A

Appendix

A.1 – Formal Power Series

An **indeterminate** is a symbol, such as X, that serves as a scaffold for algebraic structures but has no meaning by itself. That is, X is an abstract object that does not represent a variable, a numerical quantity either known or unknown, or any other specific mathematical entity. A mathematical expression constructed using one or more indeterminates is called a **formal expression**. For example an expression of the form

$$a_0 + a_1X + a_2X^2 + \cdots + a_nX^n,$$

where $a_0, \ldots, a_n \in \mathbb{C}$, is called a **formal polynomial in X over $\mathbb{C}$**.

Let X be an indeterminate, and let R be a commutative ring with additive identity 0 and multiplicative identity 1. Define a **formal power series in X** to be the expression

$$f(X) = \sum_{n=0}^{\infty} a_nX^n = a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots$$

with **coefficients** $a_n \in R$ for all $n \geq 0$. For any $n \geq 0$ we call a_n the ***n*th-order coefficient** of $f(X)$, and a_nX^n the ***n*th-order term**. The 0th-order term is also called the **constant term**. Some special formal power series are

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!}X^n, \quad \sin(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}X^{2n+1}, \quad \cos(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}X^{2n}.$$

Note in particular that $\sin(X)$ has nth-order coefficients equal to 0 if n is even, and $\cos(X)$ has nth-order coefficients equal to 0 if n is odd. An alternate symbol for $\exp(X)$ is e^X. Another special series is the **zero power series**,

$$\mathbf{0}(X) = 0 + 0X + 0X^2 + 0X^3 + \cdots,$$

also denoted simply by $\mathbf{0}$.

Remark. If the indeterminate X is understood, the symbol f may be used to denote $f(X)$. We will largely adhere to this convention from now on.

Definition A.1. *The **order** of $f = \sum_{n=0}^{\infty} a_nX^n$, denoted by $\operatorname{ord}(f)$, is defined to be the lowest n for which $a_n \neq 0$. That is,*

$$\operatorname{ord}(f) = \min\{n : a_n \neq 0\}.$$

We define $\mathrm{ord}(\mathbf{0}) = \infty$.

Given another formal power series

$$g = \sum_{n=0}^{\infty} b_n X^n,$$

we define the **sum** of f and g to be

$$f + g = \sum_{n=0}^{\infty} (a_n + b_n) X^n,$$

the **product** to be

$$fg = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n,$$

the **constant multiple** to be

$$cf = \sum_{n=0}^{\infty} (ca_n) X^n$$

for any $c \in R$, and the **difference** to be

$$f - g = f + (-1)g.$$

With the sum and product operations we can construct the **ring of formal power series in X with coefficients in R**, denoted by $R[[X]]$. If $f = \sum_{n=0}^{\infty} a_n X^n$ is such that $a_n = 0$ for all but finitely many $n \geq 0$ such as $n_1, \ldots, n_k$, then f reduces to a formal polynomial,

$$f = a_{n_1} X^{n_1} + \cdots + a_{n_k} X^{n_k},$$

and we see that the ring of formal polynomials in X over R is a subring of $R[[X]]$. By definition the arithmetic of formal polynomials in X over R conforms exactly with the rules of standard algebra in which X is a variable that represents a complex number. In particular we have the usual properties of exponents such as $X^m X^n = X^{m+n}$, $(X^m)^n = X^{mn}$, and $(aX)^n = a^n X^n$ for any $a \in R$.

It is a fact that $R[[X]]$ is a unitary ring, for we may define

$$\mathbf{1} = \sum_{n=0}^{\infty} \delta_{0n} X^n = 1 + 0X + 0X^2 + \cdots = 1,$$

where δ_{ij} represents the Kronecker delta, and see that

$$f = \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \delta_{0,n-k} \right) X^n = f\mathbf{1}$$

and

$$f = \sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \delta_{0k} a_{n-k} \right) X^n = \mathbf{1}f.$$

Remark. If $f \in R[[X]]$ happens to be a polynomial, then it is customary to represent f by the polynomial. Thus in particular we have $\mathbf{1} = 1$ and $\mathbf{0} = 0$, which is to say the formal power series $\mathbf{1}$ and $\mathbf{0}$ may be represented by the polynomials 1 and 0.

We say that two formal power series

$$f = \sum_{n=0}^{\infty} a_n X^n \quad \text{and} \quad g = \sum_{n=0}^{\infty} b_n X^n,$$

are **equal** if and only if $a_n = b_n$ for all $n \geq 0$.

Definition A.2. *Two formal power series $f = \sum_{n=0}^{\infty} a_n X^n$ and $g = \sum_{n=0}^{\infty} b_n X^n$ are* ***congruent modulo*** *X^k, written*

$$f \equiv g \mod (X^k),$$

if $a_n = b_n$ for all $0 \leq n \leq k-1$.

It is easy to verify that the relation $\equiv$ is an equivalence relation, which is to say for any $f, g, h \in R[[X]]$ we have $f \equiv f$,

$$f \equiv g \quad \Leftrightarrow \quad g \equiv f,$$

and

$$f \equiv g,\ g \equiv h \quad \Rightarrow \quad f \equiv h.$$

It is also easy to see that $f = g$ if and only if $f \equiv g \mod (X^k)$ for all $k \geq 1$.

Proposition A.3. *Let $f_1, f_2, g_1, g_2 \in R[[X]]$ such that $f_1 \equiv f_2 \mod (X^k)$ and $g_1 \equiv g_2 \mod (X^k)$.*

1. *For any $c \in R$, $cf_1 \equiv cf_2 \mod (X^k)$*
2. *For any $n \in \mathbb{N}$, $f_1^n \equiv f_2^n \mod (X^k)$*
3. $f_1 + g_1 \equiv f_2 + g_2 \mod (X^k)$
4. $f_1 g_1 \equiv f_2 g_2 \mod (X^k)$

The following proposition gives a few properties of formal power series arithmetic. The first part in particular establishes that the ring $R[[X]]$ is commutative.

Proposition A.4. *Let $a, b \in R$. For any $f, g, h \in R[[X]]$,*

1. $fg = gf$
2. $f(g + h) = fg + fh$
3. $(af)(bg) = (ab)(fg)$

Definition A.5. *Let $f \in R[[X]]$. If $g \in R[[X]]$ is such that*

$$fg = \mathbf{1},$$

then f is said to be ***invertible in*** *$R[[X]]$, and g is called the* ***multiplicative inverse*** *(or* ***reciprocal****) of f in $R[[X]]$. The multiplicative inverse of f is denoted by $1/f$ or f^{-1}.*

We will usually refer to the multiplicative inverse of any $f \in R[[X]]$ as simply the "inverse" of f whenever it does not lead to ambiguity.

Theorem A.6. *$f = \sum_{n=0}^{\infty} a_n X^n$ is invertible in $R[[X]]$ if and only if $a_0 \neq 0$.*

Proof. First suppose that $f = \sum_{n=0}^{\infty} a_n X^n$ with $a_0 = 1$. If

$$h = \sum_{n=0}^{\infty} (\delta_{0n} - 1) a_n X^n = -a_1 X - a_2 X^2 - a_3 X^3 - \cdots ,$$

then by the definition of the constant multiple and sum operations

$$\begin{aligned}
\mathbf{1} - h &= \sum_{n=0}^{\infty} \delta_{0n} X^n - \sum_{n=0}^{\infty} (\delta_{0n} - 1) a_n X^n = \sum_{n=0}^{\infty} \delta_{0n} a_n X^n - \sum_{n=0}^{\infty} (\delta_{0n} - 1) a_n X^n \\
&= \sum_{n=0}^{\infty} \delta_{0n} a_n X^n + \sum_{n=0}^{\infty} (1 - \delta_{0n}) a_n X^n = \sum_{n=0}^{\infty} [\delta_{0n} + (1 - \delta_{0n})] a_n X^n \\
&= \sum_{n=0}^{\infty} (1) a_n X^n = f. \qquad (1)
\end{aligned}$$

Now, define

$$\varphi = \sum_{n=0}^{\infty} h^n = 1 + h + h^2 + h^3 + \cdots . \qquad (2)$$

We first show that $\varphi \in R[[X]]$. By the definition of the multiplication operation we easily discover that, for each $m \geq 1$,

$$h^m = \sum_{n=0}^{\infty} b_{mn} X^n$$

with $b_{mn} = 0$ for all $n < m$. Hence h^m is a formal power series in X for which $\operatorname{ord}(h^m) \geq m$, and

$$\varphi = 1 + \sum_{n=0}^{\infty} b_{1n} X^n + \sum_{n=0}^{\infty} b_{2n} X^n + \sum_{n=0}^{\infty} b_{3n} X^n + \cdots$$

If we define $b_{0n} = \delta_{0n}$ for all $n \geq 0$, then

$$\begin{aligned}
\varphi &= \sum_{n=0}^{\infty} b_{0n} X^n + \sum_{n=1}^{\infty} b_{1n} X^n + \sum_{n=2}^{\infty} b_{2n} X^n + \sum_{n=3}^{\infty} b_{3n} X^n + \cdots \\
&= b_{00} + (b_{01} + b_{11}) X + (b_{02} + b_{12} + b_{22}) X^2 + (b_{03} + b_{13} + b_{23} + b_{33}) X^3 + \cdots
\end{aligned}$$

by the definition of the sum operation. Setting

$$c_n = \sum_{k=0}^{n} b_{kn},$$

we may write

$$\varphi = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_{kn} X^n \right) = \sum_{n=0}^{\infty} c_n X^n,$$

and so $\varphi \in R[[X]]$ since $c_n \in R$ for each n.

