Complex Analysis

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The Complex Numbers

1.1 – The Field of Complex Numbers

We give first the formal definition of a field, if only so we may give each of the field axioms a convenient label for later reference.

Definition 1.1. *A **field** is a triple $(\mathbb{F}, +, \cdot)$ involving a set $\mathbb{F}$ together with binary operations $+, \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the following axioms:*

F1. $a + b = b + a$ *for any* $a, b \in \mathbb{F}$.
F2. $a + (b + c) = (a + b) + c$ *for any* $a, b, c \in \mathbb{F}$.
F3. *There exists some* $0 \in \mathbb{F}$ *such that* $a + 0 = a$ *for any* $a \in \mathbb{F}$.
F4. *For each* $a \in \mathbb{F}$ *there exists some* $-a \in \mathbb{F}$ *such that* $-a + a = 0$.
F5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *for any* $a, b, c \in \mathbb{F}$.
F6. $a \cdot (b + c) = a \cdot b + a \cdot c$ *for any* $a, b, c \in \mathbb{F}$.
F7. $a \cdot b = b \cdot a$ *for all* $a, b \in \mathbb{F}$.
F8. *There exists some* $0 \neq 1 \in \mathbb{F}$ *such that* $a \cdot 1 = a$ *for any* $a \in \mathbb{F}$.
F9. *For each* $0 \neq a \in \mathbb{F}$ *there exists some* $a^{-1} \in \mathbb{F}$ *such that* $aa^{-1} = 1$.

Let $\mathbb{R}$ denote the field of real numbers $(\mathbb{R}, +, \cdot)$. Formally we define the **complex numbers** to be the elements of the set

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

together with binary operations of **addition** and **multiplication** defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1), \tag{1.2}$$

respectively. (Often the $\cdot$ in (1.2) is suppressed.) The triple $(\mathbb{R}^2, +, \cdot)$ is easily shown to satisfy all the properties of a field, and we henceforth denote this field of complex numbers by the symbol $\mathbb{C}$, often called the **complex plane**. In particular the complex numbers $(0, 0)$ and $(1, 0)$ are the additive and multiplicative identity elements of $\mathbb{C}$, respectively, thereby satisfying Axioms F3 and F8 in Definition 1.1.

The statement and proof of our first proposition will show how $\mathbb{C}$ satisfies Axiom F9. In general, if $z \in \mathbb{C}$ is such that $z \neq (0,0)$, the symbol z^{-1} will denote the (unique) multiplicative inverse of z; that is, z^{-1} denotes the (unique) complex number for which $zz^{-1} = (1,0)$.

Proposition 1.2. *If $z = (x,y) \neq (0,0)$, then*

$$z^{-1} = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right).$$

Proof. Suppose $z = (x,y)$ is a nonzero complex number, so in particular $x^2 + y^2 > 0$. We find $(u,v) \in \mathbb{C}$ such that

$$(x,y) \cdot (u,v) = (ux - vy, vx + uy) = (1,0).$$

This gives us a nonlinear system of equations:

$$\begin{cases} ux - vy = 1 \\ vx + uy = 0 \end{cases}$$

From the first equation we have $vy = ux - 1$. Multiplying the second equation by y, we obtain

$$vxy + uy^2 = 0 \quad \Rightarrow \quad x(ux-1) + uy^2 = 0 \quad \Rightarrow \quad u = \frac{x}{x^2+y^2}.$$

From the first equation we also have $ux = vy + 1$. Multiplying the second equation by x, we obtain

$$vx^2 + uxy = 0 \quad \Rightarrow \quad vx^2 + y(vy+1) = 0 \quad \Rightarrow \quad v = -\frac{y}{x^2+y^2}.$$

It is straightforward to check that $z \cdot (u,v) = (1,0)$ using the values of u and v acquired, and therefore $z^{-1} = (u,v)$. ∎

Definition 1.3. *Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Then the* ***nth power*** *of z is*

$$z^n = \prod_{k=1}^{n} z.$$

If $z \neq (0,0)$, then we also define $z^0 = (1,0)$ and $z^{-n} = (z^{-1})^n$.

The set $\{(x,0) : x \in \mathbb{R}\}$ is a subfield of $\mathbb{C}$ that is isomorphic to the field $\mathbb{R}$ via the canonical correspondence $x \mapsto (x,0)$, and so we naturally identify $\{(x,0) : x \in \mathbb{R}\}$ with $\mathbb{R}$ itself. That is, we view $\mathbb{R}$ as a subfield of $\mathbb{C}$, and denote each $(x,0) \in \mathbb{C}$ as simply x. Then we see that, for any $c \in \mathbb{R}$,

$$(c,0) \cdot (x,y) = (cx - 0y, cy + x0) = (cx, cy),$$

which together with Axiom F7 informs us that

$$c(x,y) = (cx,cy) = (x,y)c,$$

showing $\mathbb{C}$ to be a vector space over $\mathbb{R}$. Defining $i = (0,1)$, so $iy = (0,y)$ for any $y \in \mathbb{R}$, we have

$$(x,y) = (x,0) + (0,y) = x + iy$$

for all $x, y \in \mathbb{R}$, which enables us to eschew the ordered pair notation for complex numbers completely, and gives us the more customary representation of $\mathbb{C}$ as

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

We note in particular that, for $x, y \in \mathbb{R}$,

$$x = (x, 0) = (x, 0) + (0, 0) = x + i0 \quad \text{and} \quad iy = (0, y) = (0, 0) + (0, y) = 0 + iy.$$

The number i is the **imaginary unit** (whereas 1 may be called the **real unit**). It has the following celebrated property:

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

It is because of this property that another symbol for i is $\sqrt{-1}$. Any number of the form $iy = (0, y)$ for $y \in \mathbb{R}$ is known as an **imaginary number**. A complex number (x, y) may now be regarded as a formal sum $x + iy$ of a real number x and an imaginary number iy. We call x the **real part** of $x + iy$, written $\text{Re}(x + iy) = x$; and we call y the **imaginary part** of $x + iy$, written $\text{Im}(x + iy) = y$. Clearly $z \in \mathbb{R}$ if and only if $\text{Im}(z) = 0$.

The set of imaginary numbers we denote by $\mathbb{I}$, so that

$$\mathbb{I} = \{(0, y) : y \in \mathbb{R}\} = \{iy : y \in \mathbb{R}\}.$$

Since $\mathbb{R} \cap \mathbb{I} = \{0\}$, 0 is the only number that is both real and imaginary. Also we see that $z \in \mathbb{I}$ if and only if $\text{Re}(z) = 0$.

We take a closer look now at complex number arithmetic. Using (1.1) it is easy to check that

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Also, using Axioms F1, F2, and F6, as well as (1.2),

$$\begin{aligned}
(x_1 + iy_1)(x_2 + iy_2) &= (x_1, y_1)[(x_2, 0) + (0, y_2)] \\
&= (x_1, y_1)(x_2, 0) + (x_1, y_1)(0, y_2) \\
&= (x_1x_2, x_2y_1) + (-y_1y_2, x_1y_2) \\
&= (x_1x_2 + ix_2y_1) + (-y_1y_2 + ix_1y_2) \\
&= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2
\end{aligned}$$

which shows that in the formal sum notation $x + iy$ the multiplication of two complex numbers is patterned precisely in accordance with the manner in which two binomials are multiplied:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Exercise 1.4. Show that the field $\mathbb{C}$ is isomorphic to the field $(M, +, \cdot)$, where

$$M = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} : \; x, y \in \mathbb{R} \right\}$$

Solution. Define $\varphi : \mathbb{C} \to M$ by

$$\varphi(a + ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

It must be shown that φ is a field isomorphism, which is to say φ is a bijection, and $\varphi(z_1) = Z_1$ and $\varphi(z_2) = Z_2$ imply that

$$\varphi(z_1 z_2) = Z_1 Z_2 \quad \text{and} \quad \varphi(z_1 + z_2) = Z_1 + Z_2.$$

From $\varphi(a + ib) = \varphi(c + id)$ we have

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix},$$

which immediately implies that $a = c$ and $b = d$, so $a + ib = c + id$ and φ is injective. That φ is surjective is even more trivial, and thus φ is bijective.

Let $z_1 = a + ib$ and $z_2 = c + id$. Then

$$\begin{aligned} \varphi(z_1 + z_2) = \varphi((a + c) + i(b + d)) &= \begin{bmatrix} a + c & b + d \\ -(b + d) & a + c \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ -b - d & a + c \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \varphi(z_1) + \varphi(z_2), \end{aligned}$$

and

$$\begin{aligned} \varphi(z_1 z_2) = \varphi((ac - bd) + i(ad + bc)) &= \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{bmatrix} \\ &= \begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \varphi(z_1)\varphi(z_1), \end{aligned}$$

and therefore φ is a field isomorphism. ■

Exercise 1.5. Show that the field $\mathbb{C}$ cannot be ordered, which is to say there exists no set $P \subseteq \mathbb{C}$ of "positive elements" such that: (a) P is closed under complex number addition and multiplication; and (b) for each $z \in \mathbb{C}$ exactly one of the relations $z \in P$, $z = 0$, $-z \in P$ holds.

Solution. We must have $1 \in P$, since (b) implies either 1 or -1 is in P, and by (a) it follows that $1^2 = 1$ or $(-1)^2 = 1$ is in P. We also must have $-1 \in P$, since (b) implies either i or $-i$ is in P, and by (a) it follows that $i^2 = -1$ or $(-i)^2 = -1$ is in P. Now, $1 \in P$ and $-1 \in P$ plainly contradicts (b). ■

A sequence of complex numbers we may denote most generically by (z_n). For integers $m \geq k$ the symbol $(z_n)_{n=k}^m$ denotes the finite sequence $z_k, z_{k+1}, \ldots, z_m$, whereas $(z_n)_{n=m}^\infty$ signifies the infinite sequence $z_m, z_{m+1}, z_{m+2}, \ldots$. For any sequence (z_n) we define $\Delta z_k = z_{k+1} - z_k$ for all applicable integers k. In the course of doing complex-number arithmetic, one formula that is sometimes quite useful is the following.

Proposition 1.6 (Summation by Parts). *If (z_k) and (w_k) are sequences of complex numbers, then*

$$\sum_{k=m}^{n} z_k \Delta w_k = z_{n+1}w_{n+1} - z_m w_m - \sum_{k=m}^{n} w_{k+1}\Delta z_k$$

for all integers $n \geq m$.

We pause to introduce some notation for other important sets of numbers. The set of **nonzero complex numbers** is

$$\mathbb{C}_* = \mathbb{C} \setminus \{0\} = \mathbb{C} \setminus (\mathbb{R} \cap \mathbb{I}).$$

We define $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$ to be the set of **whole numbers**, $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of **natural numbers**, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ the set of **integers**, and

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

the set of **rational numbers**. Also we denote the set of positive, negative, and nonzero real numbers by the symbols $\mathbb{R}_+$, $\mathbb{R}_-$, and $\mathbb{R}_*$, respectively.

Proposition 1.7. *For all $n \in \mathbb{W}$ and $z \neq 1$,*

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}. \tag{1.3}$$

Proof. Clearly the equation holds when $n = 0$. Let $n \in \mathbb{W}$ be arbitrary and suppose that (1.3) holds. Then

$$\begin{aligned}
1 + z + \cdots + z^{n+1} &= (1 + z + \cdots + z^n) + z^{n+1} \\
&= \frac{z^{n+1} - 1}{z - 1} + z^{n+1} = \frac{z^{n+1} - 1}{z - 1} + \frac{z^{n+1}(z-1)}{z - 1} \\
&= \frac{z^{n+1} - 1}{z - 1} + \frac{z^{n+2} - z^{n+1}}{z - 1} \\
&= \frac{z^{n+2} - 1}{z - 1},
\end{aligned}$$

and thus by induction we conclude that (1.3) holds for all n. ∎

1.2 – Moduli and Conjugates

Recall that if $z = x + iy$ for some $x, y \in \mathbb{R}$, then the **real part** of z is $\mathrm{Re}(z) = x$ and the **imaginary part** of z is $\mathrm{Im}(z) = y$. We may also write $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ as $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$.

Definition 1.8. *Let $z = x + iy$ for $x, y \in \mathbb{R}$. The **conjugate** of z is*

$$\bar{z} = x - iy,$$

*and the **modulus** of z is*

$$|z| = \sqrt{x^2 + y^2}.$$

The conjugate of $\bar{z}$ we naturally denote by $\bar{\bar{z}}$. The modulus of a complex number z is also known as the **absolute value** of z.

