \frac{Ce^{re^{i\theta}nz}}{e^{nz}}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n}, \qquad (12.1)
\end{aligned}$$

where the constant term 1 is neglected in the right grouping since $|\varphi(re^{i\theta}z/b)| > 1$. Now,

$$\left|\frac{e^{re^{i\theta}nz}}{e^{nz}}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right| = \left|\frac{e^{re^{i\theta}nz}}{e^{nz}}\left(\frac{re^{i\theta}z}{b}e^{1-re^{i\theta}z/b}\right)^{bn}\right| = \left(\frac{r|z|}{b}\left|e^{1-z/b}\right|\right)^{bn},$$

and since $|\varphi(z/b)| > 1/r$ by the definition of Ω_1, with

$$\left|\varphi\left(\frac{z}{b}\right)\right| > \frac{1}{r} \quad \Leftrightarrow \quad \frac{|z|}{b}\left|e^{1-z/b}\right| > \frac{1}{r} \quad \Leftrightarrow \quad \frac{r|z|}{b}\left|e^{1-z/b}\right| > 1,$$

the 1 in (12.1) may be neglected to obtain

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \lim_{n\to\infty} \left| e^{nz}\varphi^{an}\left(\frac{z}{a}\right) + Ce^{re^{i\theta}nz}\varphi^{bn}\left(\frac{re^{i\theta}z}{b}\right)\right|^{1/n}$$

$$= \left|e^{re^{i\theta}z}\right| \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right|^b \lim_{n\to\infty} \left|\frac{e^{nz}\varphi^{an}\left(\dfrac{z}{a}\right)}{Ce^{re^{i\theta}nz}\varphi^{bn}\left(\dfrac{re^{i\theta}z}{b}\right)} + 1\right|^{1/n}.$$

Since

$$\lim_{n\to\infty} \left|\frac{e^{nz}\varphi^{an}\left(\dfrac{z}{a}\right)}{Ce^{re^{i\theta}nz}\varphi^{bn}\left(\dfrac{re^{i\theta}z}{b}\right)}\right|^{1/n} = \frac{|e^z|\left|\varphi\left(\dfrac{z}{a}\right)\right|^a}{\left|e^{re^{i\theta}z}\right|\left|\varphi\left(\dfrac{re^{i\theta}z}{b}\right)\right|^b} < 1$$

by Lemma 10.6, we conclude by Proposition 3.3 that

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \left|e^{re^{i\theta}z}\right| \left|\varphi\left(\frac{re^{i\theta}z}{b}\right)\right|^b = \frac{e^b r^b}{b^b}|z|^b$$

as usual.

The analysis of Ω_4 proceeds much as in the proof of Lemma 11.5, only with $r \geq \kappa_3$ we cannot rely on Lemma 10.8. However, $\Omega_4 \subseteq \mathcal{S}_4^<$ is now built into the definition of the region, so $|\varphi(z/b)| < 1/r$ still holds for any $z \in \Omega_4$. We again obtain $\lim_{n\to\infty} |P_n(z)|^{1/n} = |e^z|$. ■

The condition $\cos\theta \leq 0$ in Theorem 12.3 stems from the possibility that equation (9.8) may have multiple solutions when $\cos\theta > 0$, in which case (provided $\theta \notin I$) the region Ω_2 may alternate between being connected and disconnected multiple times as r increases. We recall, however, that uniqueness of solution for (9.8) was secured at the end of §9 for some values of θ for which $\cos\theta > 0$, namely values in the intervals $[\operatorname{arcsec} b/a, \pi/2)$ and $(-\pi/2, -\operatorname{arcsec} b/a]$, and so the condition $\cos\theta \leq 0$ may correspondingly be relaxed in the statement of the theorem.

Theorem 12.4. *If $r \geq \kappa_5$, then $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \left[\left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b\right)\right]$$

$$\Omega_3 : \quad \mathcal{S}_2^< \cap \mathcal{H}_{r\theta}^- \cap \mathbb{D}_{b/r}$$

$$\Omega_4 : \quad \mathcal{S}_1^< \cap \mathcal{S}_4^< \cap \mathcal{H}_{r\theta}^+ \cap \mathbb{D}_a.$$

Proof. By definition $\kappa_5 = \min\{r : \Omega_2 = \varnothing\}$, and the proof of Proposition 9.9 shows that Ω_2 is in fact empty for all $r > \kappa_5$. This enables a slight simplification of the expressions for Ω_1 and Ω_3. In the case of Ω_1, Definition 9.1 implies that

$$\mathcal{D}_r \subseteq \left(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\right) \cup \left(\overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r}\right)$$

for $r \geq \kappa_5$, and hence

$$\overline{\mathcal{D}}_r \subseteq \left(\overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a\right) \cup \left(\overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r}\right).$$

Suppose $z \in \overline{\mathcal{D}}_r$, so either $z \in \overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a$ or $z \in \overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r}$. Now,

$$z \in \overline{\mathcal{S}}_1^< \cap \overline{\mathbb{D}}_a \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_1^< \cap \overline{\mathcal{D}}_r \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_4^< \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b,$$

where the second implication follows from Lemma 10.9 and the third from the simple fact that $\overline{\mathbb{D}}_a \subseteq \overline{\mathbb{D}}_b$. On the other hand,

$$z \in \overline{\mathcal{S}}_3^< \cap \overline{\mathbb{D}}_{b/r} \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_3^< \cap \overline{\mathcal{D}}_r \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_2^< \quad \Rightarrow \quad z \in \overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r},$$

where again Lemma 10.9 is used for the second implication. Thus

$$\overline{\mathcal{D}}_r \subseteq \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b\right),$$

and the expression for Ω_1 in Theorem 12.3 simplifies to that in Theorem 12.4. This observation, moreover, makes clear that the asymptotic analysis of Ω_1 as expressed here will be identical to that carried out in the previous theorem. The same holds true for Ω_3, and also for Ω_4 (whose expression is unchanged in any case). ■

Finally, if $\theta \in I$ we find that Ω_2 remains connected for all $0 < r < \kappa_5$ until becoming the empty set when $r \geq \kappa_5$. The zero attractor therefore remains in the same homotopy class for r between $\kappa_2 = 1$ and κ_5 (where $\kappa_5 > 1$ was established by Proposition 9.9). The next theorem takes this into account, giving definitions for the various regions that are valid for all such r values.