From (1) we find that f as an element of $R[[h]]$ is given by $f = 1 - h$, which is to say

$$f = \sum_{n=0}^{\infty} \alpha_n h^n$$

with $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_n = 0$ for $n \geq 2$. Of course $\varphi \in R[[h]]$ is given by

$$\varphi = \sum_{n=0}^{\infty} \beta_n h^n$$

with $\beta_n = 1$ for all $n \geq 0$. Now, the product of f and φ as elements of $R[[h]]$ is

$$\begin{aligned} f\varphi &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \alpha_k \beta_{n-k}\right) h^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \alpha_k\right) h^n = \alpha_0 + (\alpha_0 + \alpha_1)h + \sum_{n=2}^{\infty}\left(\sum_{k=0}^{n} \alpha_k\right) h^n \\ &= 1 + (1-1)h + \sum_{n=2}^{\infty}\left(\sum_{k=2}^{n} \alpha_k\right) h^n = 1 + 0h + \sum_{n=2}^{\infty}(0)h^n = 1, \end{aligned}$$

and by Proposition A.15 below it follows that the product of f and φ as elements of $R[[X]]$ is also 1:

$$f\varphi = \left(\sum_{n=0}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} c_n X^n\right) = 1.$$

Therefore φ is an inverse for f in $R[[X]]$.

If $a_0 \neq 0, 1$, define

$$\hat{f} = \sum_{n=0}^{\infty} \hat{a}_n X^n = \sum_{n=0}^{\infty} \frac{a_n}{a_0} X^n = \frac{1}{a_0} f,$$

so that $\hat{f}$ is such that $\hat{a}_0 = 1$. There exists some $\hat{\varphi}$ such that $\hat{f}\hat{\varphi} = \mathbf{1}$. Let $\varphi = a_0^{-1}\hat{\varphi}$. Then $\hat{\varphi} = a_0\varphi$, and by Proposition A.4

$$\hat{f}\hat{\varphi} = \mathbf{1} \quad \Leftrightarrow \quad \left(\frac{1}{a_0} f\right)(a_0\varphi) = \mathbf{1} \quad \Leftrightarrow \quad \left(\frac{1}{a_0} \cdot a_0\right)(f\varphi) = \mathbf{1} \quad \Leftrightarrow \quad f\varphi = \mathbf{1}.$$

Therefore φ is an inverse for f.

The proof that f is not invertible in $R[[X]]$ if $a_0 = 0$ will come later. ■

Definition A.7. *The **quotient** of $f, g \in R[[X]]$ is*

$$f/g = fg^{-1},$$

provided that g^{-1} exists.

Example A.8. Find f/g for $f = \sin(X)$ and $g = \cos(X)$

With the quotient operation we define new formal power series

$$\tan(X) = \frac{\sin(X)}{\cos(X)} \quad \text{and} \quad \cot(X) = \frac{\cos(X)}{\sin(X)}.$$

A **formal Laurent series in** X is a formal expression of the form

$$\sum_{n\in\mathbb{Z}} a_n X^n$$

such that $a_n = 0$ for all but finitely many $n < 0$. Thus a formal Laurent series is much like a formal power series, only a finite number of terms of negative order may be included. If $f \in R[[X]]$ does not have an inverse in $R[[X]]$, there may nevertheless be a formal Laurent series ℓ such that $f\ell = \mathbf{1}$, in which case ℓ is called the **inverse** for f. Suppose that

$$f = \sum_{n=m}^{\infty} a_n X^n$$

for $m > 1$, where $a_m \neq 0$. We have

$$f = a_m X^m + a_{m+1} X^{m+1} + a_{m+2} X^{m+2} + \cdots = a_m X^m g,$$

where

$$\psi = 1 + \frac{a_{m+1}}{a_m} X + \frac{a_{m+2}}{a_m} X^2 + \cdots$$

has inverse $\psi^{-1} \in R[[X]]$ by Theorem A.6. The formal power series $a_m X^m$ is not invertible in $R[[X]]$, nevertheless we define

$$(a_m X^m)^{-1} = a_m^{-1} X^{-m}.$$

Now, observing that

$$f \cdot a_m^{-1} X^{-m} \psi^{-1} = a_m X^m \psi \cdot a_m^{-1} X^{-m} \psi^{-1} = a_m a_m^{-1} X^m X^{-m} \psi \psi^{-1} = \mathbf{1},$$

we obtain a reasonable definition for an inverse of $f = a_m X^m \psi$:

$$f^{-1} = a_m^{-1} X^{-m} \psi^{-1}.$$

With this expansion of the notion of the inverse of a formal power series, we are in a position to determine a greater variety of quotients.

Example A.9. Let $f = X$ and $g = e^X - 1$. Find the terms of order ≤ 4 for f/g.

Solution. We have

$$e^X - 1 = \sum_{n=0}^{\infty} \frac{1}{n!} X^n - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} X^n = X + \frac{1}{2} X^2 + \frac{1}{6} X^3 + \frac{1}{24} X^4 + \cdots,$$

which by Theorem A.6 has no inverse in $R[[X]]$ since it has constant term equal to 0. We write

$$e^X - 1 = X\left(1 + \frac{1}{2} X + \frac{1}{6} X^2 + \frac{1}{24} X^3 + \cdots\right) = X\psi,$$

where

$$\psi = 1 + \frac{1}{2} X + \frac{1}{6} X^2 + \frac{1}{24} X^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} X^n.$$

Now, we have $\psi = 1 - h$ for

$$h = -\frac{1}{2} X - \frac{1}{6} X^2 - \frac{1}{24} X^3 + \cdots = -\sum_{n=1}^{\infty} \frac{1}{(n+1)!} X^n,$$

and by the proof of Theorem A.6

$$\psi^{-1} = \sum_{n=0}^{\infty} h^n.$$

Up to order 4 we have

$$\begin{aligned} h^2 &= \frac{1}{4}X^2 + \frac{1}{6}X^3 + \frac{7}{144}X^4 + \cdots, \\ h^3 &= -\frac{1}{8}X^3 - \frac{1}{8}X^4 + \cdots, \\ h^4 &= \frac{1}{16}X^4 + \cdots, \end{aligned}$$

and so

$$\psi^{-1} = 1 + h + h^2 + h^3 + h^4 + \cdots = 1 - \frac{1}{2}X + \frac{1}{12}X^2 + 0X^3 - \frac{1}{45}X^4 + \cdots$$

We now compute the quotient f/g:

$$f/g = fg^{-1} = X(e^X - 1)^{-1} = X(X^{-1}\psi^{-1}) = (XX^{-1})\psi^{-1} = \psi^{-1}.$$

That is,

$$f/g = 1 - \frac{1}{2}X + \frac{1}{12}X^2 - \frac{1}{45}X^4 + \cdots$$

■

Exercise A.10 (La2.1.6). Let $a_0, a_1, u_1, u_2 \in \mathbb{C}$ (with $u_2 \neq 0$), and define

$$a_n = u_1 a_{n-1} + u_2 a_{n-2} \tag{3}$$

for $n \geq 2$. If $T^2 - u_1T - u_2 = (T - \alpha)(T - \beta)$ for $\alpha \neq \beta$, show that there exist numbers A, B such that

$$a_n = A\alpha^n + B\beta^n \tag{4}$$

for all $n \geq 0$. Also show that the formal power series

$$\sum_{n=0}^{\infty} a_n T^n$$

is a formal rational expression, and give its partial fraction decomposition.

Solution. Suppose $T^2 - u_1T - u_2 = (T - \alpha)(T - \beta)$ for $\alpha \neq \beta$. Then

$$\alpha^2 - u_1\alpha - u_2 = 0 \quad \text{and} \quad \beta^2 - u_1\beta - u_2 = 0,$$

and so

$$u_2 = \alpha^2 - u_1\alpha \quad \text{and} \quad u_2 = \beta^2 - u_1\beta.$$

(Note: $u_2 \neq 0$ implies that $\alpha, \beta \neq 0$.)

We start by showing that there exist numbers A and B such that (4) holds for $n = 0, 1$. This entails showing that the system

$$\begin{cases} A + B = a_0 \\ A\alpha + B\beta = a_1 \end{cases}$$

has a solution, which we easily find to be the case:

$$A = \frac{a_1 - a_0\beta}{\alpha - \beta} \quad \text{and} \quad B = a_0 - \frac{a_1 - a_0\beta}{\alpha - \beta}. \tag{5}$$

For each $k \geq 0$ define the statement

$$P(k) = \text{“The numbers } A \text{ and } B \text{ satisfy (4) for all } 0 \leq n \leq k.\text{”}$$

We have found that $P(1)$ is true. Fix $k \geq 1$ and suppose that $P(k)$ is true. We endeavor to show that $P(k+1)$ must be true. Since $k+1 \geq 2$, we employ the difference equation (3) to obtain

$$\begin{aligned}
a_{k+1} &= u_1 a_k + u_2 a_{k-1} \\
&= u_1(A\alpha^k + B\beta^k) + u_2(A\alpha^{k-1} + B\beta^{k-1}) \\
&= Au_1\alpha^k + Bu_1\beta^k + Au_2\alpha^{k-1} + Bu_2\beta^{k-1} \\
&= Au_1\alpha^k + Bu_1\beta^k + A(\alpha^2 - u_1\alpha)\alpha^{k-1} + B(\beta^2 - u_1\beta)\beta^{k-1} \\
&= Au_1\alpha^k + Bu_1\beta^k + A\alpha^{k+1} - Au_1\alpha^k + B\beta^{k+1} - Bu_1\beta^k \\
&= A\alpha^{k+1} + B\beta^{k+1}.
\end{aligned}$$

From this it follows that A and B satisfy (4) when $n = k+1$, and thus $P(k+1)$ is true. By the principle of induction we conclude that $P(k)$ is true for all $k \geq 1$, and therefore A and B satisfy (4) for all $n \geq 0$.