The proof of the following proposition is straightforward. For instance, if $z = x + iy$, then

$$z - \bar{z} = (x + iy) - (x - iy) = x - x + iy + iy = 2iy = 2i \,\mathrm{Im}\, z \quad \Rightarrow \quad \mathrm{Im}\, z = \frac{z - \bar{z}}{2i},$$

and

$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x| = |\,\mathrm{Re}\, z|.$$

Proposition 1.9. *For $z, w \in \mathbb{C}$ the following hold.*

1. $\bar{\bar{z}} = z$
2. $|\bar{z}| = |z|$
3. $z\bar{z} = |z|^2$
4. $|zw| = |z||w|$
5. $\overline{z + w} = \bar{z} + \bar{w}$ *and* $\overline{zw} = \bar{z}\bar{w}$
6. $\mathrm{Re}\, z = \frac{1}{2}(z + \bar{z})$ *and* $\mathrm{Im}\, z = \frac{1}{2i}(z - \bar{z})$
7. $|z| \geq |\,\mathrm{Re}\, z|$ *and* $|z| \geq |\,\mathrm{Im}\, z|$
8. $\bar{z} = z$ *if and only if* $z \in \mathbb{R}$

Simple inductive arguments will show that

$$\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \bar{z}_k \quad \text{and} \quad \overline{\prod_{k=1}^{n} z_k} = \prod_{k=1}^{n} \bar{z}_k.$$

Defining

$$\bar{z}^n := (\bar{z})^n \quad \text{and} \quad \overline{z^n} := \overline{(z^n)},$$

for any $n \in \mathbb{N}$, it is immediate that

$$\bar{z}^n = \overline{z^n}.$$

Proposition 1.10 (Triangle Inequality). *If $z, w \in \mathbb{C}$, then*

$$|z + w| \leq |z| + |w|.$$

Moreover, in the case when $z, w \neq 0$, we have $|z + w| = |z| + |w|$ if and only if $z = cw$ for some $c > 0$.

Proof. Using properties in Proposition 1.9, we have

$$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$$
$$= z\bar{z} + z\bar{w} + \overline{z\bar{w}} + w\bar{w} = z\bar{z} + 2\,\mathrm{Re}(z\bar{w}) + w\bar{w}$$

and

$$(|z|+|w|)^2 = |z|^2 + 2|z||w| + |w|^2 = z\bar{z} + 2|zw| + w\bar{w},$$

and so

$$(|z|+|w|)^2 - |z+w|^2 = 2|zw| - 2\,\mathrm{Re}(z\bar{w}) = 2\big[|z\bar{w}| - \mathrm{Re}(z\bar{w})\big] \geq 0 \tag{1.4}$$

by Proposition 1.9(7). Therefore

$$|z+w|^2 \leq (|z|+|w|)^2,$$

and taking square roots yields the desired inequality.

Next, assume $z, w \neq 0$. Suppose $|z+w| = |z| + |w|$. From (1.4) it follows that

$$\mathrm{Re}(z\bar{w}) = |z\bar{w}| = \sqrt{[\mathrm{Re}(z\bar{w})]^2 + [\mathrm{Im}(z\bar{w})]^2},$$

and thus $\mathrm{Im}(z\bar{w}) = 0$. This shows that $z\bar{w} \in \mathbb{R}$, and indeed $z\bar{w} = \mathrm{Re}(z\bar{w}) = |z\bar{w}| > 0$. Now,

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2} > 0,$$

which is to say $z/w = c$ for some $c \in \mathbb{R}_+$, and finally $z = cw$.

Conversely, if $z = cw$ for some $c \in \mathbb{R}_+$, then

$$|z+w| = |cw+w| = |(c+1)w| = (c+1)|w| = c|w| + |w| = |cw| + |w| = |z| + |w|$$

since $c + 1 \in \mathbb{R}_+$. ■

Proposition 1.11 (Parallelogram Law). *For any $z, w \in \mathbb{C}$,*

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$

Proof. We use the property $z\bar{z} = |z|^2$ in Proposition 1.9 to obtain

$$\begin{aligned}
|z+w|^2 + |z-w|^2 &= (z+w)(\overline{z+w}) + (z-w)(\overline{z-w}) \\
&= (z+w)(\overline{z}_1 + \overline{w}) + (z-w)(\overline{z}_1 - \overline{w}) \\
&= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} + z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} \\
&= z\overline{z} + w\overline{w} + z\overline{z} + w\overline{w} = 2z\overline{z} + 2w\overline{w} \\
&= 2|z|^2 + 2|w|^2
\end{aligned}$$

for any $z, w \in \mathbb{C}$. ■

Theorem 1.12 (Conjugate Zeros Theorem). *Let f be a polynomial function with real coefficients. If z is a zero for f, then $\bar{z}$ is also a zero for f.*

Proof. Let
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$
with $a_n, \ldots, a_0 \in \mathbb{R}$. Let $w \in \mathbb{C}$ be such that $f(w) = 0$, which is to say
$$a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0 = 0.$$
Now, since $\overline{a}_i = a_i$ for all $0 \leq i \leq n$,
$$\begin{aligned} f(\overline{w}) &= a_n \overline{w}^n + a_{n-1} \overline{w}^{n-1} + \cdots + a_1 \overline{w} + a_0 \\ &= a_n \overline{w^n} + a_{n-1} \overline{w^{n-1}} + \cdots + a_1 \overline{w} + a_0 \\ &= \overline{a_n w^n} + \overline{a_{n-1} w^{n-1}} + \cdots + \overline{a_1 w} + \overline{a}_0 \\ &= \overline{a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0} \\ &= \overline{f(w)} = \overline{0} = 0. \end{aligned}$$
Therefore $\overline{w}$ is a zero for f. ■

In the proof above note that if $\deg(f) < 1$, so that $f(z) = a_0$, then the only way to have $f(w) = 0$ for some $w \in \mathbb{C}$ is to have $a_0 = 0$, in which case $f \equiv 0$ and the conclusion of the theorem follows trivially.

Exercise 1.13. Find the real and imaginary parts of $z = (1 + i)^{100}$.

Solution. First we have $(1 + i)^2 = 2i$. Now,
$$(1 + i)^{100} = \left[(1 + i)^2\right]^{50} = (2i)^{50} = 2^{50} i^{50} = 2^{50}(-1)^{25} = -2^{50},$$
using appropriate laws of exponents that are understood to be applicable to complex numbers by definition. So $\operatorname{Re}(z) = -2^{50}$ and $\operatorname{Im}(z) = 0$. ■

Exercise 1.14. Prove that, for any $z, w \in \mathbb{C}$,
$$|z| \leq |z - w| + |w|, \quad |z| - |w| \leq |z - w|, \quad \text{and} \quad |z| - |w| \leq |z + w|.$$

Solution. Since $|z + w| \leq |z| + |w|$ for any $z, w \in \mathbb{C}$, we have
$$|z| = |(z - w) + w| \leq |z - w| + |w|,$$
which proves the first inequality. From this we immediately obtain
$$|z| - |w| \leq |z - w|, \tag{1.5}$$
which proves the second inequality. Finally, from
$$|z| = |(z + w) + (-w)| \leq |z + w| + |-w| = |z + w| + |w|$$
we obtain
$$|z| - |w| \leq |z + w|,$$
the third inequality.

It's worthwhile adding that

$$|w| = |(w - z) + z| \le |w - z| + |z| \quad \Rightarrow \quad |w| - |z| \le |w - z| = |z - w|,$$

and this observation together with (1.5) implies that

$$\big||z| - |w|\big| \le |z - w|,$$

another useful result. ■

Exercise 1.15. Let $\alpha = a + bi$, $z = x + iy$, and $c \in (0, \infty)$. Transform $|z - \alpha| = c$ into an equation involving only x, y, a, b, and c, and describe the set of points $z \in \mathbb{C}$ that satisfy the equation.

Solution. We have

$$|z - \alpha| = |(x + iy) - (a + bi)| = |(x - a) + (y - b)i| = \sqrt{(x - a)^2 + (y - b)^2},$$

and so the equation $|z - \alpha| = c$ becomes

$$\sqrt{(x - a)^2 + (y - b)^2} = c$$

and hence

$$(x - a)^2 + (y - b)^2 = c^2,$$

which has as its solution set the set of all points in $\mathbb{R}^2$ lying on a circle centered at (a, b) with radius c. Correspondingly the solution set of $|z - \alpha| = c$ consists of all $z \in \mathbb{C}$ lying on a circle with center α and radius c. ■

We now define some special subsets of $\mathbb{C}$ using the modulus operation. The **open ball** (or **open disc**) centered at z_0 with radius $\epsilon > 0$ is defined to be the set

$$B_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\},$$

while the **deleted neighborhood** of z_0 with radius ϵ is

$$B'_\epsilon(z_0) = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\},$$

which is $B_\epsilon(z_0)$ with the point z_0 removed. The **closed ball** (or **closed disc**) at z_0 with radius ϵ is

$$\overline{B}_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \le \epsilon\}.$$

The open unit disc with center at the origin,

$$\mathbb{B} = B_1(0) = \{z \in \mathbb{C} : |z| < 1\},$$

arises especially frequently in complex analysis, as does the closed unit disc at the origin,

$$\overline{\mathbb{B}} = \{z \in \mathbb{C} : |z| \le 1\},$$

and the unit circle at the origin,

$$\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}.$$

Don't miss the bus: $\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S}$.

1.3 – Polar Form

Let $z = x + iy$ be a nonzero complex number, so that $r = |z| > 0$, and let $o = 0 + i0$. If θ is a measure of the angle, in radians, the ray $\overrightarrow{oz}$ makes with $\mathbb{R}_+$ (i.e. the positive real axis in $\mathbb{C}$), then basic trigonometry informs us that $x = r\cos\theta$ and $y = r\sin\theta$, and hence

$$z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta).$$

Defining

$$e^{it} = \cos t + i\sin t \tag{1.6}$$

for any $t \in \mathbb{R}$, which is known as **Euler's Formula**, we may pass from the **rectangular form** $x + iy$ of z to its **polar form**:

$$z = re^{i\theta}.$$

Much as the polar coordinate system may at times be more convenient to work with than the rectangular coordinate system in the course of an analysis, so too is it often much easier to work with the polar forms of complex numbers rather than their rectangular forms. We always have $r = |z|$ (so r is never negative), and the real-valued parameter θ is called the **argument** of z, denoted by $\arg(z)$. Note that the argument of z is not a unique number: if θ is an angle between $\overrightarrow{op}$ and $\mathbb{R}_+$, then so too is $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

Proposition 1.16. *For any $s, t \in \mathbb{R}$,*

$$e^{is+it} = e^{is}e^{it}.$$

Proof. Using established trigonometric identities, we have

$$\begin{aligned}
e^{is+it} = e^{(s+t)i} &= \cos(s+t) + i\sin(s+t)\\
&= (\cos s\cos t - \sin s\sin t) + i(\sin s\cos t + i\cos s\sin t)\\
&= \cos t(\cos s + i\sin s) + \sin t(i\cos s - \sin s)\\
&= \cos t(\cos s + i\sin s) + i\sin t(\cos s + i\sin s)\\
&= (\cos s + i\sin s)(\cos t + i\sin t) = e^{is}e^{it}
\end{aligned}$$

for any $s, t \in \mathbb{R}$. ■

A simple application of induction will show that

$$\prod_{k=1}^{n} e^{it_k} = e^{\sum_{k=1}^{n} it_k}$$

for $n \in \mathbb{N}$ and $t_k \in \mathbb{R}$, and in particular

$$(e^{it})^n = \prod_{k=1}^{n} e^{it} = e^{\sum_{k=1}^{n} it} = e^{int}.$$

Proposition 1.17 (**De Moivre's Formula**)**.** *For all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$,*

$$(\cos t + i\sin t)^n = \cos nt + i\sin nt.$$

Proof. Suppose $n \in \mathbb{N}$. Then

$$(\cos t + i \sin t)^n = (e^{it})^n = e^{int} = \cos nt + i \sin nt,$$

thereby confirming the formula for positive integers. Next, with the aid of Proposition 1.16 it is easy to see that $(e^{it})^{-1} = e^{-it}$, and so, applying Definition 1.3,

$$(\cos t + i \sin t)^{-n} = (e^{it})^{-n} = [(e^{it})^{-1}]^n = (e^{-it})^n = e^{-int} = \cos(-nt) + i \sin(-nt),$$

thereby confirming the formula for negative integers. The formula yields the identity $1 = 1$ when $n = 0$, finishing the proof. ∎

Having assigned a meaning to e^{it} for $t \in \mathbb{R}$, it is only natural to go further and define e^z for any $z \in \mathbb{C}$.

Definition 1.18. *For any $x, y \in \mathbb{R}$,*

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

Euler's Formula can be seen to be a special case of this definition for which $x = 0$, since $e^0 = 1$.

Proposition 1.19. *For any $x, y \in \mathbb{R}$, $e^{x+iy} = e^x e^{iy}$.*

Proof. For any $x, y \in \mathbb{R}$ we obtain

$$e^{x+iy} = e^x(\cos y + i \sin y) = e^x e^{iy}$$

using Definition 1.18. ∎

Propositions 1.16 and 1.19 can be used to obtain the following more general result, which is proved again using a different technique in §4.4.