Theorem 12.5. *If $\theta \in I$ and $1 < r < \kappa_5$, then $\mathcal{A}$ is the union of the boundaries of the connected regions*

$$\Omega_1 : \quad \mathbb{C} \setminus \left[\overline{\mathcal{D}}_r \cup \left(\overline{\mathcal{S}}_2^< \cap \overline{\mathbb{D}}_{b/r}\right) \cup \left(\overline{\mathcal{S}}_4^< \cap \overline{\mathbb{D}}_b\right)\right]$$

$$\Omega_2: \quad \mathcal{D}_r \setminus \left[\,\overline{\mathcal{S}}_3^{<} \cup \left(\overline{\mathcal{S}}_1^{<} \cap \overline{\mathbb{D}}_a\right)\right]$$

$$\Omega_3: \quad \Omega_3^{*} \cup \Omega_3^{**}$$

$$\Omega_4: \quad \mathcal{S}_1^{<} \cap \mathcal{S}_4^{<} \cap \mathcal{H}_{r\theta}^{+} \cap \mathbb{D}_a.$$

Proof. The asymptotic analysis of each region proceeds along lines already traced in the course of proving Theorems 12.2 and 12.3, the sole difference being the present assumption that $\theta \in I$ and therefore Ω_2 never becomes disconnected. ■

Section 13: The Zero Attractor of Perturbed Chebyshev Polynomials

Let $T_n(z)$ be the nth Chebyshev polynomial of the first kind, and for fixed integer $\ell \geq 2$ define the perturbation

$$\tilde{T}_n(z) = T_n(z) - z^{\ell n}.$$

Also, for $r(z) = \sqrt{z^2 - 1}$ define the sets

$$C_1 = \left\{z \notin \overline{\mathbb{D}} : z \in (\mathbb{C} \setminus \overline{\mathbb{H}}) \cup (-i\infty, 0) \text{ and } |z - r(z)| = |z|^\ell\right\}$$

and

$$C_2 = \left\{z \notin \overline{\mathbb{D}} : z \in \mathbb{H} \cup (0, i\infty) \text{ and } |z + r(z)| = |z|^\ell\right\},$$

so that $C = C_1 \cup C_2$ is a simple closed curve in the domain $|z| > 1$. Shown in Figure 19 are the solutions to $\tilde{T}_{40}(z) = 0$ when $\ell = 4$, and also the graphs of $|z \pm r(z)| = |z|^4$. The curves C_1 and C_2 form the left and right halves of the outer loop, respectively. If $i\mathbb{R}$ is the imaginary axis, then $C_1 \cap i\mathbb{R} = \{-i\beta\}$ and $C_2 \cap i\mathbb{R} = \{i\beta\}$ for some $\beta > 1$. In explicit

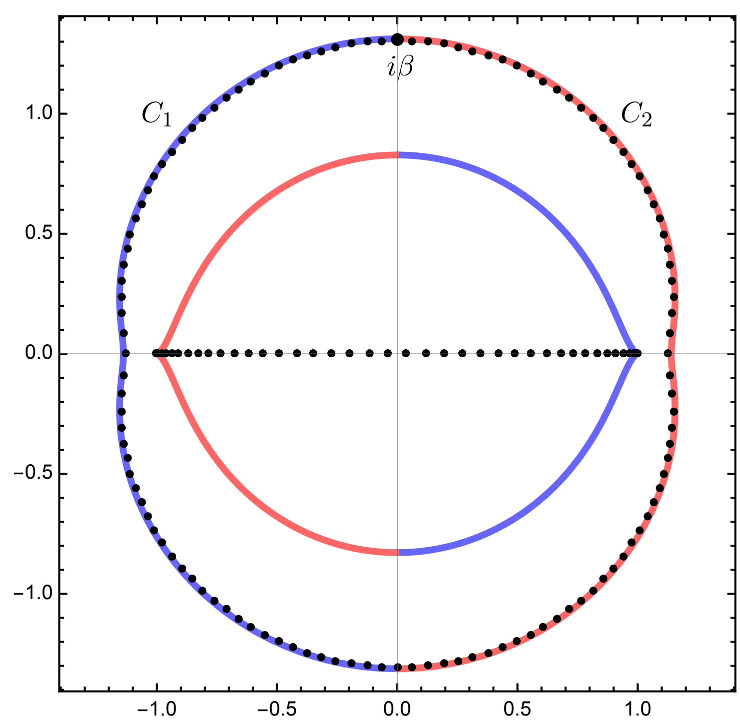

FIGURE 19. The graphs of $|z \pm r(z)| = |z|^4$ and zeros of $\tilde{T}_{40}(z)$ for $\ell = 4$.

terms β is such that

$$\beta + \sqrt{\beta^2 + 1} = \beta^{\ell}. \tag{13.1}$$

It is the disposition of the zeros of $\tilde{T}_{40}(z)$ in the figure that motivates the following.

Theorem 13.1. *The zero attractor of the sequence $\tilde{T}_n(z)$ is $C \cup [-1, 1]$.*

As has been our approach in the past, the proof of the theorem will follow from a series of lemmas together with Theorem 2.5.

Our first lemma will establish that $\tilde{T}_n$ satisfies the two hypotheses stated in Theorem 2.5 concerning the boundedness of $\bigcup_n Z(\tilde{T}_n)$ and the uniform boundedness of the family of functions $|\tilde{T}_n|^{1/n}$ on compacta. Part of the lemma's proof will employ the following result, a form of which is found in [6, p.74].

Proposition 13.2. *For each $1 \leq k \leq m$ define the polynomial*

$$p_k(z) = z^{n_k} + a_{k,1} z^{n_k - 1} + \cdots + a_{k,n_k},$$

and suppose D_k is an open disc such that $Z(p_k) \subseteq D_k$. For fixed $\lambda_1, \ldots, \lambda_m \in \mathbb{C} \setminus \{0\}$ let

$$p(z) = \sum_{k=1}^{m} \lambda_k p_k(z).$$

If $\zeta \in Z(p)$, then there exist $\alpha_k \in D_k$ such that ζ is a root of

$$\sum_{k=1}^{m} \lambda_k (z - \alpha_k)^{n_k} = 0.$$

Also needed for the proof of the first lemma is the fact that if sequences $|p_n|^{1/n}$ and $|q_n|^{1/n}$ are uniformly bounded on $S \subseteq \mathbb{C}$, then so too is $|p_n + q_n|^{1/n}$. To see this, define $\sigma, \tau \in \mathbb{R}$ by

$$\sigma = \sup\left\{\left\||p_n|^{1/n}\right\|_S : n \in \mathbb{N}\right\} \quad \text{and} \quad \tau = \sup\left\{\left\||q_n|^{1/n}\right\|_S : n \in \mathbb{N}\right\}.$$