With these values for A and B we have $a_n = A\alpha^n + B\beta^n$ for all $n \geq 0$. This implies that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n T^n &= \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) T^n = \sum_{n=0}^{\infty} A\alpha^n T^n + \sum_{n=0}^{\infty} B\beta^n T^n \\
&= A\sum_{n=0}^{\infty} (\alpha T)^n + B\sum_{n=0}^{\infty} (\beta T)^n = A \cdot \frac{1}{1 - \alpha T} + B \cdot \frac{1}{1 - \beta T},
\end{aligned}$$

recalling that in general $(1 - X)^{-1} = \sum_{n=0}^{\infty} X^n$. Therefore $\sum_{n=0}^{\infty} a_n T^n$ is a formal rational expression with partial fraction decomposition given by

$$\sum_{n=0}^{\infty} a_n T^n = \frac{A}{1 - \alpha T} + \frac{B}{1 - \beta T},$$

where A and B are as given by (5). ■

Exercise A.11 (La2.1.7). Let $b_0, \ldots, b_{r-1}, u_1, \ldots, u_r \in \mathbb{C}$ with $u_r \neq 0$, and define

$$b_n = u_1 b_{n-1} + \cdots + u_r b_{n-r} \tag{6}$$

for $n \geq r$. If

$$T^r - u_1 T^{r-1} - \cdots - u_{r-1} T - u_r = (T - \alpha_1) \cdots (T - \alpha_r) \tag{7}$$

such that $\alpha_i \neq \alpha_j$ whenever $i \neq j$, show that there exist numbers $A_1, \cdots, A_r$ such that

$$b_n = A_1 \alpha_1^n + \cdots + A_r \alpha_r^n$$

for all $n \geq 0$.

Solution. Suppose (7) is the case for distinct numbers $\alpha_1, \ldots, \alpha_r$. Then for each $1 \leq i \leq r$,

$$\alpha_i^r - u_1\alpha_i^{r-1} - \cdots - u_{r-1}\alpha_i - u_r = 0,$$

whence

$$u_r = \alpha_i^r - u_1\alpha_i^{r-1} - \cdots - u_{r-1}\alpha_i.$$

Observe that $u_r \neq 0$ implies that $\alpha_1, \ldots, \alpha_r \neq 0$.

We start by showing that there exist constants $A_1, \ldots, A_r$ such that

$$b_n = \sum_{j=1}^{r} A_j\alpha_j^n \tag{8}$$

holds for all $0 \leq n \leq r-1$. Consider the system

$$\begin{cases} x_1 + \quad x_2 + \cdots + \quad x_r = b_0 \\ \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = b_1 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ \alpha_1^{r-1}x_1 + \alpha_2^{r-1}x_2 + \cdots + \alpha_r^{r-1}x_r = b_{r-1} \end{cases}$$

Letting

$$\mathbf{A} = \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{r-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_r & \cdots & \alpha_r^{r-1} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{r-1} \end{bmatrix},$$

the system may be written as $\mathbf{A}^T\mathbf{x} = \mathbf{b}$. Now, $\det(\mathbf{A})$ is an $r \times r$ Vandermonde determinant, and by the results of an example in §5.2 of the *Linear Algebra Notes* [LAN] we have

$$\det(\mathbf{A}) = \prod_{1 \leq i < j \leq r} (\alpha_j - \alpha_i) \neq 0,$$

since $\alpha_i \neq \alpha_j$ whenever $i \neq j$. Now, by a theorem in §7.2 of [LAN],

$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \neq 0,$$

and so by a proposition in §5.4 of [LAN] we conclude that the system of equations has a unique solution. That is, there exist numbers $A_1, \ldots, A_r$ such that (8) holds for $n = 0, \ldots, r-1$.

For each $k \geq 0$ define the statement

$$P(k) = \text{“The numbers } A_1, \ldots, A_r \text{ satisfy (8) for all } 0 \leq n \leq k.\text{”}$$

We have found that $P(r-1)$ is true. Fix $k \geq r-1$ and suppose that $P(k)$ is true. We endeavor to show that $P(k+1)$ must be true. Since $k+1 \geq r$, we employ the difference equation (6) to obtain

$$b_{k+1} = \sum_{i=1}^{r} u_i b_{k+1-i} = \sum_{i=1}^{r}\left(u_i \sum_{j=1}^{r} A_j\alpha_j^{k+1-i}\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{r-1}\left(u_i \sum_{j=1}^{r} A_j \alpha_j^{k+1-i}\right) + \sum_{j=1}^{r} A_j u_r \alpha_j^{k+1-r} \\
&= \sum_{i=1}^{r-1}\sum_{j=1}^{r} A_j u_i \alpha_j^{k+1-i} + \sum_{j=1}^{r} A_j \left(\alpha_j^r - \sum_{i=1}^{r-1} u_i \alpha_j^{r-i}\right)\alpha_j^{k+1-r} \\
&= \sum_{i=1}^{r-1}\sum_{j=1}^{r} A_j u_i \alpha_j^{k+1-i} + \sum_{j=1}^{r} A_j \alpha_j^{k+1} - \sum_{j=1}^{r}\sum_{i=1}^{r-1} A_j u_i \alpha_j^{k+1-i} \\
&= \sum_{j=1}^{r} A_j \alpha_j^{k+1}.
\end{aligned}
$$

From this it follows that $A_1, \ldots, A_r$ satisfy (8) when $n = k+1$, and thus $P(k+1)$ is true. By the principle of induction we conclude that $P(k)$ is true for all $k \geq r-1$, and therefore $A_1, \ldots, A_r$ satisfy (8) for all $n \geq 0$. ■

Suppose that $f, g \in R[[X]]$ are given by

$$
f = \sum_{n=0}^{\infty} a_n X^n \quad \text{and} \quad g = \sum_{n=1}^{\infty} b_n X^n,
$$

so that in particular the constant term for g is $b_0 = 0$. If we regard g to be itself an indeterminate, then we have

$$
f(g) = a_0 + a_1 g + a_2 g^2 + \cdots = \sum_{n=0}^{\infty} a_n g^n \in R[[g]].
$$

The argument that $f(g) \in R[[X]]$ is much the same as that used to show the power series (2) is in $R[[X]]$. For each $n \geq 0$ we find that $a_n g^n$ is itself a formal power series in $R[[X]]$ such that $\operatorname{ord}(a_n g^n) \geq n$, which is to say

$$
a_n g^n = a_n\big(b_1 X + b_2 X^2 + b_3 X^3 + \cdots\big)^n = \sum_{k=n}^{\infty} c_{nk} X^k
$$

for appropriate values $c_{nk} \in R$, $k \geq n$. Thus the nth-order X coefficient of $f(g)$ is

$$
c_n = \sum_{k=0}^{n} c_{kn}
$$

for $n \geq 0$, which is to say

$$
f(g) = \sum_{n=0}^{\infty} c_n X^n \in R[[X]].
$$

We have now demonstrated a new kind of binary operation $(f, g) \mapsto f \circ g$ for formal power series that is closed on $R[[X]]$ provided that $\operatorname{ord}(g) > 0$.

Definition A.12. *Let* $f = \sum_{n=0}^{\infty} a_n X^n$ *and* $g = \sum_{n=1}^{\infty} b_n X^n$ *be series in* $R[[X]]$*. The* ***composition*** *of* f *with* g *is the series* $f \circ g \in R[[X]]$ *given by*

$$f \circ g = f(g) = \sum_{n=0}^{\infty} a_n g^n.$$

Proposition A.13. *Let* $f_1, f_2, g_1, g_2 \in R[[X]]$ *such that* $\mathrm{ord}(g_1), \mathrm{ord}(g_2) > 0$*. If*

$$f_1 \equiv f_2 \mod (X^k) \quad \text{and} \quad g_1 \equiv g_2 \mod (X^k),$$

then

$$f_1 \circ g_1 \equiv f_2 \circ g_2 \mod (X^k).$$

Proposition A.14. *Let* $f, g, h \in R[[X]]$ *such that* $\mathrm{ord}(h) > 0$*.*
1. $(f + g) \circ h = f \circ h + g \circ h$
2. $(fg) \circ h = (f \circ h)(g \circ h)$
3. *If* $\mathrm{ord}(g) = 0$*, then* $(f/g) \circ h = (f \circ h)/(g \circ h)$
4. *If* $\mathrm{ord}(g) > 0$*, then* $f \circ (g \circ h) = (f \circ g) \circ h$*.*