Theorem 1.20. *For any $z, w \in \mathbb{C}$, $e^z e^w = e^{z+w}$.*

Proof. Let $z, w \in \mathbb{C}$, so $z = a + ib$ and $w = c + id$ for some $a, b, c, d \in \mathbb{R}$. Now,

$$\begin{aligned} e^z e^w &= e^{a+ib} e^{c+id} = (e^a e^{ib})(e^c e^{id}) = (e^a e^c)(e^{ib} e^{id}) \\ &= e^{a+c} e^{ib+id} = e^{a+c} e^{i(b+d)} = e^{(a+c)+i(b+d)} \\ &= e^{(a+ib)+(c+id)} = e^{z+w}, \end{aligned}$$

where $e^a e^c = e^{a+c}$ is an established fact in real analysis. ∎

Exercise 1.21. Show that

$$|z + w| = |z| + |w|$$

if and only if z or w is a nonnegative multiple of the other.

Solution. Suppose that $|z+w| = |z| + |w|$. If either $z = 0$ or $w = 0$ it follows trivially that one number is a nonnegative multiple of the other. Assume, then, that $z = a + ib$ and $w = c + id$ are two nonzero complex numbers. Identifying z and w with points (a, b) and (c, d) in $\mathbb{R}^2$, respectively, by the Law of Cosines we have

$$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w| \cos \varphi. \tag{1.7}$$

(See Figure 1, which gives a geometric motivation.) On the other hand, by the Parallelogram Law,

$$\begin{aligned} 2|z|^2 + 2|w|^2 = |z+w|^2 + |z-w|^2 &= (|z| + |w|)^2 + |z-w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 + |z-w|^2, \end{aligned} \tag{1.8}$$

and so

$$|z - w|^2 = |z|^2 - 2|z||w| + |w|^2.$$

Combining (1.7) and (1.8) yields

$$|z|^2 - 2|z||w| + |w|^2 = |z|^2 + |w|^2 - 2|z||w| \cos \varphi,$$

and thus

$$2|z||w| \cos \varphi = 2|z||w|.$$

Since $w, z \neq 0$, we obtain $\cos \varphi = 1$, and therefore $\varphi = 0$. This implies that z and w are expressible in polar form as $z = r_1 e^{i\theta}$ and $w = r_2 e^{i\theta}$ for some common argument θ, where $r_1, r_2 > 0$. In particular we have

$$w = \frac{r_2}{r_1} z$$

for $r_2/r_1 > 0$, so that w is seen to be a nonnegative multiple of z.

The converse is straightforward to show. If, say, $w = cz$ for some $c \geq 0$, then

$$|z + w| = |z + cz| = |(1 + c)z| = (1 + c)|z| = |z| + c|z| = |z| + |w|,$$

as desired. ■

Thus the Triangle Inequality only admits the possibility of equality in the case when two complex numbers lie on the same ray with initial point at the origin. We can extend the result of Exercise 1.21 as follows.

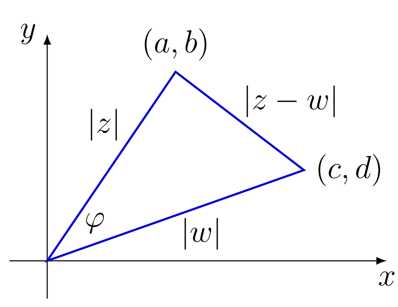

Figure 1.

Proposition 1.22. *Suppose that* $z_1, \ldots, z_n \in \mathbb{C}$. *Then*

$$\left|\sum_{k=1}^{n} z_k\right| = \sum_{k=1}^{n} |z_k|$$

if and only if there exists some $\theta \in [0, 2\pi)$ *and* $r_1, \ldots, r_n \geq 0$ *such that* $z_k = r_k e^{i\theta}$ *for all* $1 \leq k \leq n$.

Proof. The proposition is trivially true in the case when $n = 1$, and it's been shown to be true when $n = 2$ in Exercise 1.21. Suppose the proposition is true for some arbitrary $n \in \mathbb{N}$. Let $z_1, \ldots, z_{n+1} \in \mathbb{C}$ be such that

$$\left|\sum_{k=1}^{n+1} z_k\right| = \sum_{k=1}^{n+1} |z_k|. \tag{1.9}$$

Suppose that

$$\left|\sum_{k=1}^{n} z_k\right| \neq \sum_{k=1}^{n} |z_k|.$$

Then by the Triangle Inequality we obtain

$$\left|\sum_{k=1}^{n} z_k\right| < \sum_{k=1}^{n} |z_k|,$$

whence

$$\left|\sum_{k=1}^{n} z_k\right| + |z_{n+1}| < \sum_{k=1}^{n+1} |z_k| \quad \Rightarrow \quad \left|\sum_{k=1}^{n+1} z_k\right| < \sum_{k=1}^{n+1} |z_k|,$$

which contradicts (1.9). Thus it must be that

$$\left|\sum_{k=1}^{n} z_k\right| = \sum_{k=1}^{n} |z_k|. \tag{1.10}$$

By our inductive hypothesis it follows that there exists some $\theta \in [0, 2\pi)$ and $r_1, \ldots, r_n \geq 0$ such that $z_k = r_k e^{i\theta}$ for all $1 \leq k \leq n$. Now, by (1.9) and (1.10),

$$\left|\sum_{k=1}^{n} z_k + z_{n+1}\right| = \left|\sum_{k=1}^{n} z_k\right| + |z_{n+1}|,$$

and so by Exercise 1.21 we conclude that z_{n+1} must be a nonnegative multiple of $\sum_{k=1}^{n} z_k$ or vice-versa. Defining $r = r_1 + \cdots + r_n$, we have

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} r_k e^{i\theta} = re^{i\theta}.$$

If $re^{i\theta} = 0$ then we have $r_1 = \cdots = r_n = 0$, in which case if $z_{n+1} = se^{i\varphi}$ then we may write $z_k = 0e^{i\varphi}$ for all $1 \leq k \leq n$ and we are done. If $z_{n+1} = 0$, then we may write $z_{n+1} = 0e^{i\theta}$ and again we are done. Finally, if $re^{i\theta} \neq 0$ and $z_{n+1} \neq 0$, then by Exercise 1.21 one number must be a positive multiple of the other. That is, there exists some $c > 0$ such that

$$z_{n+1} = cre^{i\theta} \quad \text{or} \quad re^{i\theta} = cz_{n+1}$$

the latter implying that $z_{n+1} = c^{-1}re^{i\theta}$. That is, we once again have $z_k = r_k e^{i\theta}$ for all $1 \leq k \leq n+1$, where either $r_{n+1} = cr > 0$ or $r_{n+1} = r/c > 0$.

The proof of the converse statement of the proposition is easily proven directly for any $k \in \mathbb{N}$, and so the proposition is seen to hold in the $n+1$ case. ■

Exercise 1.23. Let $\alpha \neq 0$ be a complex number. Show there there are two distinct complex numbers whose square is α.

Solution. We can write α in polar form as $re^{i\theta}$, where $\alpha \neq 0$ implies that $r > 0$. One complex number whose square is α is $\beta_1 = \sqrt{r}e^{i\theta/2}$, and another is $\beta_2 = \sqrt{r}e^{i(\theta/2+\pi)}$. We have

$$\beta_2 = \sqrt{r}e^{i\pi}e^{i\theta/2} = -\sqrt{r}e^{i\theta/2} = -\beta_1,$$

and so it is clear that $\beta_1 \neq \beta_2$. ■

Exercise 1.24. Let $\alpha \neq 0$ be a complex number, and $n \in \mathbb{N}$. Show there exist n distinct complex numbers z such that $z^n = \alpha$.

Solution. In polar form we have $\alpha = re^{i\theta}$ for some $r > 0$. Now, for each $k \in \{0, 1, \ldots, n-1\}$ define

$$z_k = \sqrt[n]{r}e^{i(\theta/n+2\pi k/n)}.$$

Now, for any k we have

$$z_k^n = \left(\sqrt[n]{r}e^{i(\theta/n+2\pi k/n)}\right)^n = re^{i(\theta/n+2\pi k/n)n} = re^{i\theta+2\pi ik} = re^{i\theta}e^{2\pi ik} = \alpha\left(e^{2\pi i}\right)^k = \alpha \cdot 1^k = \alpha,$$

where the fourth equality follows from Proposition 1.16. ■

Exercise 1.25. Find $\text{Re}(i^{1/4})$ and $\text{Im}(i^{1/4})$, taking $i^{1/4} = re^{i\theta}$ for $0 < \theta < \pi/2$.

Solution. We must find $z = re^{i\theta}$ with $r > 0$ and $\theta \in (0, \pi/2)$ such that $z^4 = i$. From $|z|^4 = |z^4| = |i| = 1$ we have $|z| = 1$, so that $|re^{i\theta}| = |r| \cdot |e^{i\theta}| = |r| = 1$ and thus $r = 1$. At this stage we have $z = e^{i\theta}$ with

$$z^4 = \left(e^{i\theta}\right)^4 = e^{i\cdot 4\theta} = \cos(4\theta) + i\sin(4\theta) = i,$$

which requires $\cos(4\theta) = 0$ and $\sin(4\theta) = 1$. These conditions can be satisfied if $4\theta = \pi/2$, and hence $\theta = \pi/8$. We therefore have

$$i^{1/4} = z = e^{i\cdot\pi/8} = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right),$$

which by the appropriate half-angle trigonometric identities yields

$$\begin{aligned} i^{1/4} &= \sqrt{\frac{1+\cos(\pi/4)}{2}} + i\sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1+1\sqrt{2}}{2}} + i\sqrt{\frac{1-1\sqrt{2}}{2}} \\ &= \sqrt{\frac{2+\sqrt{2}}{4}} + i\sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

Therefore

$$\text{Re}(i^{1/4}) = \frac{1}{2}\sqrt{2+\sqrt{2}} \quad \text{and} \quad \text{Im}(i^{1/4}) = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

■

Exercise 1.26. Show that if $e^{z_1} = e^{z_2}$, then $z_2 = z_1 + 2\pi ki$ for some $k \in \mathbb{Z}$.

Solution. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, and suppose that $e^{z_1} = e^{z_2}$. We obtain

$$e^{x_1}e^{iy_1} = e^{x_2}e^{iy_2} \quad\Rightarrow\quad e^{x_1-x_2} = e^{i(y_2-y_1)} \quad\Rightarrow\quad \cos(y_2-y_1) + i\sin(y_2-y_1) = e^{x_1-x_2},$$

whence

$$\sin(y_2 - y_1) = 0 \quad\text{ and }\quad \cos(y_2 - y_2) = e^{x_1-x_2}.$$

From the first equation it follows that $y_2 - y_1 = n\pi$ for some $n \in \mathbb{Z}$, whereupon the second equation yields

$$e^{x_1-x_2} = \cos(n\pi) = \pm 1.$$

Of course $e^{x_1-x_2} = -1$ is impossible since $x_1 - x_2 \in \mathbb{R}$, which leads us to conclude that $\cos(n\pi) = 1$ and therefore $n = 2k$ for some $k \in \mathbb{Z}$ (i.e. n must be even). Now,

$$e^{x_1-x_2} = 1 \quad\Rightarrow\quad x_1 - x_2 = 0 \quad\Rightarrow\quad x_1 = x_2,$$

and

$$y_2 - y_1 = n\pi \quad\Rightarrow\quad y_2 = y_1 + n\pi \quad\Rightarrow\quad y_2 = y_1 + 2k\pi.$$

Finally,

$$z_2 = x_2 + iy_2 = x_1 + i(y_1 + 2\pi k) = (x_1 + iy_1) + 2\pi ki = z_1 + 2\pi ki$$

as was to be shown. ■

Exercise 1.27. Prove that

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)}$$

for any $\theta \in (0, 2\pi)$.

Solution. Let $0 < \theta < 2\pi$. Letting $z = \cos\theta + i\sin\theta = e^{i\theta}$, from Proposition 1.7 we have

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = 1 + e^{i\theta} + \left(e^{i\theta}\right)^2 + \cdots + \left(e^{i\theta}\right)^n = \frac{\left(e^{i\theta}\right)^{n+1} - 1}{e^{i\theta} - 1}. \tag{1.11}$$

The left-hand side of (1.11) becomes

$$1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \cdots + (\cos n\theta + i\sin n\theta),$$

and hence

$$(1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta) + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta). \tag{1.12}$$

As for the right-hand side of (1.11), we have

$$\begin{aligned}
\frac{\left(e^{i\theta}\right)^{n+1} - 1}{e^{i\theta} - 1} &= \frac{e^{(n+1/2)i\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \\
&= \frac{\cos(n+1/2)\theta + i\sin(n+1/2)\theta - [\cos(-\theta/2) + i\sin(-\theta/2)]}{[\cos(\theta/2) + i\sin(\theta/2)] - [\cos(-\theta/2) + i\sin(-\theta/2)]} \\
&= \frac{[\cos(n+1/2)\theta - \cos(\theta/2)] + i\,[\sin(n+1/2)\theta + \sin(\theta/2)]}{2i\sin(\theta/2)}
\end{aligned}$$

$$= \frac{\sin(n+1/2)\theta + \sin(\theta/2)}{2\sin(\theta/2)} - \frac{\cos(n+1/2)\theta - \cos(\theta/2)}{2\sin(\theta/2)}\, i \tag{1.13}$$

Equating the real part of (1.12) with the real part of (1.13) then gives

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin(n+1/2)\theta + \sin(\theta/2)}{2\sin(\theta/2)},$$

which immediately leads to the desired result. ■

1.4 – Roots of Complex Numbers

For $n \in \mathbb{N}$, an ***n*th root** of a complex number z is some $w \in \mathbb{C}$ such that $w^n = z$. We shall see that every nonzero complex number has precisely n distinct nth roots. The following example is illustrative of the $n = 2$ case.