Fix n. Then

$$\left\||p_n + q_n|^{1/n}\right\|_S = \sup_{z \in S} |p_n(z) + q_n(z)|^{1/n} \leq \sup_{z \in S} \left(|p_n(z)| + |q_n(z)|\right)^{1/n}. \tag{13.2}$$

If $z \in S$ is such that $|p_n(z)| \geq |q_n(z)|$, then

$$\big(|p_n(z)| + |q_n(z)|\big)^{1/n} \leq \big(2|p_n(z)|\big)^{1/n} = 2^{1/n}|p_n(z)|^{1/n} \leq 2\big\||p_n|^{1/n}\big\|_S \leq 2\sigma;$$

and if $|p_n(z)| < |q_n(z)|$, then

$$\big(|p_n(z)| + |q_n(z)|\big)^{1/n} \leq 2\tau.$$

Thus

$$\big(|p_n(z)| + |q_n(z)|\big)^{1/n} \leq 2(\sigma + \tau)$$

for all $z \in S$, and (13.2) implies

$$\big\||p_n + q_n|^{1/n}\big\|_S \leq 2(\sigma + \tau)$$

for all n. It follows that

$$\sup\big\{\big\||p_n + q_n|^{1/n}\big\|_S : n \in \mathbb{N}\big\} \in \mathbb{R},$$

and therefore $|p_n + q_n|^{1/n}$ is uniformly bounded on S.

Finally, we will need the well-known identity

$$T_n(z) = \frac{1}{2}\left[\left(z - \sqrt{z^2-1}\right)^n + \left(z + \sqrt{z^2-1}\right)^n\right] \tag{13.3}$$

for $z \in \mathbb{C}$, given in [8, p. 5]. We now state and prove our first lemma.

Lemma 13.3. *The set $\bigcup_n Z(\tilde{T}_n)$ is bounded, and the family $\{|\tilde{T}_n(z)|^{1/n}\}$ is uniformly bounded on compact sets.*

Proof. Fix n, and let $p_1(z) = z^{\ell n}$ and $p_2(z) = T_n(z)$. For $\delta = 0.1$, say, we have $Z(p_1) \subseteq D_1 := \mathbb{D}_\delta$ and $Z(p_2) \subseteq D_2 := \mathbb{D}_{1+\delta}$. We apply Proposition 13.2 to $p = p_1 - p_2 = -\tilde{T}_n$ to conclude that any zero ζ of $p(z)$ must be a root of

$$(z - \alpha_1)^{\ell n} - (z - \alpha_2)^n = 0 \tag{13.4}$$

for some $\alpha_1 \in D_1$ and $\alpha_2 \in D_2$, which implies

$$|\zeta - \alpha_1|^{\ell} = |\zeta - \alpha_2| \tag{13.5}$$

for some $|\alpha_1| < 0.1$ and $|\alpha_2| < 1.1$.

Suppose $|\zeta| \geq 3$. Some algebra shows $(|\zeta| - 0.1)^2 > 2|\zeta|$, with $||\zeta| - 0.1| > \sqrt{6} > 1$ in particular, and thus $(|\zeta| - 0.1)^\ell > 2|\zeta|$ since $\ell \geq 2$. It follows that

$$(|\zeta| - 0.1)^\ell > |\zeta| + 1.1,$$

and since

$$|\zeta - \alpha_1|^\ell \geq (|\zeta| - |\alpha_1|)^\ell > (|\zeta| - 0.1)^\ell$$

while

$$|\zeta - \alpha_2| \leq |\zeta| + |\alpha_2| < |\zeta| + 1.1,$$

we see (13.5) cannot be satisfied if $|\zeta| > 3$, and thus (13.4) has no roots outside $\mathbb{D}_3$ for any choice of $\alpha_k \in D_k$. Therefore $Z(\tilde{T}_n) = Z(p) \subseteq \mathbb{D}_3$ for all n, and $\bigcup_n Z(\tilde{T}_n)$ is bounded.

Next, let $K \subseteq \mathbb{C}$ be compact. Knowing that $\sqrt{z^2 - 1}$ is bounded on K, we find that

$$\sup_{n\in\mathbb{N}} \left\| \left| \frac{\left(z \pm \sqrt{z^2-1}\right)^n}{2} \right|^{1/n} \right\|_K = \sup_{n\in\mathbb{N}} \left\| \frac{\left|z \pm \sqrt{z^2-1}\right|}{2^{1/n}} \right\|_K = \left\| z \pm \sqrt{z^2-1} \right\|_K \in \mathbb{R},$$

and also

$$\sup_{n\in\mathbb{N}} \left\| \left| z^{\ell n} \right|^{1/n} \right\|_K = \sup_{n\in\mathbb{N}} \|z^\ell\|_K = \|z^\ell\|_K \in \mathbb{R}.$$

Therefore the family of functions given by

$$|\tilde{T}_n(z)|^{1/n} = \left| \frac{\left(z - \sqrt{z^2-1}\right)^n}{2} + \frac{\left(z + \sqrt{z^2-1}\right)^n}{2} - z^{\ell n} \right|^{1/n}$$

is uniformly bounded on K. ■

Next, define the sets

$$E_1 = \left[\{z : \operatorname{Re} z < 0\} \cup (-i\infty, 0)\right] \setminus [-1, 1]$$

and

$$E_2 = \left[\{z : \operatorname{Re} z > 0\} \cup (0, i\infty)\right] \setminus [-1, 1],$$

with $E := E_1 \cup E_2 = \mathbb{C} \setminus [-1, 1]$.

Lemma 13.4. *The inequality*

$$\left|z - \sqrt{z^2 - 1}\right| > \left|z + \sqrt{z^2 - 1}\right| \tag{13.6}$$

holds throughout E_1, while

$$\left|z - \sqrt{z^2 - 1}\right| < \left|z + \sqrt{z^2 - 1}\right| \tag{13.7}$$

holds throughout E_2.