Proof. *Proof of Part (4).* Let

$$f = \sum_{n=0}^{\infty} a_n X^n, \quad g = \sum_{n=1}^{\infty} b_n X^n, \quad h = \sum_{n=1}^{\infty} c_n X^n,$$

so that $\mathrm{ord}(g), \mathrm{ord}(h) > 1$. Fix $k \geq 1$. Define the formal polynomials

$$F = \sum_{n=0}^{k-1} a_n X^n, \quad G = \sum_{n=0}^{k-1} b_n X^n, \quad H = \sum_{n=0}^{k-1} c_n X^n,$$

so we have

$$F \equiv f \mod (X^n), \quad G \equiv g \mod (X^n), \quad H \equiv h \mod (X^n).$$

Since the algebra of formal polynomials is defined to be the same as the algebra of polynomial functions, we obtain

$$F \circ (G \circ H) = (F \circ G) \circ H. \tag{9}$$

Now, by Proposition A.13 $G \equiv g$ and $H \equiv h$ imply $G \circ H \equiv g \circ h$, and then since $F \equiv f$ it follows that

$$F \circ (G \circ H) \equiv f \circ (g \circ h) \mod (X^k). \tag{10}$$

Next, from $F \equiv f$ and $G \equiv g$ comes $F \circ G \equiv f \circ g$, and then since $H \equiv h$ it follows that

$$(F \circ G) \circ H \equiv (f \circ g) \circ h \mod (X^k). \tag{11}$$

Combining (9), (10), and (11), we obtain

$$f \circ (g \circ h) \equiv F \circ (G \circ H) = (F \circ G) \circ H \equiv (f \circ g) \circ h \mod (X^k),$$

and hence

$$f \circ (g \circ h) \equiv (f \circ g) \circ h \mod (X^k). \tag{12}$$

Since $k \geq 1$ is arbitrary, we conclude that (12) holds for all $k \geq 1$.

Therefore $f \circ (g \circ h) = (f \circ g) \circ h$. ∎

In general if $Y = \sum_{n=1}^{\infty} c_n X^n$ and we define $Y_k = \sum_{n=1}^{k} c_n X^n$ for all $k \geq 1$, then to write

$$\sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} b_n Y^n$$

means simply

$$\sum_{n=0}^{\infty} a_n X^n \equiv \sum_{n=0}^{k} b_n Y_k^n \mod (X^k)$$

for all $k \geq 1$.

Let the symbols $\cdot_X$ and $\cdot_Y$ denote the multiplication operations for $R[[X]]$ and $R[[Y]]$, respectively. One last result we should like to establish is that the multiplication operation for formal power series is "well-defined" in the following sense.

Proposition A.15. *Given* $Y = \sum_{n=1}^{\infty} c_n X^n$, *if*

$$\sum_{n=0}^{\infty} a_n X^n = \sum_{n=0}^{\infty} \alpha_n Y^n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} \beta_n Y^n,$$

then

$$\left(\sum_{n=0}^{\infty} a_n X^n\right) \cdot_X \left(\sum_{n=0}^{\infty} b_n X^n\right) = \left(\sum_{n=0}^{\infty} \alpha_n Y^n\right) \cdot_Y \left(\sum_{n=0}^{\infty} \beta_n Y^n\right) \tag{13}$$

The crux of the proof depends on the known fact that multiplication of two formal polynomials in X, say $p(X)$ and $q(X)$, yields the same polynomial (in terms of X) regardless of how p and q may be expressed in terms of some other polynomial $Y = r(X)$. That is, given $Y = \sum_{n=0}^{j} e_n X^n$, if

$$p(X) = \sum_{n=0}^{k_1} a_n X^n = \sum_{n=0}^{k_2} b_n Y^n$$

and

$$q(X) = \sum_{n=0}^{\ell_1} c_n X^n = \sum_{n=0}^{\ell_2} d_n Y^n,$$

then

$$\left(\sum_{n=0}^{k_1} a_n X^n\right)\left(\sum_{n=0}^{\ell_1} c_n X^n\right) = \left(\sum_{n=0}^{k_2} b_n Y^n\right)\left(\sum_{n=0}^{\ell_2} d_n Y^n\right)$$

holds.

Proof. Let $k \geq 1$ be arbitrary. Define

$$Y_k = \sum_{n=1}^{k} c_n X^n.$$

Then

$$\sum_{n=0}^{\infty} a_n X^n \equiv \sum_{n=0}^{k} \alpha_n Y_k^n \mod (X^k) \tag{14}$$

and

$$\sum_{n=0}^{\infty} b_n X^n \equiv \sum_{n=0}^{k} \beta_n Y_k^n \mod (X^k). \tag{15}$$

Noting that $Y \equiv Y_k \mod (X^k)$, Proposition A.3(2) implies that

$$Y^n \equiv Y_k^n \mod (X^k)$$

for all $n \geq 0$, whence Proposition A.3(1) gives

$$\alpha_n Y^n \equiv \alpha_n Y_k^n \mod (X^k)$$

for all $n \geq 0$, and finally by Proposition A.3(3) we obtain

$$\sum_{n=0}^{k} \alpha_n Y^n \equiv \sum_{n=0}^{k} \alpha_n Y_k^n \mod (X^k)$$

and similarly

$$\sum_{n=0}^{k} \beta_n Y^n \equiv \sum_{n=0}^{k} \beta_n Y_k^n \mod (X^k).$$

These results, together with (14) and (15), imply that

$$\sum_{n=0}^{\infty} a_n X^n \equiv \sum_{n=0}^{k} \alpha_n Y^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n X^n \equiv \sum_{n=0}^{k} \beta_n Y^n.$$

Hence

$$\left(\sum_{n=0}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} b_n X^n\right) \equiv \left(\sum_{n=0}^{k} \alpha_n Y^n\right)\left(\sum_{n=0}^{k} \beta_n Y^n\right) \mod (X^k)$$

by Proposition A.3(4), from which the desired conclusion (13) readily follows. ■

A.2 – Relations Between Formal and Convergent Series

Given a formal power series

$$f(X) = \sum_{n=0}^{\infty} a_n X^n,$$

we may replace the indeterminate X with the complex-valued variable z to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{16}$$

If the power series $\sum a_n z^n$ converges on some set $S \subseteq \mathbb{C}$, then we can naturally regard (16) as defining a function $f : S \to \mathbb{C}$ and pass from a formal power series interpretation to that of a complex-valued convergent power series. Then f is usually used as the symbol for the power series, and $f(z)$ represents the value of the power series at any $z \in S$.

Recall from §A.1 that two formal power series $f(X) = \sum a_n X^n$ and $g(X) = \sum b_n X^n$ are equal (we might now say **formally equal**) if and only if $a_n = b_n$ for all $n \geq 0$. It should be clear that if $f = g$ in the formal sense, then if f and g are absolutely convergent power series on some set $S \subseteq \mathbb{C}$, then $f(z) = g(z)$ for all $z \in S$. What is not clear is the converse: if $f(z) = g(z)$ for all $z \in S$, then is it necessarily so that $a_n = b_n$ for all $n \geq 0$? It is true, but we will not assume this fact in this section except in the exercises at the end. The proof that equal absolutely convergent series are necessarily equal formal series would have come in §A.5 if the Lang treatment were pursued that far here.

Recall also from §A.1 that if f and g are formal power series given in term of an indeterminate X by $f(X) = \sum a_n X^n$ and $g(X) = \sum b_n X^n$, then the sum of f and g is the formal power series $f + g$ given by

$$(f+g)(X) = \sum (a_n + b_n) X^n,$$

the product of f and g is the formal power series fg given by

$$(fg)(X) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n,$$

and the constant multiple of f by a constant $c \in R$ (where R is any ring) is the formal power series cf given by

$$(cf)(X) = \sum_{n=0}^{\infty} (c a_n) X^n.$$

Theorem A.16. *Let $\alpha \in \mathbb{C}$. If f and g are power series which converge absolutely on $B_r(0) \subseteq \mathbb{C}$, then $f+g$, fg, and αf also converge absolutely on $B_r(0)$, with*

$$(f+g)(z) = f(z) + g(z), \quad (fg)(z) = f(z)g(z), \quad \text{and} \quad (\alpha f)(z) = \alpha f(z)$$

for all $z \in B_r(0)$.