Example 1.28. For any $a, b \in \mathbb{R}$ find $x, y \in \mathbb{R}$ such that $(x + iy)^2 = a + bi$.

Solution. Since

$$(x + iy)^2 = (x^2 - y^2) + 2xyi,$$

we must have $x^2 - y^2 = a$ and $2xy = b$. From these equations we obtain $y^2 = x^2 - a$ and $4x^2y^2 = b^2$, leading to

$$4x^2(x^2 - a) = b^2$$

and thus

$$4x^4 - 4ax^2 - b^2 = 0.$$

By the quadratic formula we then obtain

$$x^2 = \frac{4a \pm \sqrt{16a^2 - 4(4)(-b^2)}}{2(4)} = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$

If x is to be a real number we must have $x^2 \geq 0$, so it follows that we must have

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2},$$

and therefore we can let

$$x = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}}.$$

Now, since $b = 2xy$ and our choice for x is nonnegative, we must find a value for y that has the same sign as b. With this in mind we use $y^2 = x^2 - a$ to obtain

$$y = (\operatorname{sgn} b)\sqrt{x^2 - a} = (\operatorname{sgn} b)\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2} - a} = (\operatorname{sgn} b)\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}.$$

Note that the opposite values for x and y would also be suitable. ■

1.5 – Complex-Valued Functions

We consider here complex-valued functions of a single complex-valued variable, which is to say $f : S \to \mathbb{C}$ for some $S \subseteq \mathbb{C}$. If it happens that $S \subseteq \mathbb{R}$, then of course f is a complex-valued function of a single *real* variable. In either case it is convenient to write

$$f(z) = u(z) + iv(z),$$

where $u : S \to \mathbb{R}$ and $v : S \to \mathbb{R}$ are the **real** and **imaginary** parts of f, respectively. We define $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$, so that

$$(\operatorname{Re} f)(z) = u(z) = \operatorname{Re}[f(z)] \quad \text{and} \quad (\operatorname{Im} f)(z) = v(z) = \operatorname{Im}[f(z)] \tag{1.14}$$

For brevity the symbols $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ may be used instead of $(\operatorname{Re} f)(z)$ and $(\operatorname{Im} f)(z)$.

A notational device that will be used now and again is as follows. If $f_k : S_k \to \mathbb{C}$ are functions for $1 \le k \le n$, where $S_k \subseteq \mathbb{C}$ for each k and $f_j(z) = f_k(z)$ whenever $z \in S_j \cap S_k$, then $f_1 \cup \cdots \cup f_n$ (also written as $\bigcup_{k=1}^n f_k$) is the function given by

$$(f_1 \cup \cdots \cup f_n)(z) = f_k(z)$$

if $z \in S_k$. In particular if $t_0 < t_1 < \cdots < t_n$, and $f_k : [t_{k-1}, t_k] \to \mathbb{C}$ with $f_k(t_k) = f_{k+1}(t_k)$ for all k, then $f = \bigcup_{k=1}^n f_k$ is the piecewise-defined function given by

$$f(t) = \begin{cases} f_1(t), & t \in [t_0, t_1] \\ \quad\vdots & \\ f_n(t), & t \in [t_{n-1}, t_n]. \end{cases}$$

The most important complex-valued function, and indeed arguably the most powerful function in all of mathematics, is the **exponential function** $\exp : \mathbb{C} \to \mathbb{C}$ given by

$$\exp(x + iy) = e^x(\cos y + i \sin y).$$

From Definition 1.18 we see that

$$e^z = \exp(z)$$

for any $z \in \mathbb{C}$.

We introduce here the **big-oh** notation that is prevalent in the literature and will be used here in the later chapters. Let f and g be two functions, let $D = \operatorname{Dom}(f) \cap \operatorname{Dom}(g)$, and let $z_0 \in \mathbb{C}$ be a limit point of D. We make the following definition:

$$f(z) = O(g(z)) \text{ as } z \to z_0 \quad \Leftrightarrow \quad \exists c, \delta > 0\, \forall z \in D \cap B_\delta(z_0)\, \big(|f(z)| \le c|g(z)|\big).$$

If there is some $r > 0$ such that $z \in D$ whenever $|z| > r$, then we make the definition

$$f(z) = O(g(z)) \text{ as } z \to \infty \quad \Leftrightarrow \quad \exists c, \rho > 0\, \forall z \in \mathbb{C} \setminus B_\rho(0)\, \big(|f(z)| \le c|g(z)|\big).$$

Exercise 1.29. Let $f(z) = 1/z$. Describe the image under f of $B_1'(0)$, $\mathbb{C} \setminus \mathbb{B}$ and $\mathbb{S}$.

Solution. Given $z = x + yi \neq 0$, we have

$$f(z) = \frac{1}{z} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i = \frac{1}{x^2+y^2}(x - yi),$$

and so f can be seen to effect a reflection about the real axis

$$x + yi \mapsto x - yi = \rho e^{i\varphi},$$

followed by translation along the ray

$$R = \{re^{i\varphi} : r > 0\}$$

in a direction depending on whether $x^2 + y^2 > 1$ or $x^2 + y^2 < 1$.

If $x^2 + y^2 = |z|^2 < 1$, so that $z \in B_1'(0)$, then

$$|f(z)| = \frac{1}{x^2+y^2}|x - yi| = \frac{1}{\sqrt{x^2+y^2}} > 1,$$

so $f(z) \in \mathbb{C} - B_1'(0)$ and translation along R proceeds away from the origin. If $x^2+y^2 = |z|^2 > 1$, so that $z \in \mathbb{C} \setminus B_1'(0)$, then

$$|f(z)| = \frac{1}{x^2+y^2}|x - yi| = \frac{1}{\sqrt{x^2+y^2}} < 1,$$

so $f(z) \in B_1'(0)$ and translation along R proceeds toward the origin. In the special case when $x^2 + y^2 = 1$ we have $z \in \mathbb{S}$, and then $|f(z)| = 1$ implies that $f(z) \in \mathbb{S}$ also and no translation along R occurs.

Thus f generally can be regarded as performing a reflection about the real axis followed by a radial dilation or contraction, so that

$$f(B_1'(0)) = \mathbb{C} \setminus B_1'(0), \quad f(\mathbb{C} \setminus B_1'(0)) = B_1'(0),$$

and $f(\mathbb{S}) = \mathbb{S}$. ■

Exercise 1.30. Let $f(z) = 1/\bar{z}$. Describe the image under f of $B_1'(0)$, $\mathbb{C} \setminus \mathbb{B}$ and $\mathbb{S}$.

Solution. Given $z = x + yi = \rho e^{i\varphi} \neq 0$, we have

$$f(z) = \frac{1}{\bar{z}} = \frac{1}{x-yi} \cdot \frac{x+yi}{x+yi} = \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} i = \frac{1}{x^2+y^2}(x + yi),$$

and so $f(z) = z$ if $x^2 + y^2 = 1$ (i.e. $z \in \mathbb{S}$), otherwise f effects a translation along the ray

$$R = \{re^{i\varphi} : r > 0\}$$

in a direction depending on whether $x^2 + y^2 > 1$ or $x^2 + y^2 < 1$

If $x^2 + y^2 = |z|^2 < 1$, so that $z \in B_1'(0)$, then

$$|f(z)| = \frac{1}{x^2+y^2}|x + yi| = \frac{1}{\sqrt{x^2+y^2}} > 1,$$

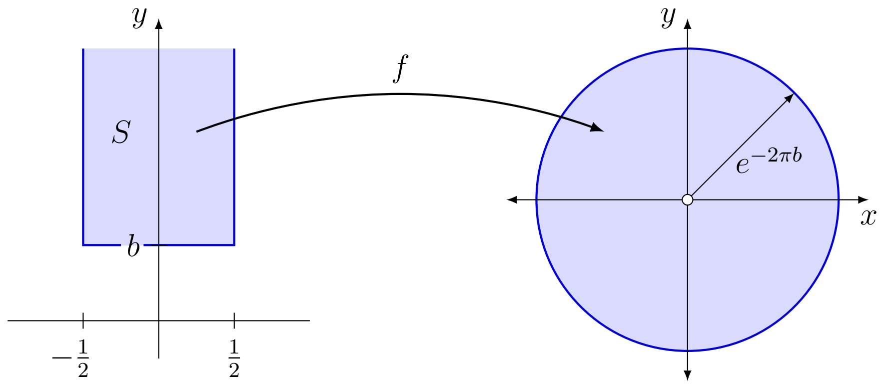

FIGURE 2.

so $f(z) \in \mathbb{C} - B_1'(0)$ and translation along R proceeds away from the origin. If $x^2 + y^2 = |z|^2 > 1$, so that $z \in \mathbb{C} \setminus B_1'(0)$, then

$$|f(z)| = \frac{1}{x^2+y^2}\,|x+yi| = \frac{1}{\sqrt{x^2+y^2}} < 1,$$

so $f(z) \in B_1'(0)$ and translation along R proceeds toward the origin.

Thus f generally performs a radial dilation or contraction such that

$$f(B_1'(0)) = \mathbb{C} \setminus B_1'(0), \quad f(\mathbb{C} \setminus B_1'(0)) = B_1'(0),$$

and $f(\mathbb{S}) = \mathbb{S}$. ■

Exercise 1.31. Let $f(z) = e^{2\pi i z}$. Describe the image under f of the set

$$S = \{x + yi : -1/2 \le x \le 1/2 \text{ and } y \ge b\}$$

shown at left in Figure 2.

Solution. For any $z = x + yi$ we have

$$f(z) = e^{2\pi i(x+yi)} = e^{2\pi i x - 2\pi y} = e^{-2\pi y}e^{2\pi i x} = e^{-2\pi y}\left[\cos(2\pi x) + i\sin(2\pi x)\right].$$

Now, the set

$$C_y = \left\{e^{-2\pi y}\left[\cos(2\pi x) + i\sin(2\pi x)\right] : -1/2 \le x \le 1/2\right\}$$

forms a circle centered at the origin with radius $e^{-2\pi y}$, and since $e^{-2\pi y} \le e^{-2\pi b}$ for all $y \ge b$, and $e^{-2\pi y} \to 0$ as $y \to \infty$, it can be seen that

$$f(S) = \bigcup_{y \ge b} C_y = \overline{B}'_{e^{-2\pi b}}(0),$$

as shown at right in Figure 2. ■

Exercise 1.32. Let $f(z) = e^z$. Describe the image under f of the following sets.

(a) The set $S_1 = \{x + iy : x \le 1 \text{ and } 0 \le y \le \pi\}$.

(b) The set $S_2 = \{x + iy : 0 \le y \le \pi\}$.

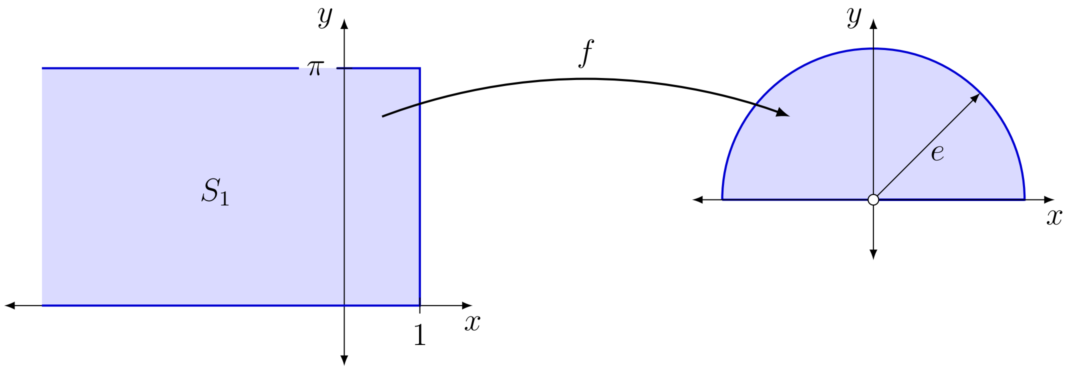

FIGURE 3.

Solution.
(a) For any $x + iy \in S_1$ we have

$$f(x + iy) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y),$$

which is a point on the upper semicircle with center 0 and radius e^x; and since $e^x \to 0$ as $x \to -\infty$, it follows that

$$f(S_1) = \{x + iy : y \geq 0\} \cap \overline{B}_e(0) - \{0\}.$$

See Figure 3.