Proof. For the principal branch of the square root,

$$\sqrt{a + ib} = u + iv$$

if and only if

$$u = \sqrt{\frac{(a^2 + b^2)^{1/2} + a}{2}} \quad \text{and} \quad v = (\operatorname{sgn} b)\sqrt{\frac{(a^2 + b^2)^{1/2} - a}{2}},$$

where we set $\operatorname{sgn} b = 1$ if $b = 0$ and $a < 0$, and $\operatorname{sgn} b = 0$ if $b = 0$ and $a \geq 0$.

Let $z = x + iy \in \mathbb{C} \setminus [-1, 1]$ with $x < 0$. We have

$$\sqrt{z^2 - 1} = \sqrt{(x^2 - y^2 - 1) + i(2xy)} := u + iv,$$

with

$$u = \sqrt{\frac{[(x^2 - y^2 - 1)^2 + (2xy)^2]^{1/2} + (x^2 - y^2 - 1)}{2}}$$

and

$$v = \operatorname{sgn}(2xy)\sqrt{\frac{[(x^2 - y^2 - 1)^2 + (2xy)^2]^{1/2} - (x^2 - y^2 - 1)}{2}}.$$

Now,

$$\begin{aligned} \left|z - \sqrt{z^2 - 1}\right| > \left|z + \sqrt{z^2 - 1}\right| \quad &\Leftrightarrow \quad (x - u)^2 + (y - v)^2 > (x + u)^2 + (y + v)^2 \\ &\Leftrightarrow \quad xu < -yv. \end{aligned}$$

We examine cases, observing that $z \neq \pm 1$ and so $u = v = 0$ is impossible.

Suppose $y \neq 0$. Then $\operatorname{sgn}(2xy) = -1$, so $-yv = -y \operatorname{sgn}(2xy)|v| = |yv| \geq 0$. If $v = 0$, then $u > 0$ must hold and we have $xu < 0 = -yv$. If $v \neq 0$, then $xu \leq 0 < |yv| = -yv$.

Next suppose $y = 0$. Then $z = x < -1$, and

$$u + iv = \sqrt{x^2 - 1}$$

implies that $u = \sqrt{x^2 - 1} > 0$ and $v = 0$. Now it follows that $xu < 0 = -yv$, and we see $xu < -yv$, and hence (13.6), holds on the half-plane $\operatorname{Re} z < 0$ outside $[-1, 1]$.

Finally, let $z = iy$. Then

$$u + iv = \sqrt{-y^2 - 1} = i\sqrt{y^2 + 1},$$

indicating that $u = 0$ and $v = \sqrt{y^2 + 1}$. From this we see $xu = 0 < -yv$ if and only if $y < 0$.

It is now clear that (13.6) holds for all $z \in E_1$. The treatment of (13.7) for $z \in E_2$ is similar. ∎

Lemma 13.5. *The function $h : E \to \mathbb{C}$ given by*

$$h(z) = \begin{cases} -\sqrt{z^2 - 1}, & z \in E_1 \\ \sqrt{z^2 - 1}, & z \in E_2 \end{cases}$$

is analytic on E.

Proof. Define $\tilde{h} : iE \to \mathbb{C}$ by $\tilde{h}(z) = ih(-iz)$. Thus

$$\tilde{h}(z) = \begin{cases} -i\sqrt{-z^2 - 1}, & z \in iE_1 \\ i\sqrt{-z^2 - 1}, & z \in iE_2. \end{cases}$$

Clearly h is analytic on E if and only if $\tilde{h}$ is analytic on iE. Define $f : iE_1 \to \mathbb{C}$ by

$$f(z) = -i\sqrt{-z^2 - 1},$$

so that $\tilde{h}(z) = f(z)$ for $z \in iE_1$. Suppose $z \in (iE_2)^\circ$, so $\operatorname{Im} z > 0$ with $z \notin [0, i]$, and hence $\operatorname{Arg} w \in (-\pi, \pi)$ for $w := -z^2 - 1$. From this it follows that $\operatorname{Arg} \overline{w} = -\operatorname{Arg} w$, and subsequently

$$\overline{\sqrt{w}} = \overline{|w|^{1/2} \exp[(i/2) \operatorname{Arg} w]} = |w|^{1/2} \exp[(-i/2) \operatorname{Arg} w]$$

$$= |\overline{w}|^{1/2} \exp[(i/2) \operatorname{Arg} \overline{w}] = \sqrt{\overline{w}}.$$

That is,

$$\overline{\sqrt{-z^2-1}} = \sqrt{-\overline{z}^2-1},$$

and since $\overline{z} \in (iE_1)^\circ$,

$$\overline{f(\overline{z})} = \overline{-i\sqrt{-\overline{z}^2-1}} = i\sqrt{-z^2-1} = \tilde{h}(z).$$

Thus

$$\tilde{h}(z) = \begin{cases} f(z), & z \in iE_1 \\ \overline{f(\overline{z})}, & z \in (iE_2)^\circ. \end{cases}$$

The mapping $z \mapsto \sqrt{z^2-1}$ is known to be continuous and analytic on E_1 and E_1°, respectively, implying that f is continuous and analytic on

$$iE_1 = \big[\{z : \operatorname{Im} z < 0\} \cup (0,\infty)\big] \setminus [-i, 0]$$

and $(iE_1)^\circ$. Moreover, for $x > 0$ we have

$$f(x) = -i\sqrt{-x^2-1} = \sqrt{x^2+1},$$

so f is real-valued on $(0,\infty)$. By the Schwarz reflection principle it follows that $\tilde{h}$ is analytic on $iE_1 \cup (iE_2)^\circ$. Since a symmetrical argument shows that analyticity holds on $iE_2 \cup (iE_1)^\circ$, we conclude that $\tilde{h}$ is analytic on $iE = iE_1 \cup iE_2 = \mathbb{C} \setminus [-i, i]$, and therefore h is analytic on $E = \mathbb{C} \setminus [-1, 1]$. ■

Let U be the unbounded domain outside the curve C, and let $B = \mathbb{C} \setminus \overline{U}$ be the bounded domain inside. Furthermore, set $B' = B \setminus [-1, 1]$, and define the regions $B_1 = E_1 \cap B$ and $B_2 = E_2 \cap B$, so that $B' = B_1 \cup B_2$. More explicitly we have

$$B_1 = \{z \in B' : \operatorname{Re} z < 0\} \cup (-i\beta, 0)$$

and

$$B_2 = \{z \in B' : \operatorname{Re} z > 0\} \cup (0, i\beta).$$

Figure 20 illustrates these regions, as well as the points $\pm i\beta$ mentioned at the beginning of the section.