Proof. Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$. Then formally we have

$$(f+g)(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n.$$

Let $z \in B_r(0)$. Then $f(z), g(z) \in \mathbb{C}$, which is to say the limits

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=0}^{n} b_k z^k$$

exist in $\mathbb{C}$. Now, by a well-established law of limits,

$$\begin{aligned}(f+g)(z) &= \sum_{n=0}^{\infty}(a_n+b_n)z^n = \lim_{n\to\infty}\sum_{k=0}^{n}(a_k+b_k)z^k = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k z^k + \sum_{k=0}^{n} b_k z^k\right) \\ &= \lim_{n\to\infty}\sum_{k=0}^{n} a_k z^k + \lim_{n\to\infty}\sum_{k=0}^{n} b_k z^k = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n \\ &= f(z) + g(z).\end{aligned}$$

The part of the proof concerning αf is equally straightforward, and the arguments for fg are already made in Lang. ■

Theorem A.17. *Let $f(z) = \sum a_n z^n$ be a non-constant power series with radius of convergence $r > 0$. If $f(0) = 0$, then there exists some $\delta > 0$ such that $f(z) \neq 0$ for all $z \in \overline{B}'_\delta(0)$.*

Proof. Suppose that $f(0) = 0$. Then $a_0 = 0$, and so $\operatorname{ord}(f) = m$ for some $m \geq 1$ and we may write

$$f(z) = \sum_{n=m}^{\infty} a_n z^n.$$

Now, since $a_m \neq 0$ we have

$$f(z) = \lim_{k\to\infty} \sum_{n=m}^{m+k} a_n z^n = \lim_{k\to\infty}\left(a_m z^m \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n\right). \tag{17}$$

For $0 < |z| < r$ it must be that

$$g(z) = \sum_{n=0}^{\infty} \frac{a_{m+n}}{a_m} z^n$$

converges. To see this, suppose there exists some z for which $0 < |z| < r$ and $g(z)$ diverges. This means the limit

$$\lim_{k\to\infty} \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n$$

does not exist in $\mathbb{C}$, and since $a_m z^m \neq 0$ it follows that the limit

$$\lim_{k\to\infty}\left(a_m z^m \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n\right)$$

likewise does not exist in $\mathbb{C}$ and so by (17) we conclude that $f(z)$ diverges—contradicting the hypothesis that f has radius of convergence r. Therefore g has radius of convergence $\rho \geq r$.[1]

[1] Indeed $\rho = r$, but we do not need this fact.

For any $|z| < r$, then, an established law of limits gives

$$f(z) = a_m z^m \lim_{k\to\infty} \sum_{n=0}^{k} \frac{a_{m+n}}{a_m} z^n = a_m z^m \sum_{n=0}^{\infty} \frac{a_{m+n}}{a_m} z^n.$$

If we let $b_n = a_{m+n}/a_m$ for $n \geq 1$, and define

$$h(z) = \sum_{n=1}^{\infty} b_n z^n,$$

then

$$f(z) = a_m z^m g(z) = a_m z^m (1 + h(z)). \tag{18}$$

It is clear that h, like g, has radius of convergence at least r, and so

$$\sum_{n=1}^{\infty} |b_n|(r/2)^n$$

is a convergent series. Let $S = \overline{B}_{r/2}(0)$. If we define $f_n : S \to \mathbb{C}$ by

$$f_n(z) = b_n z^n$$

for all $n \geq 1$, then

$$\|f_n\|_S = \sup_{z\in S} |f_n(z)| = \sup_{z\in S} |b_n||z|^n \leq |b_n|(r/2)^n$$

for all $n \geq 1$, and by the Weierstrass M-Test we conclude that the series $h = \sum f_n$ converges uniformly on S. That is, the sequence of partial sums

$$(s_n)_{n=1}^{\infty} = \left(\sum\nolimits_{k=1}^{n} f_k\right)_{n=1}^{\infty}$$

converges uniformly on S, with the limit function being h. Since each s_n is a polynomial function and therefore continuous on S, it follows by Theorem 2.13 that the limit function h is itself continuous on S.

Finally, since $h(0) = 0$ and h is continuous at 0, there exists some $0 < \delta < r/2$ such that $|h(z)| < 1/2$ for all $z \in \overline{B}_\delta(0)$, whence $1 + h(z) \neq 0$ on $\overline{B}_\delta(0)$ and by (18) we conclude that $f(z) \neq 0$ for all $z \in \overline{B}'_\delta(0)$. ■

In the course of the proof above it was necessary to show that the power series h is continuous at 0. Indeed we have the following proposition.

Proposition A.18. *If $f(z) = \sum a_n z^n$ has radius of convergence r, then $\sum a_n z^n$ is continuous on $B_r(0)$.*

Proof. Suppose that $f(z) = \sum a_n z^n$ has radius of convergence r. If $r = 0$ then the domain of f is simply $\{0\}$ and continuity follows trivially. Suppose that $r > 0$. Let $0 < \rho < r$ be arbitrary. For each $n \geq 0$ define $f_n : \overline{B}_\rho(0) \to \mathbb{C}$ by $f_n(z) = a_n z^n$. We have

$$\|f_n\| = \sup_{z\in\overline{B}_\rho(0)} |f_n(z)| = \sup_{z\in\overline{B}_\rho(0)} |a_n||z|^n \leq |a_n|\rho^n,$$

and since the series $\sum |a_n|\rho^n$ is convergent it follows by the Weierstrass M-Test that $\sum f_n$ converges uniformly to f on $\overline{B}_\rho(0)$. That is, the sequence

$$(s_n)_{n=0}^{\infty} = \left(\sum\nolimits_{k=0}^{n} f_k\right)_{n=0}^{\infty}$$

converges uniformly to $f|_{\overline{B}_\rho(0)}$, and since each s_n is the restriction of a polynomial function to $\overline{B}_\rho(0)$ and therefore is continuous, Theorem 2.13 implies that $f|_{\overline{B}_\rho(0)}$ is continuous.

Hence f is continuous on $\overline{B}_\rho(0)$ for all $s < r$, and we conclude that $\sum a_n z^n$ is continuous on $\overline{B}_r(0)$. ■

Theorem A.19. *Suppose $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ converge absolutely on $B_\rho(0)$ for some $\rho > 0$. Let*

$$A = \{z \in B_\rho(0) : f(z) = g(z)\}\,.$$

If A is an infinite set having 0 as a limit point, then $f \equiv g$ on $B_\rho(0)$.

Proof. By Theorem A.16 $f - g$ converges absolutely on $B_\rho(0)$, with

$$(f-g)(z) = \sum_{n=0}^{\infty}(a_n - b_n)z^n = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} b_n z^n$$

for all $z \in B_\rho(0)$. From this we have

$$(f-g)(0) = f(0) - g(0) = a_0 - b_0.$$

Suppose 0 is a limit point of A. Then for each $\epsilon > 0$ we have $B'_\epsilon(0) \cap A \neq \varnothing$, which is to say there exists some $a \in A$ such that $0 < |a| < \epsilon$ and $f(a) = g(a)$, and so we obtain $(f-g)(a) = 0$. Since $f - g$ is continuous at 0 by Proposition A.18, it follows that $(f-g)(0) = 0$ and hence $a_0 = b_0$.

Suppose that $f - g$ is non-constant on $B_\rho(0)$. Since $f - g$ has radius convergence $r \geq \rho > 0$ and $(f-g)(0) = 0$, by Theorem A.17 there exists some $\delta > 0$ such that $(f-g)(z) \neq 0$ for all $z \in \overline{B}'_\delta(0)$. However this entails that $B'_\delta(0) \cap A = \varnothing$, so that 0 is not a limit point of A and we have arrived at a contradiction.

Therefore $f - g$ must be constant on $B_\rho(0)$, and since $(f-g)(0) = 0$ we conclude that $f - g \equiv 0$. That is, $f \equiv g$ on $B_\rho(0)$. ■

While two *formal* power series $\sum a_n X^n$ and $\sum b_n X^n$ are defined to be equal if and only if $a_n = b_n$ for all $n \geq 0$, it is not at all clear at this point (Lang's blithe declaration at the end of his version of Theorem A.19 notwithstanding) that coefficients must match if $\sum a_n z^n$ and $\sum b_n z^n$ are numerically equal for all z in some open disc $B_r(0) \subseteq \mathbb{C}$ on which both series converge absolutely. It happens to be so, but a proof will have to wait until it is established that convergent power series have derivatives of all orders.

Define power series E, S, and C by

$$E(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \text{and} \quad C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

It is easy to show, using the Ratio Test, that the power series all have radius of convergence ∞; that is, they all define functions on $\mathbb{C}$.

In Chapter 10 of the *Calculus Notes* are developed the theorems to show that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

for all $x \in \mathbb{R}$ (and in fact this is done for the sine function in an example). Thus, in particular, $\exp(x) = E(x)$ for all $x \in (-1, 1)$, and so if $\hat{E}(z)$ is any power series extension of $\exp : \mathbb{R} \to \mathbb{R}$ to $\mathbb{C}$, then certainly $\hat{E}(x) = E(x)$ for all $x \in (-1, 1)$ and by Theorem A.19 we conclude that $\hat{E} \equiv E$ on $\mathbb{C}$. Therefore $E(z)$ is the *only* possible power series extension of the exponential function to the complex plane. There can be no other! We define

$$\exp(z) := E(z) \tag{19}$$

for all $z \in \mathbb{C}$. Similar arguments establish that $S(z)$ and $C(z)$ are the unique power series extensions of the sine and cosine functions to the complex plane, and so we define

$$\sin(z) := S(z) \quad \text{and} \quad \cos(z) := C(z)$$

for all $z \in \mathbb{C}$.

In Chapter 1 we established the formula

$$e^z = e^x(\cos y + i \sin y). \tag{20}$$

for all $z = x + iy$. We should like to verify that the value of e^z as defined without power series by (20) is equal to $\exp(z)$ as defined by (19) for all $z \in \mathbb{C}$. To do this we use two facts:

$$E(iz) = C(z) + iS(z)$$

for any $z \in \mathbb{C}$, and

$$E(z_1 + z_2) = E(z_1) \cdot E(z_2). \tag{21}$$

for any $z_1, z_2 \in \mathbb{C}$. Recalling that $E(x) = \exp(x) = e^x$ for all $x \in \mathbb{R}$, we have, for any $z = x+iy$,

$$\begin{aligned}\exp(z) = E(z) = E(x + iy) = E(x) \cdot E(iy) &= e^x\big(C(y) + iS(y)\big)\\ &= e^x(\cos y + i \sin y) = e^{x+iy} = e^z.\end{aligned}$$

Thus the equivalency of e^z and $\exp(z)$ is established.