(b) For any $x + iy \in S_2$ we have

$$f(x + iy) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y),$$

which is a point on the upper semicircle with center 0 and radius e^x; now, since $e^x \to 0$ as $x \to -\infty$, and $e^x \to \infty$ as $x \to \infty$, it follows that

$$f(S_2) = \{x + iy : y \geq 0\} - \{0\}.$$

See Figure 4. ■

Exercise 1.33 (Eneström's Theorem). Suppose that

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $n \geq 1$ and $a_0 \geq a_1 \geq \cdots \geq a_n > 0$. Prove that the zeros of P lie outside $\mathbb{B}$.

Solution. Suppose that $P(z) = 0$. It is clear that $z \neq 1$ since $P(1) = a_0 + \cdots + a_n > 0$, and so

$$P(z) = 0 \quad \Leftrightarrow \quad (1 - z)P(z) = 0 \quad \Leftrightarrow \quad \sum_{k=0}^{n} a_k(z^k - z^{k+1}) = 0$$

$$\Leftrightarrow \quad a_0 = \sum_{k=1}^{n}(a_{k-1} - a_k)z^k + a_n z^{n+1} \tag{1.15}$$

Suppose that $|z| < 1$. Then

$$\left|\sum_{k=1}^{n}(a_{k-1}-a_k)z^k + a_n z^{n+1}\right| \le \sum_{k=1}^{n}(a_{k-1}-a_k)|z|^k + a_n|z|^{n+1} < \sum_{k=1}^{n}(a_{k-1}-a_k) + a_n = a_0$$

since $a_{k-1} - a_k \ge 0$ for $1 \le k \le n$ and $a_n > 0$. Hence z cannot satisfy (1.15), and it follows that z cannot satisfy $P(z) = 0$. That is, if $z \in \mathbb{B}$ then z cannot be a zero for P. ■

Exercise 1.34. Suppose that

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $n \ge 1$ and $a_0 > a_1 > \cdots > a_n > 0$. Prove that the zeros of P lie outside $\overline{\mathbb{B}}$.

Solution. As with the previous exercise, $P(z) = 0$ if and only if

$$a_0 = \sum_{k=1}^{n}(a_{k-1}-a_k)z^k + a_n z^{n+1}.$$

We found that this equation cannot be satisfied for $z \in \mathbb{B}$, so suppose $z \in \mathbb{S}$. Observe that if z is a zero for P, then it must be that

$$a_0 = \sum_{k=1}^{n}(a_{k-1}-a_k) + a_n = \sum_{k=1}^{n}(a_{k-1}-a_k)|z|^k + a_n|z|^{n+1} = \left|\sum_{k=1}^{n}(a_{k-1}-a_k)z^k + a_n z^{n+1}\right|$$

since $a_0 > 0$ and $|z| = 1$. By Proposition 1.22 we have

$$\left|\sum_{k=1}^{n}(a_{k-1}-a_k)z^k + a_n z^{n+1}\right| = \sum_{k=1}^{n}(a_{k-1}-a_k)|z|^k + a_n|z|^{n+1}$$

if and only if

$$w_1 = (a_0 - a_1)z\ , \ldots,\ w_n = (a_{n-1} - a_n)z^n,\ \ w_{n+1} = a_n z^{n+1}$$

lie on a common ray in $\mathbb{C}$, and since $z \neq 0$ and

$$a_0 - a_1, \ldots, a_{n-1} - a_n, a_n > 0,$$

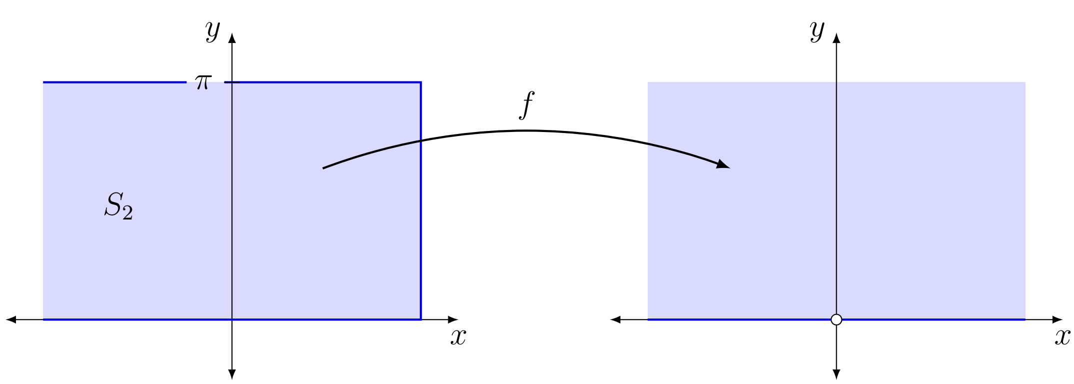

FIGURE 4.

this means there exists some $\theta \in [0, 2\pi)$ such that for each $1 \le k \le n+1$

$$w_k = r_k e^{i\theta}$$

for some $r_k > 0$. Now,

$$\left(\sum_{k=1}^{n+1} r_k\right) e^{i\theta} = \sum_{k=1}^{n} r_k e^{i\theta} + r_{n+1} e^{i\theta} = \sum_{k=1}^{n} (a_{k-1} - a_k) z^k + a_n z^{n+1} = a_0 > 0$$

implies that $\theta = 0$, and since $z \in \mathbb{S}$ and

$$(a_0 - a_1) z = w_1 = r_1 \quad \Rightarrow \quad z = \frac{r_1}{a_0 - a_1} > 0,$$

we conclude that $z = 1$. However, it is clear that 1 is not a zero for P, and we must conclude that there exists no $z \in \mathbb{S}$ such that $P(z) = 0$. This observation, together with the result of the previous exercise, shows that any zero for P must lie outside $\overline{B}_1(0)$. ■

2

Topology on the Complex Plane

2.1 – Metric Spaces

Definition 2.1. *A **metric space** is a pair (X,d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function, called a **metric** on X, for which the following properties hold for all $x, y, z \in X$:*

1. $d(x,y) \geq 0$, *with* $d(x,y) = 0$ *if and only if* $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,z) \leq d(x,y) + d(y,z)$.

The third property in Definition 2.1 is called the **Triangle Inequality**. A metric is also known as a **distance function**, and the real number $d(x,y)$ is said to be the **distance** between $x, y \in X$. It is clear that if (X,d) is a metric space and $S \subseteq X$, then (S,d) is likewise a metric space.

The **open ball** in (X,d) with center $a \in X$ and radius $\epsilon > 0$ is the set

$$B_\epsilon(a) = \{x \in X : d(x,a) < \epsilon\}.$$

A set $S \subseteq X$ is **bounded** if there exists some $M \in \mathbb{R}$ such that $d(x,y) \leq M$ for every $x, y \in S$. It is straightforward to verify that a set is bounded if and only if it is a subset of an open ball of finite radius. In particular, a set $S \subseteq \mathbb{R}$ is bounded if and only if there exist $a, b \in \mathbb{R}$ such that $S \subseteq (a,b)$.

A **topology** on a metric space (X,d) is a family of subsets of X that are called **open** sets. Unless said otherwise, we take the topology on (X,d) to be the set of all $U \subseteq X$ with the property that, for each $x \in U$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. That is, the topology on (X,d) is assumed to be the set

$$\mathcal{T}_d = \big\{U \subseteq X : \forall x \in U\, \exists \epsilon > 0\, (B_\epsilon(x) \subseteq U)\big\},$$

called the **topology on X induced by** d. It is always the case that $\varnothing, X \in \mathcal{T}_d$.

Given a subset S of a metric space (X,d), there are two common topologies on S. One is the topology on S induced by d, $\mathcal{T}_d(S)$, in which we take S to be the metric space (S,d). Letting

$$B_\epsilon(S,a) = B_\epsilon(a) \cap S = \{x \in S : d(x,a) < \epsilon\},$$

we have

$$\mathcal{T}_d(S) = \big\{U \subseteq S : \forall x \in U\, \exists \epsilon > 0\, \big(B_\epsilon(S,x) \subseteq U\big)\big\}.$$

The other common topology on S is the **subspace topology on** S,

$$\mathcal{T}_{\text{sub}}(S) = \{U \subseteq S : U = V \cap S \text{ for some } V \text{ open in } X\},$$

also referred to as the subspace topology S "inherits" from X. It will never be necessary to specify which topology on S is being used in these notes, because fortunately the two topologies are equivalent: $\mathcal{T}_d(S) = \mathcal{T}_{\text{sub}}(S)$. Which will be used, therefore, will be dictated solely by considerations of convenience.

Example 2.2. Recall the modulus function $|\cdot| : \mathbb{C} \to \mathbb{R}$ introduced in the previous chapter. The mapping $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by

$$(z, w) \mapsto |z - w| \tag{2.1}$$

is easily verified to be a metric on $\mathbb{C}$, which is to say $(\mathbb{C}, |\cdot|)$ is a metric space if it's understood that the symbol $|\cdot|$ now denotes the function (2.1). In these notes (2.1) will occasionally be referred to as the **euclidean metric** in situations when another metric is also in play.

The topology $\mathcal{T}$ on $\mathbb{C}$ induced by the euclidean metric is called the **standard topology**. Clearly every open ball $B_\epsilon(z) \subseteq \mathbb{C}$ is itself an open set, and it is also the case that $\mathbb{C}$ and $\varnothing$ are open. So see why $\varnothing$ is open, recall that $\varnothing \in \mathcal{T}$ iff for every $z \in \varnothing$ there is some $\epsilon > 0$ such that $B_\epsilon(z) \subseteq \varnothing$, and the latter statement is vacuously true! ■

The **complement** of a set $S \subseteq X$ is defined to be the set

$$S^c = X \setminus S = \{x \in X : x \notin S\}.$$

We define a set S to be **closed** if its complement S^c is open.

A **limit point** of a set S is a point x such that, for every $\epsilon > 0$, the open ball $B_\epsilon(x)$ contains a point $y \in S$ such that $y \neq x$. That is, x is a limit point of S iff $B'_\epsilon(x) \cap S \neq \varnothing$ for all $\epsilon > 0$. An equivalent definition states that x is a limit point of S if every neighborhood of x contains an infinite number of elements of S. We may now define the **closure** of S, denoted by $\overline{S}$, to be the union of S with all of its limit points. It is a fact that S is closed iff $\overline{S} = S$; that is, a set is closed if and only if it contains all of its limit points.

We say $S \subseteq X$ is **dense** in X if every $x \in X$ is either an element of S or a limit point of S. That is, S is dense in X if and only if

$$\forall x \in X\big[(x \in S) \vee \big(\forall \epsilon > 0\, (B'_\epsilon(x) \cap S \neq \varnothing)\big)\big].$$

It follows easily that S is dense in X iff $\overline{S} = X$.

A **boundary point** of S is a point x such that, for all $\epsilon > 0$, both $B_\epsilon(x) \cap S \neq \varnothing$ and $B_\epsilon(x) \cap S^c \neq \varnothing$ hold. We denote the set of boundary point of S by ∂S, called the **boundary** of S. It can be shown that $\overline{S} = S \cup \partial S$.

We say $x \in S$ is an **interior point** of S if there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq S$. Let S° denote the **interior** of S, which is the set of all interior points of S so that

$$S^\circ = \{x \in S : \exists \epsilon > 0 (B_\epsilon(x) \subseteq S)\}.$$

Another symbol for S° is $\mathrm{Int}(S)$. It can be shown that $\partial S = \overline{S} \setminus S^\circ$.

Given a set $S \subseteq \mathbb{C}$, let $\mathcal{B}(S)$ denote the set of all bounded functions $f : S \to \mathbb{C}$. For any $f \in \mathcal{B}(S)$ define the **uniform norm** of f to be

$$\|f\|_S = \sup\{|f(z)| : z \in S\}.$$

Note that, since the function f is given to be bounded on S, $\|f\|_S$ will exist as a real number by the Completeness Axiom of $\mathbb{R}$. Next, define the **uniform distance** function on $\mathcal{B}(S)$ to be the function $\mathcal{B}(S) \times \mathcal{B}(S) \to \mathbb{R}$ given by

$$(f, g) \mapsto \|f - g\|_S. \tag{2.2}$$

This is in fact a metric on $\mathcal{B}(S)$, which is to say $(\mathcal{B}(S), \|\cdot\|_S)$ is a metric space if it's understood that the symbol $\|\cdot\|_S$ denotes the function (2.2).

In a metric space (X, d), for a set $A \subseteq X$ and point $x \in X$, the **distance** between x and A is defined to be

$$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

For sets $A, B \subseteq X$, the **distance** between A and B is defined to be

$$\mathrm{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Neither of these distances has any hope of being a metric. In the metric space $(\mathbb{C}, |\cdot|)$ consider the sets $A = \{0, 1\}$ and $B = \{1, 2\}$. Clearly $A \neq B$, and yet $\mathrm{dist}(A, B) = 0$, which violates the first axiom of a metric in Definition 2.1.