Lemma 13.6. *The inequality*

$$\left|z - \sqrt{z^2 - 1}\right| > |z|^\ell \tag{13.8}$$

holds throughout B_1, while

$$\left|z + \sqrt{z^2 - 1}\right| > |z|^\ell \tag{13.9}$$

holds throughout B_2.

Proof. We will verify the inequalities for the relevant intervals on the imaginary axis first. Define

$$g(z) = \frac{\left|z - \sqrt{z^2 - 1}\right|}{|z|^\ell} - 1.$$

For $x > 0$,

$$g(-ix) = \frac{x + \sqrt{x^2 + 1}}{x^\ell} - 1,$$

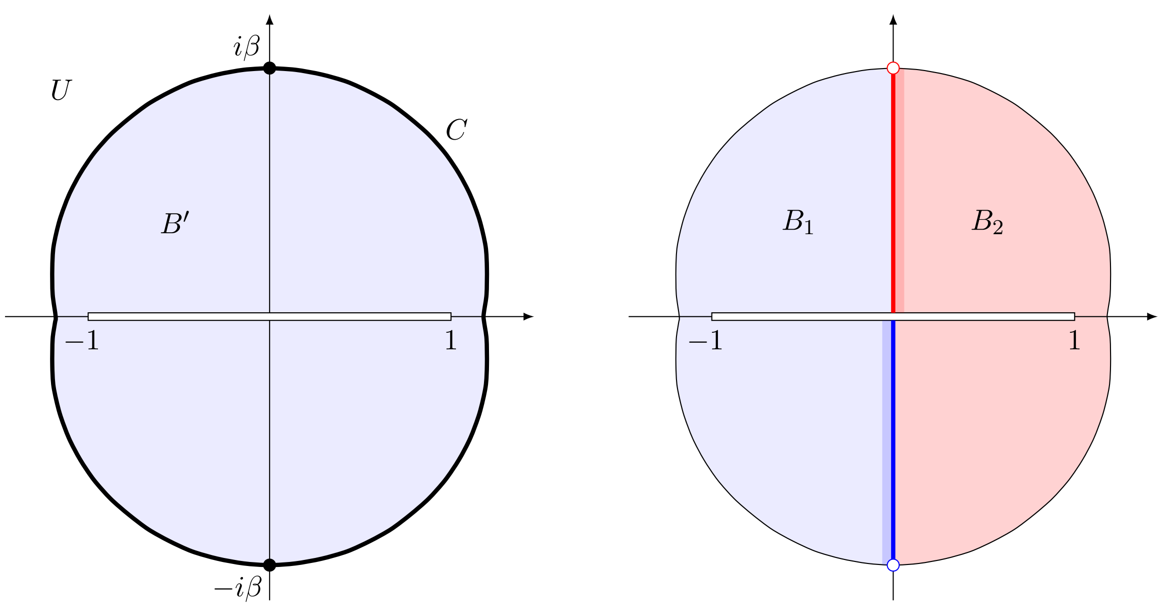

FIGURE 20. Left: regions U and B'. Right: regions B_1 and B_2, which contain the portions of the negative and positive imaginary axes, respectively, that lie in B'.

so that

$$g(-ix) > 0 \quad \Leftrightarrow \quad x - x^\ell + \sqrt{x^2+1} > 0. \tag{13.10}$$

Since $\ell \geq 2$, it is clear the right-hand inequality holds for $x \in (0,1]$, and so $g(-ix) > 0$ for $x \in (0,1]$. Let

$$G(x) = x - x^\ell + \sqrt{x^2+1},$$

where $G(\beta) = 0$ by (13.1). Then

$$G'(x) = 1 - \ell x^{\ell-1} + \frac{x}{\sqrt{x^2+1}},$$

so that $G'(x) < 0$ if and only if

$$\frac{x}{\sqrt{x^2+1}} + 1 < \ell x^{\ell-1}. \tag{13.11}$$

The left-hand side of (13.11) is always less than 2, whereas $\ell x^{\ell-1} > 2$ whenever $x > 1$. This implies G is decreasing on $(1,\infty)$, so that $G(x) > 0$ for $x \in (1,\beta)$ since $G(1) = \sqrt{2}$ and $G(\beta) = 0$. It follows that $g(-ix) > 0$ for $x \in (1,\beta)$ by (13.10), and hence $g(z) > 0$ for $z \in (-i\beta, 0)$. Therefore (13.8) holds on $(-i\beta, 0)$, and by a similar argument (13.9) holds on $(0, i\beta)$.

Let Q_k be the interior of the kth quadrant in $\mathbb{C}$, and define

$$f(z) = \frac{z + \sqrt{z^2-1}}{z^\ell}.$$

We will show that (13.9), or equivalently $|f(z)| > 1$, holds on $B_2 \cap Q_1$ as well as on $B_2 \cap \mathbb{R}$. Since $\ell \geq 2$, there exists $0 < \delta < 1$ sufficiently small that (13.9) holds for all $z \in \partial\mathbb{D}_\delta$. That (13.9) holds on the interval $(i\delta, i\beta)$ follows from the foregoing analysis, whereas $|z + r(z)| = |z|^\ell$ holds on $C_2 \cap \overline{Q}_1$ by the definition of C_2. To complete the circuit and apply the minimum modulus principle, it is necessary to verify that the curve C_2 intersects $\mathbb{R}$ at a unique $\alpha > 1$, and that $|z + r(z)| \geq |z|^\ell$ holds on $(\delta, 1]$ while $|z + r(z)| > |z|^\ell$ holds on $(1, \alpha) \subseteq B_2$.