Definition A.20. *Let $f(X) = \sum a_n X^n$ and $\varphi(X) = \sum c_n X^n$ be two formal power series such that $a_n \in \mathbb{C}$ and $c_n \in [0, \infty)$ for all n. We say f is **dominated by** φ, written $f \prec \varphi$, if there exists some $N \in \mathbb{Z}$ such that $|a_n| \leq c_n$ for all $n \geq N$.*

It is easy to verify that if $f \prec \varphi$ and $g \prec \psi$, then

$$f + g \prec \varphi + \psi \quad \text{and} \quad fg \prec \varphi\psi.$$

Theorem A.21. *Suppose $f(z) = \sum a_n z^n$ has nonzero radius of convergence and constant term $a_0 \neq 0$. Let $g(z) = \sum b_n z^n$ be the formal multiplicative inverse of f, so that formally $fg = 1$. Then g also has a nonzero radius of convergence.*

Proof. The proof will consist of formal manipulations, which is to say we will operate in $R[[z]]$. Let $r > 0$ be the radius of convergence of f. Define $\hat{f} = f/a_0$, so that

$$\hat{f}(z) = \sum_{n=0}^{\infty} \hat{a}_n z^n$$

with $\hat{a}_n = a_n/a_0$, and in particular $\hat{a}_0 = 1$. Clearly $\hat{f}$ has nonzero radius of convergence $r > 0$ also. Let $\hat{g}$ be the formal multiplicative inverse of $\hat{f}$. Much as in the proof of Theorem A.6, let

$$h(z) = 1 - \hat{f}(z) = \sum_{n=0}^{\infty} (\delta_{0n} - 1)\hat{a}_n z^n$$

so that, formally,

$$\hat{g}(z) = \frac{1}{\hat{f}(z)} = \frac{1}{1 - h(z)} = \sum_{n=0}^{\infty} h^n(z). \tag{22}$$

By Corollary 2.24 there exists some $A > 0$ such that $|a_n| \leq A^n$ for all $n \geq 0$, so that

$$h(z) \prec \sum_{n=1}^{\infty} A^n z^n = \frac{Az}{1 - Az} \tag{23}$$

and hence

$$\hat{g}(z) \prec \sum_{n=0}^{\infty} \left(\frac{Az}{1 - Az} \right)^n \tag{24}$$

At this stage it should be noted that h and the dominating series in (23) and (24) all have nonzero radii of convergence. In particular the dominating series in (23) will converge if

$$\left| \frac{Az}{1 - Az} \right| < 1,$$

which implies $|z| < |z - 1/A|$, and so convergence is assured for all z such that $|z| < 1/(2A)$. Therefore $\hat{g}$ must have radius of convergence at least equal to $1/(2A) > 0$.

Now,

$$\hat{f}\hat{g} = 1 \quad \Leftrightarrow \quad (f/a_0)\hat{g} = 1 \quad \Leftrightarrow \quad f(\hat{g}/a_0) = 1,$$

so $g = \hat{g}/a_0$ is the (unique) multiplicative inverse of f. Since $\hat{g}$ has nonzero radius of convergence, it follows that g also has nonzero radius of convergence. ■

Alternate Proof. We proceed as before to obtain the formal equation (22). We then observe that h has radius of convergence $r > 0$, and by Proposition A.18 h is continuous on $B_r(0)$. Hence

$$\lim_{z \to 0} h(z) = h(0) = 0,$$

and so there exists some $0 < \delta < r$ such that $|h(z)| < 1$ for all $z \in B_\delta(0)$. Thus for any z for which $|z| < \delta$ we find that

$$\sum_{n=0}^{\infty} [h(z)]^n$$

is a convergent geometric series, and so $\hat{g}$ has radius of convergence at least $\delta > 0$. Therefore g has radius of convergence at least δ. ■

Recall that if $f, g \in R[[X]]$ such that $f(X) = \sum_{n=0}^{\infty} a_n X^n$ and $g(X) = \sum_{n=1}^{\infty} b_n X^n$, then $f \circ g$ is the formal power series given by

$$(f \circ g)(X) = f(g(X)) = \sum_{n=0}^{\infty} a_n [g(X)]^n.$$

It was shown in §A.1 that $f \circ g \in R[[X]]$; that is, there exist coefficients $c_n \in R$ such that

$$\sum_{n=0}^{\infty} a_n [g(X)]^n = \sum_{n=0}^{\infty} c_n X^n.$$

Of course the equality here is formal. A natural question that arises is this: if $z \in \mathbb{C}$ is such that $\sum_{n=0}^{\infty} a_n [g(z)]^n$ and $\sum_{n=0}^{\infty} c_n z^n$ converge, do they equal the same number? Part of the next theorem answers this in the affirmative.

Theorem A.22. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

have radii of convergence $r_f > 0$ and $r_g > 0$, respectively. Let $0 < r < r_f$, and suppose $s > 0$ is such that

$$\sum_{n=1}^{\infty} |b_n| s^n \leq r.$$

If $h = \sum c_n z^n$ is formally given by $h = f \circ g$, then h converges absolutely for $|z| \leq s$, and $h(z) = f(g(z))$ for all $z \in \overline{B}_s(0)$.

Exercise A.23 (La2.3.1a). Let $S = \{z \in \mathbb{C} : |z-1| < 1\}$, and define the function $\log : S \to \mathbb{C}$ by

$$\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

for all $z \in S$. Recalling $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ for all $z \in \mathbb{C}$, prove that $(\exp \circ \log)(z) = z$ for all $z \in S$.

Solution. As observed earlier, the developments in Chapter 10 of the *Calculus Notes* can be used to show that $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ for $x \in (-\infty, \infty)$. The same developments will also show that

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$$

for all $x \in (0, 2)$, which is to say $\log(x) = \ln(x)$ for all $x \in \mathbb{R}$ such that $|x - 1| < 1$. In Chapter 7 of the *Calculus Notes* it is shown that $\exp : \mathbb{R} \to (0, \infty)$ and $\ln : (0, \infty) \to \mathbb{R}$ are inverse functions, and therefore for $x \in \mathbb{R}$ such that $|x - 1| < 1$ we have

$$(\exp \circ \log)(x) = \exp(\log(x)) = \exp(\ln(x)) = x. \tag{25}$$

Define $p : \mathbb{C} \to \mathbb{C}$ to be the power series

$$p(z) = z + 1,$$

The series $\exp\circ\log\circ p$ and p both converge absolutely on $B_1(0)$, and moreover for all $x \in (-1,1)$ we have

$$(\exp\circ\log\circ p)(x) = (\exp\circ\log)(p(x)) = (\exp\circ\log)(x+1) = x+1 = p(x),$$

observing that $|(x+1)-1| < 1$. Since 0 is a limit point for $(-1,1)$ and $\exp\circ\log\circ p = p$ on $(-1,1)$, by Theorem A.19 we conclude that

$$\exp\circ\log\circ p \equiv p$$

on $B_1(0)$. That is, for all $z \in B_1(0)$

$$(\exp\circ\log)(z+1) = (\exp\circ\log\circ p)(z) = p(z) = z+1,$$

or equivalently

$$(\exp\circ\log)(z) = z \tag{26}$$

for all $z \in \mathbb{C}$ such that $|z-1| < 1$. ■

Exercise A.24 (La2.3.1b). Let $S^* = \{z \in \mathbb{C} : |z - z_0| < |z_0|\}$, $z_0 \neq 0$, and $\alpha \in \mathbb{C}$ be such that $\exp(\alpha) = z_0$. Define $\log^* : S^* \to \mathbb{C}$ by

$$\log^*(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{z}{z_0} - 1\right)^n + \alpha.$$

Prove that $(\exp\circ\log^*)(z) = z$ for all $z \in S^*$.

Solution. Let $z \in S^*$. We have

$$|z - z_0| < |z_0| \;\;\Rightarrow\;\; \frac{|z-z_0|}{|z_0|} < 1 \;\;\Rightarrow\;\; \left|\frac{z-z_0}{z_0}\right| < 1 \;\;\Rightarrow\;\; \left|\frac{z}{z_0} - 1\right| < 1,$$

so $z/z_0 \in S = \{z \in \mathbb{C} : |z-1| < 1\}$ and from Exercise A.23

$$\log\left(\frac{z}{z_0}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{z}{z_0} - 1\right)^n = \log^*(z) - \alpha.$$

That is,

$$\log^*(z) = \log\left(\frac{z}{z_0}\right) + \alpha,$$

and therefore

$$\begin{aligned}(\exp\circ\log^*)(z) &= \exp\big(\log^*(z)\big) = \exp\big(\log(z/z_0) + \alpha\big)\\ &= \exp\big(\log(z/z_0)\big)\cdot\exp(\alpha) = (\exp\circ\log)(z/z_0)\cdot z_0\\ &= \frac{z}{z_0}\cdot z_0 = z\end{aligned}$$

by (21) and (26). ■

Exercise A.25 (La2.3.2a). Let $\exp(X) = \sum_{n=0}^{\infty} X^n/n!$ and $\log(1+X) = \sum_{n=1}^{\infty}(-1)^{n-1}X^n/n$. Show that

$$(\exp\circ\log)(1+X) = 1+X \quad\text{and}\quad (\log\circ\exp)(X) = X.$$

Solution. Define the power series

$$\log^*(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad p(z) = z, \quad \text{and} \quad q(z) = z - 1.$$

The Ratio Test easily shows that $\log^*$ converges absolutely on $B_1(0)$, and of course q converges absolutely on $\mathbb{C}$. Thus $\log^* \circ q$ converges absolutely on $B_1(1)$, and moreover

$$(\log^* \circ q)(z) = \log^*(z-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n = \log(z) \tag{27}$$

for all $z \in B_1(1)$.