2.2 – Numerical Sequences and Series

Let (X, d) be a metric space. A sequence (x_n) in X is **convergent in** X (or simply **convergent** if no larger metric space containing (X, d) is under consideration) if there is some point $x \in X$ for which the following is true: for every $\epsilon > 0$ there exists some $n_0 \in \mathbb{Z}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$. If such is the case, then we say (x_n) **converges to** x and write

$$\lim_{n\to\infty} x_n = x.$$

A sequence (x_n) in (X, d) is **Cauchy** if for every $\epsilon > 0$ there exists some $k \in \mathbb{Z}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$. A metric space (X, d) is **complete** if every Cauchy sequence in X is convergent in X.

Theorem 2.3. *$(\mathbb{C}, |\cdot|)$ is a complete metric space.*

Definition 2.4. *Let (a_n) be a sequence of real numbers. We define the* ***limit superior*** *of (a_n) to be*

$$\limsup_{n\to\infty} a_n = \inf_n \left(\sup_{k\geq n} a_k \right),$$

and the ***limit inferior*** *to be*

$$\liminf_{n\to\infty} a_n = \sup_n \left(\inf_{k\geq n} a_k \right).$$

Alternative symbols for limit superior and limit inferior are $\limsup$ *and* $\liminf$*, respectively.*

The limit superior and limit inferior of every sequence $(a_n)_{n=m}^{\infty}$ of reals will always exist in the set of **extended real numbers** $\overline{\mathbb{R}} = [-\infty, \infty]$. The sequence

$$\left(\sup_{k\geq n} a_k \right)_{n=m}^{\infty} \tag{2.3}$$

can easily be shown to be monotone decreasing, so if it is bounded below it will converge to some real number by the Monotone Convergence Theorem, and if it is not bounded below it will converge in $\overline{\mathbb{R}}$ to $-\infty$. In either case the limiting value will be the limit superior of (a_n); that is,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right).$$

By similar reasoning we have

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right).$$

Proposition 2.5. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of nonnegative real numbers, and let*

$$a = \limsup a_n \quad \text{and} \quad b = \limsup b_n.$$

Then the following hold:

1. $\limsup(a_n + b_n) \leq a + b$.
2. $\limsup(a_n b_n) \leq ab$ *if* $a \neq 0$.
3. *If* $c \geq 0$, *then* $\limsup(ca_n) = c \limsup a_n$.

4. *If* $\lim_{n\to\infty} a_n$ *exists in* $\mathbb{R}$*, then* $a = \lim_{n\to\infty} a_n$.

Proof.
Proof of Part (4). Suppose that $\lim_{n\to\infty} a_n = \hat{a} \in \mathbb{R}$. Fix $\epsilon > 0$. Then there exist some $N \in \mathbb{N}$ such that $|a_n - \hat{a}| < \epsilon/2$ for all $n \geq N$, whence

$$\hat{a} - \epsilon/2 < a_n < \hat{a} + \epsilon/2 \tag{2.4}$$

for all $n \geq N$ and we conclude that $\hat{a} + \epsilon/2$ is an upper bound for the set $\{a_n : n \geq N\}$. Thus

$$\sup_{k \geq N} a_k \leq \hat{a} + \epsilon/2,$$

and since the set

$$S = \left\{ \lim_{k \geq n} a_k : n \in \mathbb{N} \right\}$$

has at least one element $s \leq \hat{a} + \epsilon/2$ it follows that every lower bound β for S must be such that $\beta \leq \hat{a} + \epsilon/2$. In particular $\inf_n(S) \leq \hat{a} + \epsilon/2$, which is to say

$$\limsup a_n = \inf_n(S) = \inf_n \left(\sup_{k \geq n} a_k \right) < \hat{a} + \epsilon,$$

and since $\epsilon > 0$ is arbitrary we obtain

$$\limsup a_n \leq \hat{a}. \tag{2.5}$$

Next, from (2.4) we have $a_n > \hat{a} - \epsilon/2$ for all $n \geq N$, so

$$\sup_{k \geq n} a_k > \hat{a} - \epsilon/2 \tag{2.6}$$

for each $n \geq N$, whence we obtain (2.6) for all $n \in \mathbb{N}$ since the sequence (2.3) is monotone decreasing for any chosen value for m. This means every $\hat{a} - \epsilon/2$ is a lower bound for the set S, so that $\inf_n(S) \geq \hat{a} - \epsilon$ and we have

$$\limsup a_n = \inf_n(S) = \inf_n \left(\sup_{k \geq n} a_k \right) > \hat{a} - \epsilon.$$

Thus

$$\limsup a_n \geq \hat{a} \tag{2.7}$$

since $\epsilon > 0$ is arbitrary.

Combining (2.5) and (2.7), we conclude that

$$\limsup a_n = \hat{a} = \lim_{n\to\infty} a_n$$

as desired. ∎

Theorem 2.6 (Divergence Test). *If* $\lim_{n\to\infty} z_n \neq 0$*, then the series* $\sum z_n$ *diverges.*

Proof. Suppose that $\sum z_n$ converges. If s_k is the kth partial sum of $\sum z_n$, then there exists some $s \in \mathbb{C}$ such that $\lim_{k\to\infty} s_k = s$. We also have $\lim_{k\to\infty} s_{k-1} = s$. Now, observing that $z_k = s_k - s_{k-1}$, we obtain

$$\lim_{k\to\infty} z_k = \lim_{k\to\infty}(s_k - s_{k-1}) = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = s - s = 0.$$

Thus, if $\sum z_n$ converges, then $\lim_{n\to\infty} z_n = 0$. This implies that if $\lim_{n\to\infty} z_n \neq 0$, then $\sum z_n$ diverges. ■

Theorem 2.7 (Direct Comparison Test). *Let (z_n) be a sequence in $\mathbb{C}$, and (a_n) a sequence in $[0,\infty)$. Suppose there exists some $n_0 \in \mathbb{Z}$ such that $|z_n| \le a_n$ for all $n \ge n_0$. If $\sum a_n$ converges in $\mathbb{R}$, then $\sum z_n$ converges absolutely in $\mathbb{C}$.*

Remark. In the Direct Comparison Test the sequences involved may be $(z_n)_{n=n_1}^\infty$ and $(a_n)_{n=n_2}^\infty$, with $n_1 \neq n_2$, in which case we must have $n_0 \ge \max\{n_1, n_2\}$. To say $\sum a_n$ converges means $\sum_{n=m}^\infty a_n$ converges for any $m \ge n_2$, and so in particular $\sum_{n=n_0}^\infty a_n$ converges. The Direct Comparison Test concludes from this that $\sum_{n=n_0}^\infty z_n$ converges absolutely, from which is follows that $\sum_{n=m}^\infty z_n$ converges absolutely for any $m \ge n_1$, or simply put: $\sum z_n$ converges absolutely.

Theorem 2.8 (Root Test). *Given the series $\sum z_n$, let*

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|z_n|}.$$

1. *If $\rho \in [0,1)$, then $\sum z_n$ converges absolutely.*
2. *If $\rho \in (1,\infty]$, then $\sum z_n$ diverges.*

Proof. Suppose $\rho \in [0,1)$. Then there exists some $\epsilon > 0$ such that $\rho = 1 - 2\epsilon$, and so the sequence

$$\left(\sup_{k\ge n} \sqrt[k]{|z_k|}\right)_{n=0}^\infty$$

is monotone decreasing to the value $1 - 2\epsilon$. Thus there exists some $N \in \mathbb{N}$ such that

$$1 - 2\epsilon \le \sup_{k\ge n} \sqrt[k]{|z_k|} < 1 - \epsilon$$

for all $n \ge N$, and so in particular we have $|z_k|^{1/k} < 1 - \epsilon$ for all $k \ge N$. Hence

$$|z_k| < (1-\epsilon)^k$$

for all $k \ge N$. Now, $\sum(1-\epsilon)^k$ is a convergent geometric series since $0 < 1 - \epsilon < 1$, so by the Direct Comparison Test $\sum |z_n|$ also converges.

Next, suppose $\rho \in (1,\infty]$. Then there exists some $\alpha > 0$ such that $\rho > 1 + 2\alpha$, and so for all n we have

$$\sup_{k\ge n} \sqrt[k]{|z_k|} > 1 + 2\alpha.$$

From this it follows that for every n there exists some $k \ge n$ such that $|z_k|^{1/k} > 1 + \alpha$, whence

$$|z_k| > (1+\alpha)^k$$

obtains, and we conclude that $|z_n| > 1$ for infinitely many values of n. This implies that $\lim_{n\to\infty} z_n \neq 0$, and so $\sum z_n$ diverges by the Divergence Test. ■

Theorem 2.9 (Ratio Test). *Given the series $\sum z_n$ for which there exists some $N \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \geq N$, let*

$$\rho = \limsup_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \quad \text{and} \quad \hat{\rho} = \liminf_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|.$$

1. *If $\rho \in [0, 1)$, then $\sum z_n$ converges absolutely.*
2. *If $\hat{\rho} \in (1, \infty]$, then $\sum z_n$ diverges.*
3. *If there exists some n_0 such that $|z_{n+1}/z_n| \geq 1$ for all $n \geq n_0$, then $\sum z_n$ diverges.*

Exercise 2.10 (L1.4.1). Show that $\lim_{n\to\infty} z^n = 0$ if $|z| < 1$.

Solution. From basic analysis it is known that $\lim_{n\to\infty} x^n = 0$ for any $x \in \mathbb{R}$ such that $|x| < 1$. That is,

$$\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n > N(|x^n| < \epsilon),$$

where of course $|x^n| = |x|^n$ for all $n > 0$.

Let $\epsilon > 0$. Since $|z| < 1$, there exists some $N \in \mathbb{N}$ such that $\big||z|^n\big| < \epsilon$ whenever $n > N$, where

$$\big||z|^n\big| = \big||z|\big|^n = |z|^n.$$

Thus for any $n > N$ we have $|z^n| = |z|^n < \epsilon$, which shows that

$$\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n > N(|z^n| < \epsilon)$$

and therefore $\lim_{n\to\infty} z^n = 0$. ■

Exercise 2.11 (L1.4.2). Show that $\lim_{n\to\infty} z^n$ does not exist if $|z| > 1$.

Solution. Let $z \in \mathbb{C}$ such that $|z| > 1$. Let $w \in \mathbb{C}$. Using the fact from basic analysis that

$$\lim_{n\to\infty} x^n = +\infty$$

for any $x > 1$, we find that for any $N \in \mathbb{N}$ there exists some $n > N$ such that $|z|^n > 1 + |w|$. Since by the Triangle Inequality we have

$$\begin{aligned} |z|^n > 1 + |w| \;&\Rightarrow\; |z^n| > 1 + |w| \;\Rightarrow\; |z^n - w + w| - |w| > 1 \\ &\Rightarrow\; \big(|z^n - w| + |w|\big) - |w| > 1 \;\Rightarrow\; |z^n - w| > 1, \end{aligned}$$

it follows that for any $N \in \mathbb{N}$ there exists some $n > N$ such that $|z^n - w| > 1$. This shows that $\lim_{n\to\infty} z^n \neq w$, and since $w \in \mathbb{C}$ is arbitrary we conclude that $\lim_{n\to\infty} z^n$ does not exist. ■

Exercise 2.12 (L1.4.3). Show that if $|z| < 1$, then

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}. \tag{2.8}$$

Solution. By convention it is understood that 1 is the first term in the series (2.7), even in the case when $z = 0$. That is, we take 0^0 to be 1 whenever it arises in a series.

Clearly if $z = 0$ we have

$$\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} 0^k = 1 = \frac{1}{1-0} = \frac{1}{1-z}.$$

Suppose $0 < |z| < 1$, and set

$$c = \left| \frac{z}{z-1} \right|$$

so that $c > 0$. Let $\epsilon > 0$ be arbitrary. Since $|z|^n \to 0$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that $|z|^n < \epsilon/c$ for all $n > N$. By Proposition 1.7

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z - 1}$$

for all $n \in \mathbb{W}$, and so for any $n > N$ we have

$$\begin{aligned} \left| \sum_{k=0}^{n} z^k - \frac{1}{1-z} \right| &= \left| \frac{z^{n+1}-1}{z-1} - \frac{1}{1-z} \right| = \left| \frac{z^{n+1}-1}{z-1} + \frac{1}{z-1} \right| \\ &= \left| \frac{z^{n+1}}{z-1} \right| = |z|^n \left| \frac{z}{z-1} \right| = |z|^n c < \frac{\epsilon}{c} \cdot c = \epsilon. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sum_{k=0}^{n} z^k = \frac{1}{1-z},$$

which is equivalent to (2.8). ■

Exercise 2.13 (L1.4.4). Define the function f by

$$f(z) = \lim_{n\to\infty} \frac{1}{1+n^2 z}.$$

Show that

$$f(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0, & \text{if } z \neq 0 \end{cases}$$

Solution. This can be done by a simple application of limit laws by rewriting $1/(1 + n^2 z)$ as

$$\frac{1/n^2}{1/n^2 + z},$$

but instead we will employ an $\epsilon\delta$-argument.