Define the polynomial

$$p(x) = x^{2\ell} - 2x^{\ell+1} + 1,$$

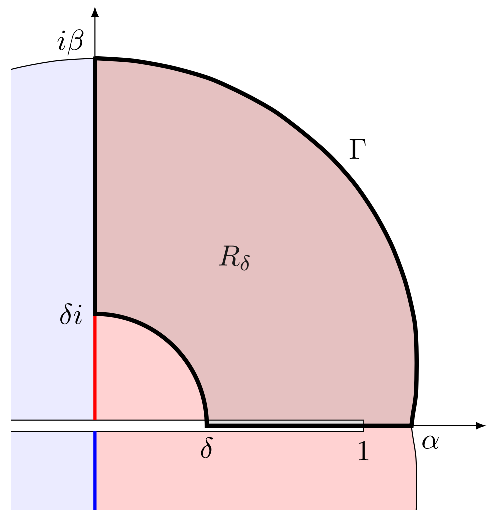

FIGURE 21. The region R_δ in B_2.

so $p'(x) = 2x^\ell(\ell x^{\ell-1} - \ell - 1)$, and for $x > 0$ we find $p'(x) > 0$ if and only if $\ell x^{\ell-1} > \ell + 1$, or equivalently

$$x > \left(\frac{\ell+1}{\ell}\right)^{1/(\ell-1)} := x_0.$$

Clearly $x_0 > 1$, so $p(x)$ is decreasing in particular on $(1, x_0)$ and increasing on (x_0, ∞). But $p(1) = 0$ implies $p(x_0) < 0$, and hence there exists a unique $\alpha > 1$ such that $p(\alpha) = 0$. This α is the sole point where C_2 intersects $\mathbb{R}$, since, for $x > 1$,

$$p(x) = 0 \quad \Leftrightarrow \quad x^2 - 1 = (x^\ell - x)^2 \quad \Leftrightarrow \quad x + \sqrt{x^2 - 1} = x^\ell.$$

Suppose $x \in (1, \alpha)$. Then $p(x) < 0$ by the immediately preceding analysis, and since

$$p(x) < 0 \quad \Leftrightarrow \quad x^2 - 1 > (x^\ell - x)^2 \quad \Leftrightarrow \quad x + \sqrt{x^2 - 1} > x^\ell,$$

we see (13.9) holds on $(1, \alpha)$. Next supposing that $x \in (\delta, 1]$, we obtain by direct calculation

$$|f(x)| = \left|\frac{x + \sqrt{x^2 - 1}}{x^\ell}\right| = \frac{|x + i\sqrt{1 - x^2}|}{x^\ell} = \frac{\sqrt{x^2 + \left(\sqrt{1 - x^2}\right)^2}}{x^\ell} = \frac{1}{x^\ell} \geq 1.$$

Now, the points in

$$(\partial\mathbb{D}_\delta \cap \overline{Q}_1) \cup [i\delta, i\alpha] \cup (C_2 \cap \overline{Q}_1) \cup [\delta, \alpha]$$

form a simple closed curve that admits a piecewise smooth parametrization Γ. Let R_δ denote the region enclosed by this parametrized curve Γ, as shown in Figure 21 Our findings thus far show that $|f(z)| \geq 1$ on ∂R_δ, with

$$\min_{z \in \partial R_\delta} |f(z)| = 1$$

since $f(1) = 1$. Noting that f is nonvanishing and analytic on R_δ, and continuous on $\overline{R}_\delta$, the minimum modulus principle implies that $|f(z)| > 1$ for all $z \in R_\delta$. Letting $\delta \to 0^+$, we conclude that $|f(z)| > 1$ holds for all $z \in B_2 \cap Q_1$. That $|f(z)| > 1$ holds for z in $B_2 \cap \mathbb{R} = (1, \alpha)$ and $B_2 \cap i\mathbb{R} = (0, i\beta)$ was shown above, and therefore (13.9) is verified on $B_2 \cap \overline{Q}_1$. The treatment in other quadrants runs along similar lines. ■

Lemma 13.7. *The inequalities*

$$\frac{\left|z - \sqrt{z^2 - 1}\right|}{|z|^\ell} < 1 \quad \text{and} \quad \frac{\left|z + \sqrt{z^2 - 1}\right|}{|z|^\ell} < 1$$

both hold throughout the domain U.

Proof. It is clear that, for some sufficiently large $R > 0$,

$$\frac{\left|z \pm r(z)\right|}{|z|^\ell} \leq \frac{1}{2}$$

for all $z \notin \mathbb{D}_R$, with the curve C lying in the interior of the disc.

Define the domain $\Omega = U \cap \mathbb{D}_R$, so that $\partial\Omega = C \cup \partial\mathbb{D}_R$. Since $\overline{\Omega} \subseteq E$, by Lemma 13.5 implies that

$$h_0(z) = \frac{z + h(z)}{z^\ell}$$

is analytic on Ω and continuous on $\overline{\Omega}$. Noting $C_k \subseteq E_k$, the definition of C_1 implies

$$|h_0(z)| = \frac{|z - r(z)|}{|z|^\ell} = 1$$

for all $z \in C_1$, and similarly

$$|h_0(z)| = \frac{|z + r(z)|}{|z|^\ell} = 1$$

for $z \in C_2$. Hence $|h_0| \equiv 1$ on C, and since $|h_0| \leq 1/2$ on $\partial\mathbb{D}_R$, it follows that

$$\max_{z\in\partial\overline{\Omega}} |h_0(z)| = 1,$$

and thus $|h_0| < 1$ on Ω by the maximum modulus principle. Therefore $|h_0(z)| < 1$ for all $z \in U$, so that

$$\frac{|z - r(z)|}{|z|^\ell} < 1$$

for $z \in U \cap E_1$, and

$$\frac{|z + r(z)|}{|z|^\ell} < 1$$

for $z \in U \cap E_2$.

Replacing h with $-h$ in the foregoing analysis leads to the conclusion that

$$\frac{|z + r(z)|}{|z|^\ell} < 1$$

for $z \in U \cap E_1$, and

$$\frac{|z - r(z)|}{|z|^\ell} < 1$$

for $z \in U \cap E_2$. Since

$$(U \cap E_1) \cup (U \cap E_2) = U \cap E = U \cap \big(\mathbb{C} \setminus [-1, 1]\big) = U,$$

the proof is done. ■

With our various lemmas in place, we are now ready to prove our theorem concerning the zero attractor of $\tilde{T}_n$.