Now, since $\exp(0) = 1$ and the function $\exp : \mathbb{C} \to \mathbb{C}$ is everywhere continuous, there exists some $r > 0$ such that $|\exp(z) - 1| < 1$ for all $z \in B_r(0)$. Since $\exp(z)$ converges absolutely on $B_r(0)$ and $(\log^* \circ q)(z)$ converges absolutely on $\exp(B_r(0)) \subseteq B_1(1)$, we conclude that $\log^* \circ q \circ \exp$ converges absolutely on $B_r(0)$.[2] From (27) we see that $\log^* \circ q = \log$ on $B_1(1)$, and so $\log \circ \exp$ and p converge absolutely on $B_r(0)$.

Observing that 0 is a limit point for $(-r, r) \subseteq B_r(0)$ and

$$(\log \circ \exp)(x) = \log(\exp(x)) = \ln(\exp(x)) = x = p(x)$$

for all $x \in (-r, r)$, by Theorem A.19 we have $\log \circ \exp \equiv p$ on $B_r(0)$. That is,

$$(\log \circ \exp)(z) = z$$

for all $z \in B_r(0)$, and so the formal result $(\log \circ \exp)(X) = X$ obtains.

The formal result $(\exp \circ \log)(1 + X) = 1 + X$ follows immediately from Exercise A.23, with z merely replaced by $1 + X$. ∎

Exercise A.26 (La2.3.2b). Let $g, h \in \mathbb{C}[[X]]$ be such that $\operatorname{ord}(g), \operatorname{ord}(h) \geq 1$. Prove that

$$\log\big((1 + g(X))(1 + h(X))\big) = \log\big(1 + g(X)\big) + \log\big(1 + h(X)\big). \tag{28}$$

Solution. We have, formally,

$$\begin{aligned}
\exp\Big(\log\big((1+g(X))(1+h(X))\big)\Big) &= (\exp\circ\log)\big((1+g(X))(1+h(X))\big) \\
&= (\exp\circ\log)\big(1+g(X)+h(X)+g(X)h(X)\big) \\
&= 1+g(X)+h(X)+g(X)h(X) = (1+g(X))(1+h(X)) \\
&= (\exp\circ\log)(1+g(X))(\exp\circ\log)(1+h(X)) \\
&= \exp\big(\log(1+g(X))\big)\exp\big(\log(1+h(X))\big) \\
&= \exp\big(\log(1+g(X)) + \log(1+h(X))\big),
\end{aligned}$$

[2]The reason for insinuating q into the analysis is simple: $\log^* \circ q \circ \exp = \log^* \circ (q \circ \exp)$ is a composition of two power series *centered at* 0 with the constant term of $q \circ \exp$ equal to 0, and so we could (if desired) invoke Theorem A.22 to argue that $\log^* \circ q \circ \exp$ converges absolutely.

where the third equality follows from the previous exercise, and the last equality makes use of (21). It is known that $\exp(z_1) = \exp(z_2)$ if and only if $z_1 = z_2 + 2\pi ki$ for some $k \in \mathbb{Z}$, and so here we have

$$\log\big((1+g(X))(1+h(X))\big) - \big[\log(1+g(X)) + \log(1+h(X))\big] = 2\pi ki \tag{29}$$

for some integer k. That is, the two formal power series on the left-hand side differ only (potentially) in the value of their constant term. Since g and h are formal power series with constant term 0, supplanting the indeterminate X with 0 gives

$$\begin{aligned} 2\pi ki &= \log\big((1+g(0))(1+h(0))\big) - \log(1+g(0)) - \log(1+h(0)) \\ &= \log\big((1+0)(1+0)\big) - \log(1+0) - \log(1+0) \\ &= \log(1) - \log(1) + \log(1) = 0 - 0 - 0 = 0, \end{aligned}$$

and therefore $k = 0$. Returning to (29), we conclude that

$$\log\big((1+g(X))(1+h(X))\big) - \big[\log(1+g(X)) + \log(1+h(X))\big] = 0,$$

from which (28) follows. ∎

Exercise A.27 (La2.3.2c). For $\alpha, \beta \in \mathbb{C}$ show that

$$\log(1+X)^\alpha = \alpha \log(1+X) \quad \text{and} \quad (1+X)^\alpha (1+X)^\beta = (1+X)^{\alpha+\beta}.$$

Solution. Recalling the binomial series, for any $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} \log(1+X)^\alpha &= \log\big((1+X)^\alpha\big) = \log\left(\sum_{n=0}^\infty \binom{\alpha}{n} X^n\right) = \log\left(1 + \sum_{n=1}^\infty \binom{\alpha}{n} X^n\right) \\ &= \sum_{n=1}^\infty \left[\frac{(-1)^{n-1}}{n}\left(\sum_{n=1}^\infty \binom{\alpha}{n} X^n\right)^n\right] = \sum_{n=1}^\infty b_n(\alpha) X^n, \end{aligned} \tag{30}$$

where it is clear that each $b_n(\alpha)$ is a polynomial in α.

For $x \in (-1, \infty)$ and $\alpha \in \mathbb{R}$ we have $\log(1+x)^\alpha = \alpha \log(1+x)$ from calculus, and thus formally (i.e. in the ring $\mathbb{C}[[X]]$) we have

$$\log(1+X)^\alpha = \alpha \log(1+X) \tag{31}$$

for any $\alpha \in \mathbb{R}$. Now,

$$\alpha \log(1+X) = \alpha \sum_{n=1}^\infty (-1)^{n-1} X^n / n = \sum_{n=1}^\infty \frac{\alpha(-1)^{n-1}}{n} X^n,$$

and so (30) and (31) imply

$$\log(1+X)^\alpha = \sum_{n=1}^\infty b_n(\alpha) X^n$$

such that

$$b_n(\alpha) = \frac{(-1)^{n-1}}{n}\alpha$$

for all $\alpha \in \mathbb{R}$, $n \geq 1$.

Define polynomial functions $a_n : \mathbb{C} \to \mathbb{C}$ by

$$a_n(\alpha) = \frac{(-1)^{n-1}}{n}\alpha$$

for all $\alpha \in \mathbb{C}$. The functions $b_n : \mathbb{C} \to \mathbb{C}$ are also polynomial, and since $a_n(\alpha) = b_n(\alpha)$ for $\alpha \in \mathbb{R}$, we conclude by Theorem A.19 that $a_n \equiv b_n$ on $\mathbb{C}$ for all $n \geq 1$. That is, for any $\alpha \in \mathbb{C}$,

$$\log(1+X)^\alpha = \sum_{n=1}^{\infty} b_n(\alpha)X^n = \sum_{n=1}^{\infty} a_n(\alpha)X^n = \sum_{n=1}^{\infty} \frac{\alpha(-1)^{n-1}}{n}X^n = \alpha\log(1+X).$$

From the first line of (30) we see that, for any $\alpha \in \mathbb{C}$,

$$(1+X)^\alpha = 1 + h_\alpha(X)$$

for some series $h_\alpha(X)$ with zero constant term, and thus for any $\beta \in \mathbb{C}$ we use Exercise A.26 to obtain

$$\begin{aligned}
\log\big((1+X)^\alpha(1+X)^\beta\big) &= \log\big((1+h_\alpha(X))(1+h_\beta(X))\big) \\
&= \log\big(1+h_\alpha(X)\big) + \log\big(1+h_\beta(X)\big) = \log(1+X)^\alpha + \log(1+X)^\beta \\
&= \alpha\log(1+X) + \beta\log(1+X) = (\alpha+\beta)\log(1+X) \\
&= \log(1+X)^{\alpha+\beta}.
\end{aligned}$$

Exponentiating then yields

$$(1+X)^\alpha(1+X)^\beta = (1+X)^{\alpha+\beta} + 2\pi k i$$

for some $k \in \mathbb{Z}$, which is to say the formal power series $(1+X)^\alpha(1+X)^\beta$ and $(1+X)^{\alpha+\beta}$ may differ only by a constant term. Substituting 0 for the indeterminate X, however, gives

$$2\pi k i = (1+0)^\alpha(1+0)^\beta - (1+0)^{\alpha+\beta} = 1 \cdot 1 - 1 = 0,$$

so $k = 0$ and the proof that

$$(1+X)^\alpha(1+X)^\beta = (1+X)^{\alpha+\beta}$$

is done. ■

Remark. The binomial series shows that $1^z = 1$ for any nonzero $z \in \mathbb{C}$:

$$1^z = (1+0)^z = \sum_{n=0}^{\infty}\binom{z}{n}0^n = 1 + \sum_{n=1}^{\infty}\binom{z}{n}0^n = 1 + 0 = 1.$$

Of course $1^0 = 1$ by definition.