It is clear that $f(0) = 1$, so suppose that $z \neq 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \sqrt{\frac{1}{|z|}\left(1 + \frac{1}{\epsilon}\right)}.$$

For any integer $n > N$ we have

$$n^2 > N^2 > \frac{1}{|z|}\left(1 + \frac{1}{\epsilon}\right),$$

whence $n^2|z| - 1 > 1/\epsilon$. Now,

$$\left|n^2|z| - 1\right| > \frac{1}{\epsilon} \quad \Rightarrow \quad |n^2 z + 1| > \frac{1}{\epsilon} \quad \Rightarrow \quad \left|\frac{1}{1 + n^2 z}\right| < \epsilon,$$

which shows that

$$f(z) = \lim_{n \to \infty} \frac{1}{1 + n^2 z} = 1$$

if $z \neq 0$. ∎

2.3 – Limits and Continuity

Definition 2.14. *Let (X, d) and (Y, ρ) be metric spaces, with $D \subseteq X$, $f : D \to Y$, and c a limit point of D. Given $y \in Y$, we say f has* ***limit*** *y at c, written*

$$\lim_{x \to c} f(x) = y,$$

if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $x \in D$,

$$0 < d(x, c) < \delta \quad \Rightarrow \quad \rho\big(f(x), y\big) < \epsilon.$$

In the metric space $(\mathbb{C}, |\cdot|)$, for a function $f : D \subseteq \mathbb{C} \to \mathbb{C}$, limit point z_0 of D, and complex number w, the definition indicates that

$$\lim_{z \to z_0} f(z) = w$$

if and only if for all $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $z \in D$,

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w| < \epsilon.$$

The following laws of limits, expressed for the metric space $(\mathbb{C}, |\cdot|)$, are proven using arguments that are nearly identical to those made to prove the corresponding laws in §2.3 of the *Calculus Notes*.

Theorem 2.15 (Laws of Limits). *For any $z_0, a, b, c \in \mathbb{C}$, if*

$$\lim_{z \to z_0} f(z) = a \quad \textit{and} \quad \lim_{z \to z_0} g(z) = b,$$

then

1. $\displaystyle\lim_{z \to z_0} c = c$
2. $\displaystyle\lim_{z \to z_0} cf(z) = ca = c \lim_{z \to z_0} f(z)$
3. $\displaystyle\lim_{z \to z_0} [f(z) \pm g(z)] = a \pm b = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$
4. $\displaystyle\lim_{z \to z_0} [f(z)g(z)] = ab = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$
5. *Provided that $b \neq 0$,*
$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{a}{b} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}.$$
6. *For any integer $n > 0$,*
$$\lim_{z \to z_0} [f(z)]^n = a^n = \left[\lim_{z \to z_0} f(z)\right]^n.$$

While the existence of $\lim_{z \to z_0}[f(z) + g(z)]$ does not necessarily imply the existence of $\lim_{z \to z_0} f(x)$ or $\lim_{z \to z_0} g(x)$, we do have the following result in the complex realm.

Proposition 2.16. *Let $f : S \to \mathbb{C}$ be given by $f(z) = u(z) + iv(z)$, where $u, v : S \to \mathbb{R}$. If*

$$\lim_{z \to z_0} f(z) = a + ib.$$

then

$$\lim_{z \to z_0} u(z) = a \quad \text{and} \quad \lim_{z \to z_0} v(z) = b.$$

Proof. Suppose that $\lim_{z \to z_0} f(z) = a + ib$. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that, for all $z \in S$ such that $0 < |z - z_0| < \delta$, we have $|f(z) - (a + ib)| < \epsilon$. That is,

$$\big|[u(z) - a] + i[v(z) - b]\big| < \epsilon,$$

whence

$$\sqrt{[u(z) - a]^2 + [v(z) - b]^2} < \epsilon.$$

From this it readily follows that $|u(z) - a| < \epsilon$ and also $|v(z) - b| < \epsilon$. Therefore $\lim_{z \to z_0} u(z) = a$ and $\lim_{z \to z_0} v(z) = b$. ■

Recalling (1.14), from Proposition 2.16 we see that

$$\operatorname{Re}\left(\lim_{z \to z_0} f(z)\right) = \lim_{z \to z_0} (\operatorname{Re} f)(z) = \lim_{z \to z_0} \operatorname{Re}[f(z)] \tag{2.9}$$

and

$$\operatorname{Im}\left(\lim_{z \to z_0} f(z)\right) = \lim_{z \to z_0} (\operatorname{Im} f)(z) = \lim_{z \to z_0} \operatorname{Im}[f(z)]. \tag{2.10}$$

Definition 2.17. *Let (X, d) and (Y, ρ) be metric spaces, and let $D \subseteq X$. A function $f : D \to Y$ is* ***continuous at*** *$x_0 \in D$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $x \in D$,*

$$d(x, x_0) < \delta \quad \Rightarrow \quad \rho\big(f(x), f(x_0)\big) < \epsilon.$$

If f is continuous at every point in a set $S \subseteq D$, then f is said to be ***continuous on*** *S. A* ***continuous function*** *is a function that is continuous on its domain.*

Clearly in order for a function f to be continuous at a point z_0, it is necessary (but not sufficient) that z_0 lie in the domain of f. Also, it is always true that a function is continuous at any isolated point in its domain.

Definition 2.18. *Let (X, d) and (Y, ρ) be metric spaces, and let $D \subseteq X$. A function $f : D \to Y$ is* ***uniformly continuous on*** *D if for every $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $x_1, x_2 \in D$,*

$$d(x_1, x_2) < \delta \quad \Rightarrow \quad \rho\big(f(x_1), f(x_2)\big) < \epsilon.$$

If $S \subseteq D$, then f is said to be ***uniformly continuous on*** *S if $f|_S$ is uniformly continuous on S.*

Theorem 2.19. *Suppose $f : (X, d) \to (Y, \rho)$. Then f is continuous on X if and only if $f^{-1}(V)$ is open in X whenever V is open in Y.*

Theorem 2.20. *Suppose $f : (X, d) \to (Y, \rho)$ and $a \in X$. Then f is continuous at a if and only if $\lim f(x_n) = f(a)$ for every sequence (x_n) in X such that $\lim x_n = a$.*

Proof. Suppose f is continuous at a. Let (x_n) be a sequence in X such that $\lim x_n = a$. Fix $\epsilon > 0$. There exists some $\delta > 0$ such that, for all $x \in X$,

$$d(x, a) < \delta \;\;\Rightarrow\;\; \rho\big(f(x), f(a)\big) < \epsilon.$$

Since $x_n \to x$, there is some $k \in \mathbb{N}$ such that $d(x_n, x) < \delta$ for all $n \geq k$. Now, for all $n \geq k$ we have $\rho(f(x_n), f(a)) < \epsilon$, and therefore $\lim f(x_n) = f(a)$.

Conversely, suppose f is not continuous at a. Then there exists some $\epsilon > 0$ such that, for each $\delta > 0$, there is some $x \in X$ for which $d(x, a) < \delta$ and yet $\rho(f(x), f(a)) \geq \epsilon$. Thus, for each $n \in \mathbb{N}$, there is some $x_n \in X$ for which $d(x_n, a) < 1/n$ and yet $\rho(f(x_n), f(a)) \geq \epsilon$. The resultant sequence (x_n) in X is such that $\lim x_n = a$ and yet $\lim f(x_n) \neq a$. ■

Theorem 2.21. *Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, let $U \subseteq X$ and $V \subseteq Y$, and let $f : U \to Y$ and $g : V \to Z$ such that $f(U) \cap V \neq \varnothing$. If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.*

Proof. Suppose f is continuous at c and g is continuous at $f(c)$. Fix $\epsilon > 0$. There exists some $\delta' > 0$ such that, for all $y \in V$,

$$d_Y(y, f(c)) < \delta' \;\;\Rightarrow\;\; d_Z(g(y), g(f(c))) < \epsilon.$$

In turn, there exists some $\delta > 0$ such that, for all $x \in U$,

$$d_X(x, c) < \delta \;\;\Rightarrow\;\; d_Y(f(x), f(c)) < \delta'.$$

Let $x \in \mathrm{Dom}(g \circ f)$ such that $d_X(x, c) < \delta$. Then we have $d_Y(f(x), f(c)) < \delta'$, and since $x \in \mathrm{Dom}(g \circ f)$ implies $f(x) \in \mathrm{Dom}(g) = V$, it follows that $d_Z(g(f(x)), g(f(c))) < \epsilon$. We conclude that for every $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $x \in \mathrm{Dom}(g \circ f)$,

$$d_X(x, c) < \delta \;\;\Rightarrow\;\; d_Z\big((g \circ f)(x), (g \circ f)(c)\big) < \epsilon.$$

Therefore $g \circ f$ is continuous at c. ■

Proposition 2.22. *Let $\lim_{z \to z_0} g(z) = w$ and let f be continuous at w. If w is an interior point of $\mathrm{Dom}(f)$, then*

$$\lim_{z \to z_0} (f \circ g)(z) = f(w).$$

Proof. Suppose w is an interior point of $\mathrm{Dom}(f)$, so $B_r(w) \subseteq \mathrm{Dom}(f)$ for some $r > 0$. Let $\epsilon > 0$. Since f is continuous at w there exists some $0 < \gamma < r$ such that

$$|z - w| < \gamma \;\;\Rightarrow\;\; |f(z) - f(w)| < \epsilon.$$

Additionally, since $\lim_{z \to z_0} g(z) = w$ there can be found some $\delta > 0$ such that, for all $z \in \mathrm{Dom}(g)$,

$$0 < |z - z_0| < \delta \;\;\Rightarrow\;\; |g(z) - w| < \gamma,$$

and hence

$$|(f \circ g)(z) - f(w)| = |f(g(z)) - f(w)| < \epsilon.$$

We have shown that for every $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $z \in \mathrm{Dom}(g)$,

$$0 < |z - z_0| < \delta \;\;\Rightarrow\;\; |(f \circ g)(z) - f(w)| < \epsilon.$$

The proof is done. ■

The conclusion of Proposition 2.22 can be written more compellingly as

$$\lim_{z\to z_0} f(g(z)) = f\left(\lim_{z\to z_0} g(z)\right).$$

The magnitude function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is clearly continuous throughout $\mathbb{C}$. If we apply Proposition 2.22 to the case when $f = |\cdot|$ we obtain

$$\lim_{z\to z_0} |g(z)| = \left|\lim_{z\to z_0} g(z)\right|$$

as a general result whenever $\lim_{z\to z_0} g(z)$ exists in $\mathbb{C}$.

Theorem 2.23. *Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. Then $f : S \to \mathbb{C}$ is continuous at z_0 if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0.*

Proof. Suppose that f is continuous at z_0, and fix $\epsilon > 0$. Then there exists some $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all $z \in S$ for which $|z - z_0| < \delta$. Thus if $z \in S \cap B_\delta(z_0)$ we have

$$\big|[\operatorname{Re} f(z) + i \operatorname{Im} f(z)] - [\operatorname{Re} f(z_0) + i \operatorname{Im} f(z_0)]\big| < \epsilon,$$

whence

$$\sqrt{[\operatorname{Re} f(z) - \operatorname{Re} f(z_0)]^2 + [\operatorname{Im} f(z) - \operatorname{Im} f(z_0)]^2} < \epsilon$$

obtains and we conclude that

$$\big|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)\big| < \epsilon \quad \text{and} \quad \big|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)\big| < \epsilon$$

Therefore $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0.

Conversely, suppose that $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0. Let $\epsilon > 0$. There exist $\delta_1, \delta_2 > 0$ such that, for any $(x, y) \in R$,

$$|z - z_0| < \delta_1 \;\; \Rightarrow \;\; \big|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)\big| < \epsilon/2$$

and

$$|z - z_0| < \delta_2 \;\; \Rightarrow \;\; \big|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)\big| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, and suppose $z \in S$ such that $|z - z_0| < \delta$. Then

$$\begin{aligned}
|f(z) - f(z_0)| &= \big|[\operatorname{Re} f(z) + i \operatorname{Im} f(z)] - [\operatorname{Re} f(z_0) + i \operatorname{Im} f(z_0)]\big| \\
&= \big|[\operatorname{Re} f(z) - \operatorname{Re} f(z_0)] + i[\operatorname{Im} f(z) - \operatorname{Im} f(z_0)]\big| \\
&\le \big|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)\big| + \big|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)\big| \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

where in general $|x + iy| \le |x| + |y|$ by the Triangle Inequality. Hence f is continuous at z_0. ■

Exercise 2.24 (L1.4.5). For $|z| \ne 1$ show that the limit

$$f(z) = \lim_{n\to\infty} \frac{z^n - 1}{z^n + 1}$$

exists. Is it possible to define $f(z)$ when $|z| = 1$ in such a way that f is continuous on $\mathbb{C}$?