Proof of Theorem 13.1. Let $K \subseteq U \cap \mathbb{H}$ be compact. From (13.3), for $z \in K$,

$$\begin{aligned}\lim_{n\to\infty} |\tilde{T}_n(z)|^{1/n} &= \lim_{n\to\infty} \left| \frac{[z - r(z)]^n}{2} + \frac{[z + r(z)]^n}{2} - z^{\ell n} \right|^{1/n} \\ &= |z|^\ell \lim_{n\to\infty} \left| \frac{[z - r(z)]^n}{2z^{\ell n}} + \frac{[z + r(z)]^n}{2z^{\ell n}} - 1 \right|^{1/n}. \end{aligned} \tag{13.12}$$

To evaluate this limit we first observe that

$$\rho(z) := \lim_{n\to\infty} \left| \left(\frac{z - r(z)}{z + r(z)} \right)^n \right|^{1/n} = \frac{|z - r(z)|}{|z + r(z)|}$$

uniformly on K. Since $r(z)$ is continuous and $z + r(z)$ nonvanishing on K, we see $\rho(z)$ is also continuous on K; moreover $\rho(z) < 1$ for all $z \in K$ by Lemma 13.4, so that $\|\rho\|_K < 1$, and Proposition 3.3 implies that

$$\lim_{n\to\infty} \left| \left(\frac{z - r(z)}{z + r(z)} \right)^n + 1 \right|^{1/n} = 1$$

uniformly on K as well. It then easily follows that

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{[z - r(z)]^n}{2z^{\ell n}} + \frac{[z + r(z)]^n}{2z^{\ell n}} \right|^{1/n} &= \lim_{n\to\infty} \frac{|z + r(z)|}{2^{1/n}|z|^{\ell}} \left| \left(\frac{z - r(z)}{z + r(z)} \right)^n + 1 \right|^{1/n} \\ &= \frac{|z + r(z)|}{|z|^{\ell}} \qquad (13.13)\end{aligned}$$

uniformly on K. By Lemma 13.7 the sup norm on K of the function at right in (13.13) is less than 1, so another application of Proposition 3.3 implies the limit in (13.12) equals 1 uniformly on K. Therefore

$$\lim_{n\to\infty} |\tilde{T}_n(z)|^{1/n} = |z|^{\ell} \qquad (13.14)$$

uniformly on K. A similar argument that reverses the roles of $z + r(z)$ and $z - r(z)$ yields the same result if $K \subseteq U \cap (\mathbb{C} \setminus \overline{\mathbb{H}})$, and since $r(z)$ restricted to $U \cap i\mathbb{R}$ is continuous, we conclude that (13.14) holds uniformly on any compact $K \subseteq U$.

Next, for $z \in B_1$ we use (13.8) to obtain

$$\begin{aligned}\lim_{n\to\infty} |\tilde{T}_n(z)|^{1/n} &= \lim_{n\to\infty} |z|^{\ell} \left| \frac{\left(z - \sqrt{z^2-1}\right)^n}{2z^{\ell n}} + \frac{\left(z + \sqrt{z^2-1}\right)^n}{2z^{\ell n}} - 1 \right|^{1/n} \\ &= \lim_{n\to\infty} |z|^{\ell} \left| \frac{\left(z - \sqrt{z^2-1}\right)^n}{2z^{\ell n}} + \frac{\left(z + \sqrt{z^2-1}\right)^n}{2z^{\ell n}} \right|^{1/n} \\ &= \left| z - \sqrt{z^2-1} \right| \lim_{n\to\infty} \left| 1 + \left(\frac{z + \sqrt{z^2-1}}{z - \sqrt{z^2-1}} \right)^n \right|^{1/n}. \qquad (13.15)\end{aligned}$$

Since

$$\lim_{n\to\infty} \left| \left(\frac{z + \sqrt{z^2-1}}{z - \sqrt{z^2-1}} \right)^n \right|^{1/n} = \left| \frac{z + \sqrt{z^2-1}}{z - \sqrt{z^2-1}} \right| < 1$$

by (13.6) in Lemma 13.4, Proposition 3.3 implies the last limit in (13.15) equals 1, and therefore

$$\lim_{n\to\infty} |\tilde{T}_n(z)|^{1/n} = \left| z - \sqrt{z^2-1} \right|$$

on B_1. For $z \in B_2$ a similar argument using (13.9) followed by (13.7) in Lemma 13.4 leads to the conclusion that

$$\lim_{n\to\infty} |\tilde{T}_n(z)|^{1/n} = \left|z + \sqrt{z^2-1}\right|$$

on B_2. To show that these limits in fact hold uniformly on compact subsets of B_1 or B_2 is straightforward with Proposition 3.3.

Observing that $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$, Lemma 13.5 implies that $h_1 : B' \to \mathbb{C}$ given by

$$h_1(z) = \begin{cases} z - \sqrt{z^2-1}, & z \in B_1 \\ z + \sqrt{z^2-1}, & z \in B_2 \end{cases}$$

is analytic on B', and thus our finding that

$$\lim_{n\to\infty} \ln |\tilde{T}_n(z)|^{1/n} = \begin{cases} \ln |h_1(z)|, & z \in B' \\ \ell \ln |z|, & z \in U \end{cases}$$

makes clear $\lim_{n\to\infty} \ln |\tilde{T}_n(z)|^{1/n}$ equals distinct harmonic functions on U and B'. By Theorem 2.5 we conclude that the zero attractor $\mathcal{A}$ of the sequence $\tilde{T}_n$ contains $C = \partial B$, and in fact $C \subseteq \mathcal{A} \subseteq C \cup [-1,1]$. We will now employ other means to determine that the interval $[-1,1]$ also lies in $\mathcal{A}$, finishing the proof.

For each n it is known that

$$T_n(z) = 2^{n-1} \prod_{k=1}^{n} \left(z - \cos\frac{(2k-1)\pi}{2n} \right),$$

so T_n has n distinct real zeros distributed uniformly on $[-1,1]$. Moreover, if $\zeta_1 < \zeta_2 < \zeta_3$ are three consecutive zeros of T_n, and $\mu_1 = (\zeta_1+\zeta_2)/2$ and $\mu_2 = (\zeta_2+\zeta_3)/2$, then $|T_n(\mu_1)| = |T_n(\mu_2)| = 1$ with $T_n(\mu_1) = -T_n(\mu_2)$.