Exercise A.28 (La2.3.3). Prove that

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

for all $z \in \mathbb{C}$.

Solution. It has already been established that

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n!)} \quad \text{and} \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

for all $z \in \mathbb{C}$. Hence

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \sum_{n=0}^{\infty}\left(\frac{i^n+(-i)^n}{2}\right)\frac{z^n}{n!}.$$

If n is odd, so that $n = 2k+1$ for some $k \geq 0$, then

$$i^n + (-i)^n = i^{2k+1} + (-i)^{2k+1} = (i^2)^k i + [(-i)^2]^k(-i) = (-1)^k i + (-1)^k(-i) = 0$$

and we obtain

$$\frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty}\left(\frac{i^{2n}+(-i)^{2n}}{2}\right)\frac{z^{2n}}{(2n)!}.$$

Now,

$$\frac{i^{2n}+(-i)^{2n}}{2} = \frac{(-1)^n+(-1)^n}{2} = \begin{cases} 1 = (-1)^n, & \text{if } n \text{ is even} \\ -1 = (-1)^n, & \text{if } n \text{ is odd} \end{cases}$$

whence

$$\frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} = \cos z.$$

The derivation for $\sin z$ is done similarly. ■

Exercise A.29 (La2.3.4). Show that $\sin z = 0$ if and only if $z = k\pi$ for some $k \in \mathbb{Z}$.

Solution. If $z = k\pi$ for some $k \in \mathbb{Z}$, then $\sin(z) = 0$ follows immediately. For the converse, suppose that $z \in \mathbb{C}$ is such that $\sin z = 0$. From the previous exercise we have

$$\frac{e^{iz}-e^{-iz}}{2i} = 0,$$

so that $e^{iz} = e^{-iz}$. By La1.2.9 we have $iz = -iz + 2\pi ki$ for some $k \in \mathbb{Z}$, which implies that $z = k\pi$. ■

A.3 – Analytic Functions

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{32}$$

is a **power series centered at** z_0. A function f is **analytic at** z_0 if there exists a series (32) that converges absolutely on $B_r(z_0)$ for some $r>0$ such that

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for all $z\in B_r(z_0)$. If such is the case, we say that f has a **power series expansion at** z_0. If U is an open set we say f is **analytic on** U if f is analytic at z for each $z\in U$. Finally, if S is an arbitrary set we say f is **analytic on** S if there exists an open set such that $S\subseteq U$ and f is analytic on U.

All the results obtained thus far about power series centered at the origin can easily be generalized to power series centered at z_0. For instance we have the following, which generalizes Theorem A.19.

Theorem A.30. *Suppose $f(z)=\sum a_n(z-z_0)^n$ and $g(z)=\sum b_n(z-z_0)^n$ converge absolutely on $B_r(z_0)$ for some $r>0$. Let*

$$A=\{z\in B_r(z_0): f(z)=g(z)\}.$$

If z_0 is a limit point of A, then $f\equiv g$ on $B_r(z_0)$.

Proof. Suppose z_0 is a limit point for A. Define $\varphi(z)=z+z_0$. Then

$$(f\circ\varphi)(z)=f(z+z_0)=\sum a_n z^n$$

converges absolutely for all $z\in B_r(0)$, since

$$z\in B_r(0) \quad\Leftrightarrow\quad |z|<r \quad\Leftrightarrow\quad |(z+z_0)-z_0|<r \quad\Leftrightarrow\quad z+z_0\in B_r(z_0),$$

and $f(z)=\sum a_n(z-z_0)^n$ is given to converge absolutely on $B_r(z_0)$. Similarly we find that

$$(g\circ\varphi)(z)=g(z+z_0)=\sum b_n z^n$$

converges absolutely on $B_r(0)$.

Define

$$A_0=\{z\in B_r(0):(f\circ\varphi)(z)=(g\circ\varphi)(z)\}.$$

Let $\epsilon>0$. Since z_0 is a limit point for A, there exists some $z\in A$ such that $z\in B'_\epsilon(z_0)$. That is, $z\in B_r(z_0)$ is such that $0<|z-z_0|<\epsilon$ and $f(z)=g(z)$. Then $z-z_0\in B_r(0)$ and

$$(f\circ\varphi)(z-z_0)=f(\varphi(z-z_0))=f(z)=g(z)=g(\varphi(z-z_0))=(g\circ\varphi)(z-z_0)$$

show that $z-z_0\in A_0$ is such that $z-z_0\in B'_\epsilon(0)$. Hence $B'_\epsilon(0)\cap A_0\neq\varnothing$ for every $\epsilon>0$, and therefore 0 is a limit point for A_0.

Since $f\circ\varphi$ and $g\circ\varphi$ converge absolutely on $B_r(0)$ and 0 is a limit point for A_0, by Theorem A.19 we conclude that $f\circ\varphi\equiv g\circ\varphi$ on $B_r(0)$.

Now, let $z \in B_r(z_0)$ be arbitrary. Then $z - z_0 \in B_r(0)$, and so

$$f(z) = (f \circ \varphi)(z - z_0) = (g \circ \varphi)(z - z_0) = g(z).$$

Therefore $f \equiv g$ on $B_r(z_0)$. ■

Before going further some formal results need to be established. If

$$f(X) = \sum_{n=0}^{\infty} a_n (X - X_0)^n \quad \text{and} \quad g(X) = \sum_{n=0}^{\infty} b_n (X - X_0)^n,$$

then we view f and g as formal power series belonging to the ring $R[[X - X_0]]$, and define $f + g, cf, fg \in R[[X - X_0]]$ (where $c \in R$) in the expected way:

$$(f + g)(X) = \sum_{n=0}^{\infty} (a_n + b_n)(X - X_0)^n, \quad (cf)(X) = \sum_{n=0}^{\infty} c a_n (X - X_0)^n,$$

and

$$(fg)(X) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (X - X_0)^n.$$

Also multiplicative inverses, quotients, and compositions are defined in a fashion analogous to the corresponding definitions for elements of $R[[X]]$.

Remark. Strictly speaking, we cannot use any fact about the arithmetic of $R[[X]]$ to "prove" anything about the arithmetic of $R[[X - X_0]]$ (such as by employing "translations"), since $R[[X]]$ and $R[[X - X_0]]$ are completely different rings! Even regarding X_0 as an element of R and instead writing $X - c_0$ (with $c_0 \in R$) leads to difficulties unless we choose $c_0 = 0$: formally f becomes the "composition" $h \circ \varphi$ with $h(X) = \sum a_n X^n$ and $\varphi(X) = -c_0 + X$, but φ has nonzero constant term!

The situation with regards to convergent power series is, of course, somewhat different since translations via compositions with, say, $\varphi(z) = z + z_0$ for $z_0 \neq 0$ is possible.[3]

Proposition A.31. *Suppose that functions f and g are analytic on U, and let $S = \{z \in U : g(z) \neq 0\}$. Then*

1. *$f + g$ and fg are analytic on U.*
2. *f/g is analytic on any open $V \subseteq S$.*

Proof. We will address only the statement concerning fg. Let $z_0 \in U$. Then there exist power series $\sum a_n (z - z_0)^n$ and $\sum b_n (z - z_0)$ that converge absolutely on $B_r(z_0)$, and moreover

$$f(z) = \sum a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum b_n (z - z_0)^n$$

for all $z \in B_r(z_0)$. By definition (see above)

$$(fg)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n. \tag{33}$$

---

[3]Lang does not make clear how (or why) results in $R[[X]]$ apply to the set of convergent power series and vice-versa, nor acknowledges how the composition operation $\circ$ may be used differently in $R[[X]]$ versus in the set of convergent power series.

Let $\varphi(z) = z + z_0$. Defining

$$f_0(z) := (f \circ \varphi)(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g_0(z) := (g \circ \varphi)(z) = \sum_{n=0}^{\infty} b_n z^n,$$

we see that f_0 and g_0 are power series that converge absolutely on $B_r(0)$. Thus if $z \in B_r(z_0)$, then $z - z_0 \in B_r(0)$ and we have by Theorem A.16

$$(f_0 g_0)(z - z_0) = f_0(z - z_0) g_0(z - z_0) = f(\varphi(z - z_0)) g(\varphi(z - z_0)) = f(z)g(z),$$

where $f(z)g(z)$ is a complex number. On the other hand we have, by the formal definition of power series multiplication,

$$(f_0 g_0)(X) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n,$$

and so

$$(f_0 g_0)(z - z_0) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n.$$

Hence

$$(fg)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n = f(z)g(z) \in \mathbb{C}$$

for any $z \in B_r(z_0)$.

Since the power series in (33) converges absolutely on $B_r(z_0)$, and since the equality given by (33) holds for all $z \in B_r(z_0)$, we conclude that fg is analytic at z_0. Since $z_0 \in U$ is arbitrary, we further conclude that fg is analytic on U. ■

Proposition A.32. *Let $U, V \subseteq \mathbb{C}$ be open. If $g : U \to V$ is analytic on U and $f : V \to \mathbb{C}$ is analytic on V, then $f \circ g$ is analytic on U.*

Exercise A.33 (La2.4.1). Find the terms of order ≤ 3 in the power series expansion at 1 of the function

$$f(z) = \frac{z^2}{z - 2}.$$