Fix $x_0 \in [-1,1]$ and $\epsilon > 0$. Defining the deleted open disc $D'_\epsilon(x_0) = D_\epsilon(x_0) \setminus \{x_0\}$, choose $a < b$ to designate a closed interval $[a,b] \subseteq D'_\epsilon(x_0) \cap (-1,1)$. There exists n_0 such that, for all $n > n_0$, $[a,b]$ contains three consecutive zeros $\zeta_{n,1} < \zeta_{n,2} < \zeta_{n,3}$ of T_n, with midpoints $\zeta_{n,1} < \mu_{n,1} < \zeta_{n,2}$ and $\zeta_{n,2} < \mu_{n,2} < \zeta_{n,3}$. For definiteness assume $T_n(\mu_{n,1}) = -1$ and $T_n(\mu_{n,2}) = 1$. There also exists n_1 such that $0 \le x^{\ell n} < 1/2$ for all $n > n_1$ and $x \in [a,b]$, so that in particular

$$\tilde{T}_n(\mu_{n,1}) = T_n(\mu_{n,1}) - x^{\ell n} = -1 - x^{\ell n} < 0$$

and

$$\tilde{T}_n(\mu_{n,2}) = T_n(\mu_{n,2}) - x^{\ell n} = 1 - x^{\ell n} > \frac{1}{2} > 0.$$

Now the intermediate value theorem ensures there is at least one zero of $\tilde{T}_n$ in $(\mu_{n,1}, \mu_{n,2}) \subseteq D'_\epsilon(x_0)$ for each $n > \max\{n_0, n_1\}$. This implies that x_0 is not only a limit point of the set of all zeros of the sequence $\tilde{T}_n$, but in fact is in the zero attractor $\mathcal{A}$. Therefore $[-1, 1] \subseteq \mathcal{A}$. ■

Section 14: Conclusion

In the early days of grappling with the zero attractor of the sequence

$$p_n(z) = As_{an}(\alpha nz) + Bs_{bn}(\beta nz)$$

it was hoped that, once the analysis was finished, the problem of determining the zero attractor of

$$p_n(z) = \sum_{k=1}^{N} A_k s_{a_k n}(\alpha_k nz)$$

could be solved via a similar analysis. However, the "two-term case" that spans sections 6 through 12 of this thesis is seen to be quite complicated, and so we turned instead to the problem of finding the zero attractor of perturbed Chebyshev polynomials in §13. As this bore fruit, and proved to be neither too difficult nor too trivial an undertaking, the problem of determining the zero attractors of more general perturbations of Chebyshev polynomials remains a possible avenue (among many others) for future research.

List of References

[1] R. Ash and W. P. Novinger, *Complex Variables*, 2nd edition, Dover Publications, 2004.

[2] P. Bleher and R. Mallison, Jr., *Zeros of sections of exponential sums*, Int. Math. Res. Not. (2006), Art. ID 38937, 49.

[3] P. B. Borwein, W. Chen, and K. Dilcher, *Zeros of iterated integrals of polynomials*, Can. J. Math. **11** (1995), 65-87.

[4] Robert P. Boyer and William M. Y. Goh, *Appell polynomials and their zero attractors*, Contemporary Mathematics, **517** (2010), 69-96.

[5] J. Erickson, *Zeros of linear combinations of partial sums of the exponential function*, Ph.D. candidacy paper, Drexel University, 2016.

[6] M. Marden, *Geometry of Polynomials*, Amer. Math. Soc. Surveys, No. 3 (1996).

[7] G. Pólya and G. Szegő, *Problems and Theorems in Analysis I*, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[8] T. J. Rivlin, *Chebyshev Polynomials*, 2nd ed. Wiley, New York (1990).

[9] A. Sokal, *Chromatic roots are dense in the whole complex plane*, Combin. Probab. Computing. **13** (2004), 221-261.

[10] G. Szegő, *Über eine Eigenschaft der Exponentialreihe*, Sitzungsber. Berl. Math. Ges. **23** (1924), 50-64.

[11] A. Vargas, *Limit curves for zeros of sections of exponential integrals*, Constructive Approximation, **40** (2014), No. 2, pp. 219-239.

Appendix A: Symbol Glossary

$\mathbb{A}_{s,t}$	Open annulus $\{z \in \mathbb{C} : s < \lvert z\rvert < t\}$	p. 13
C_{ab}	Circle containing intersection of $\lvert\varphi(z/a)\rvert = 1$ and $\lvert\varphi(z/b)\rvert = 1$	p. 21
$\mathcal{C}_r$	"Intersection circle" at 0 with radius ρ_r, containing $\mathcal{S}_2 \cap \mathcal{S}_3$	p. 37
$\mathbb{D}_s$	Open disc at 0 with radius s	p. 13
D_{ab}	Open disc with boundary C_{ab}	p. 21
$\mathcal{D}_r$	Open disc with boundary $\mathcal{C}_r$	p. 38
$D_s(z)$	Open disc at z with radius s	p. 6
$\mathbb{H}$	Open half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$	p. 30
$\mathcal{H}^+_{r\theta}$	Open half-plane $e^{(\ell_{r\theta}+\pi/2)i}\mathbb{H}$	p. 38
$\mathcal{H}^-_{r\theta}$	Open half-plane $-\mathcal{H}^+_{r\theta}$	p. 38
H_θ	Rotated open half-plane $\{e^{i\theta}z : z \in \mathbb{H}\}$	p. 30
$\ell_{r\theta}$	Angle between the line $\mathcal{L}_{r\theta}$ and the positive real axis	p. 38
$\mathcal{L}_{r\theta}$	"Intersection line" $\{z : \mathrm{Arg}(\pm z) = \ell_{r\theta}\}$ containing $\mathcal{S}_1 \cap \mathcal{S}_3$	p. 37
L_w	Open region $\{z : \lvert\varphi(z/w)\rvert < 1 \text{ and } \lvert z\rvert < \lvert w\rvert\}$	p. 12
$\overline{S}$	Closure of set S	p. 9
S°	Interior of set S	p. 9
$\mathcal{S}^>$	Exterior of Szegő curve $\mathcal{S}$	p. 37
$\mathcal{S}^<$	Interior of Szegő curve $\mathcal{S}$	p. 37
$s_n(z)$	Taylor polynomial $\sum_{k=0}^{n} z^k/k!$	p. 3
S_w	Szegő curve $\lvert\varphi(z/w)\rvert = 1$	p. 29
T_w	Open region $\{z : \lvert\varphi(z/w)\rvert < 1 \text{ and } \lvert z\rvert > \lvert w\rvert\}$	p. 21
$Z(p(z))$	Also $Z(p)$, set of zeros of the polynomial $p(z)$	p. 6
ρ_r	"Intersection radius," or radius of $\mathcal{C}_r$	p. 37
$\varphi(z)$	Expression ze^{1-z}	p. 12
$\sim$	Asymptotic equivalence	p. 31

Vita

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He began work on his Ph.D. in Mathematics at Drexel University in the fall of 2014, and completed the program in 2019. Though still teaching at BCCC, he plans to continue the mathematical research he began with his doctoral dissertation, "The Zero Attractor of Perturbed Chebyshev Polynomials and Sums of Taylor Polynomials."