Solution. When $|z| < 1$ we have $z^n \to 0$ as $n \to \infty$, and so

$$\lim_{n\to\infty} \frac{z^n - 1}{z^n + 1} = \frac{\lim_{n\to\infty}(z^n - 1)}{\lim_{n\to\infty}(z^n + 1)} = \frac{0-1}{0+1} = -1.$$

When $|z| > 1$ we have $|1/z| = 1/|z| < 1$, so that $1/z^n = (1/z)^n \to 0$ as $n \to \infty$, and thus

$$\lim_{n\to\infty} \frac{z^n - 1}{z^n + 1} = \lim_{n\to\infty} \frac{1 - 1/z^n}{1 + 1/z^n} = \frac{\lim_{n\to\infty}(1 - 1/z^n)}{\lim_{n\to\infty}(1 + 1/z^n)} = \frac{1-0}{1+0} = 1.$$

Hence

$$f(z) = \begin{cases} -1, & \text{if } |z| < 1 \\ 1, & \text{if } |z| > 1 \end{cases}$$

which makes it clear that f has no continuous extension to $\mathbb{C}$. ■

Exercise 2.25 (L1.4.6). Let

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1 + z^n}.$$

Determine the domain of f, and give the explicit values of $f(z)$ for $z \in \mathrm{Dom}(f)$.

Solution. Clearly if $|z| < 1$ we have

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1 + z^n} = \frac{0}{1+0} = 0.$$

If $|z| > 1$ we have

$$f(z) = \lim_{n\to\infty} \frac{z^n}{1 + z^n} = \lim_{n\to\infty} \frac{1}{1/z^n + 1} = \frac{1}{0+1} = 1.$$

If $z = 1$ we have

$$f(1) = \lim_{n\to\infty} \frac{1^n}{1 + 1^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

For no other $z \in \mathbb{C}$ is $f(z)$ defined. This can be verified directly in the cases when $z = \pi/2, \pi, 3\pi/2$. Suppose $z \neq 1, \pi/2, \pi, 3\pi/2$ is such that $|z| = 1$, so that $z = e^{i\theta}$ for some $\theta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$. Suppose $f(z) = \alpha$ for some $\alpha \in \mathbb{C}$. Then

$$\lim_{n\to\infty} \frac{z^n}{1 + z^n} = \lim_{n\to\infty} \frac{1}{z^{-n} + 1} = \lim_{n\to\infty} \frac{1}{e^{-i(n\theta)} + 1} = \alpha,$$

and so

$$\frac{1}{\alpha} = \lim_{n\to\infty} \left(e^{-i(n\theta)} + 1\right) = \lim_{n\to\infty} \left[\cos(n\theta) - i\sin(n\theta) + 1\right].$$

Thus,

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N \left(\left| \left[\cos(n\theta) - i\sin(n\theta)\right] - (1/\alpha - 1) \right| < \epsilon \right).$$

However, if we choose $\epsilon = 1/4$, we find that for any $N \in \mathbb{N}$ there exists some $n > N$ for which

$$\left| \left[\cos(n\theta) - i\sin(n\theta)\right] - (1/\alpha - 1) \right| \geq 1/4.$$

To see this, consider that there exists some $p, q \in \mathbb{Z}$ for which

$$\frac{p + 1/4}{q}\pi \leq \theta < \frac{p + 1/2}{q}\pi,$$

and now if we let $n = q, 2q, 3q, \ldots$ we find that $n\theta$ revisits each of the various quadrants $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, and $(3\pi/2, 2\pi)$ an infinite number of times. This places $\cos(n\theta) - i\sin(n\theta)$ in each of the four quadrants of $\mathbb{S} \subseteq \mathbb{C}$ an infinite number of times, and since $\mathbb{S}$ has radius 1 it follows that $\cos(n\theta) - i\sin(n\theta)$ will be a distance greater than $1/4$ from $1/\alpha - 1$ for infinitely many $n \in \mathbb{N}$.

Therefore $\mathrm{Dom}(f) = (\mathbb{C} \setminus \mathbb{S}) \cup \{1\}$, with values of $f(z)$ given above. ■

2.4 – Connectedness

The notion of connectedness will here be dealt with mostly in the setting of metric spaces, though there will be occasional consideration of a topological subspace rather than the equivalent metric subspace, since the former is oftentimes simpler to analyze.

Definition 2.26. *A metric space (X, d) is* ***connected*** *if there do not exist nonempty open sets $U, V \subseteq X$ such that $U \cap V = \varnothing$ and $U \cup V = X$. A set $S \subseteq X$ is connected if the metric space (S, d) is connected. A* ***region*** *is an open connected set.*

We say (X, d) is **disconnected** if it is not connected; that is, (X, d) is disconnected if there exist nonempty disjoint open sets $U, V \subseteq X$ such that $U \cup V = X$, in which case we say that U and V **disconnect** X.

Given a metric space (X, d), since the subspace topology on a set $S \subseteq X$ is the same as the topology on S induced by the metric d, we may say that S is a connected set if and only if S is a connected *subspace* of X. That is, S is a connected set if and only if there do not exist nonempty disjoint sets U, V *open in* S such that $U \cup V = S$.

Theorem 2.27. *Let (X, d) be a metric space.*

1. *Let $U, V \subseteq X$ be disjoint open sets. If $C \subseteq X$ is connected and $C \subseteq U \cup V$, then either $C \subseteq U$ or $C \subseteq V$.*
2. *If $\mathcal{F}$ is a family of connected subsets of X having a point in common, then $\cup\mathcal{F}$ is connected.*

Proof.
Proof of (1). Suppose $C \subseteq X$ is connected and $C \subseteq U \cup V$. Let $A = C \cap U$ and $B = C \cap V$, so A and B are open in the subspace topology on C, and hence are open in the metric space (C, d). Also

$$A \cup B = C \cap (U \cup V) = C \quad \text{and} \quad A \cap B = C \cap U \cap V = \varnothing.$$

Since C is connected, either $A = \varnothing$ or $B = \varnothing$ (otherwise A and B disconnect C). Therefore either $C \subseteq V$ or $C \subseteq U$.

Proof of (2). Suppose $\mathcal{F}$ is a family of connected subsets of X having a point p in common. Let $A, B \subseteq \cup\mathcal{F}$ be disjoint sets, open in $\cup\mathcal{F}$, with $A \cup B = \cup\mathcal{F}$. Thus $C \subseteq A \cup B$ for each $C \in \mathcal{F}$, and since C is connected, by Part (1) either $C \subseteq A$ or $C \subseteq B$ (but not both). Now, $p \in \cap\mathcal{F} \subseteq A \cup B$, so either $p \in A$ or $p \in B$. If $p \in A$, then since $p \in C$ for all $C \in \mathcal{F}$ and $A \cap B = \varnothing$, we find that no $C \in \mathcal{F}$ can be a subset of B, and thus $C \subseteq A$ for all $C \in \mathcal{F}$. It follows that $\cup\mathcal{F} \subseteq A$, and since $A \cup B = \cup\mathcal{F}$, we obtain $B = \varnothing$. If $p \in B$, a similar argument concludes that $\cup\mathcal{F} \subseteq B$ and $A = \varnothing$. Therefore A and B do not disconnect $\cup\mathcal{F}$, and since A and B are arbitrary disjoint sets that are open in $\cup\mathcal{F}$ and have union $\cup\mathcal{F}$, we conclude that $\cup\mathcal{F}$ is connected. ∎

Remark. Suppose (X, d) is a metric space and $S \subseteq Y \subseteq X$. Then S is a connected subset of X if and only if S is a connected subset of Y. This stems from the fact that the subspace topology that S inherits from Y is the same as the subspace topology S inherits from X.

We take as a fact from elementary analysis that intervals in $\mathbb{R}$ are connected. Thus in particular the interval $[0,1]$ is connected, which we make use of in an upcoming proof.

Definition 2.28. *A metric space (X,d) is* ***path-connected*** *if for any $x_0, x_1 \in X$ there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. A set $S \subseteq X$ is path-connected if the metric space (S,d) is path-connected.*

Proposition 2.29. *If a metric space (X,d) is path-connected, then it is connected.*

Proof. Suppose that (X,d) is path-connected. Suppose that (X,d) is not connected. Then there exist nonempty open sets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \varnothing$ and $U_1 \cup U_2 = X$. Let $x_1 \in U_1$ and $x_2 \in U_2$. Then there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x_1$ and $f(1) = x_2$. By Theorem 2.19 the sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open in $[0,1]$, and since they are disjoint and $f^{-1}(U_1) \cup f^{-1}(U_2) = [0,1]$, it follows that $[0,1]$ is not connected—a contradiction. Therefore (X,d) is connected. ■

Given $z, w \in \mathbb{C}$, the **closed line segment** (or simply **line segment**) joining z and w is the set of complex numbers

$$[z,w] = \big\{(1-t)z + tw : t \in [0,1]\big\}.$$

Clearly $[z,w] = [w,z]$, although in Chapter 3 we will take $[z,w]$ and $[w,z]$ to represent two different "orientations" of the line segment.

Given an ordered set of complex numbers $(z_0, \ldots, z_n)$, the **polygonal curve** with vertices $(z_0, \ldots, z_n)$ is the set

$$[z_0, \ldots, z_n] = \bigcup_{k=1}^{n} [z_{k-1}, z_k].$$

A line segment is the simplest nonconstant polygonal curve possible.

Definition 2.30. *A set $S \subseteq \mathbb{C}$ is* ***polygonally connected*** *if each pair of points in S is joined by a polygonal curve that lies in S.*

Polygonal connectedness is a special kind of path-connectedness since, as we will see presently, any polygonal curve P joining points $z, w \in \mathbb{C}$ has corresponding to it a continuous function $f : [0,1] \to \mathbb{C}$ such that $f(0) = z$, $f(1) = w$, and $f([0,1]) = P$. Thus, in general, polygonal connectedness implies path-connectedness, and path-connectedness implies connectedness.

A set $S \subseteq \mathbb{C}$ is **convex** if $[z,w] \subseteq S$ for all $z, w \in S$. This immediately implies that a convex set is polygonally connected, and hence path-connected, and therefore is connected by Proposition 2.29. Any open or closed ball in $\mathbb{C}$ is convex and hence connected.

A set $S \subseteq \mathbb{C}$ is **starlike** if there exists some $c \in S$ such that $[c,z] \subseteq S$ for all $z \in S$. The point c is called the **star center**. Clearly any convex set is also starlike, with any point in the set qualifying as a star center.

Proposition 2.31. *If $S \subseteq \mathbb{C}$ is starlike, then S is connected.*

Proof. Suppose $S \subseteq \mathbb{C}$ is starlike, and let $c \in S$ be the star center. Let $z, w \in S$. Then $[c, z], [c, w] \subseteq S$, and hence the polygonal curve $[z, c, w]$ lies in S. Define $f : [0, 1] \to S$ by

$$f(t) = \begin{cases} (1-2t)z + 2tc, & t \in [0, 1/2] \\ (2-2t)c + (2t-1)w, & t \in [1/2, 1]. \end{cases}$$

Then f is a continuous function with $f(0) = z$ and $f(1) = w$. Since $z, w \in S$ are arbitrary, it follows that S is path-connected, and therefore S is connected by Proposition 2.29. ■

The function f defined in the proof above gives an indication of how any polygonal curve joining points $z, w \in \mathbb{C}$ may be cast as the range of a continuous function $[0, 1] \to \mathbb{C}$. Given a polygonal curve $P = [z_0, z_1, z_2, z_3]$ with four vertices, we may define $f : [0, 1] \to P$ by

$$f(t) = \begin{cases} (1-3t)z_0 + 3tz_1, & t \in [0, 1/3] \\ (2-3t)z_1 + (3t-1)z_2, & t \in [1/3, 2/3] \\ (3-3t)z_2 + (3t-2)z_3, & t \in [2/3, 1], \end{cases}$$

which is a continuous function with $f(0) = z_0$, $f(1) = z_3$, and $f([0, 1]) = P$.

Theorem 2.32. *Let $\Omega \subseteq \mathbb{C}$ be open. Then Ω is connected if and only if it is polygonally connected.*

Proof. Suppose Ω is connected. Fix $a \in \Omega$, define Ω_1 to be the set of all $z \in \Omega$ for which there exists a polygonal curve joining a and z, and define $\Omega_2 = \Omega \setminus \Omega_1$. Fix $\zeta \in \Omega_1$, so there exist points $z_1, \ldots, z_n \in \Omega$ such that the polygonal path $[a, z_1, \ldots, z_n, \zeta]$ lies in Ω. Let $r > 0$ be such that $B_r(\zeta) \subseteq \Omega$. Since $B_r(\zeta)$ is convex, $[\zeta, z] \subseteq B_r(\zeta)$ for any $z \in B_r(\zeta)$, and then $[a, z_1, \ldots, z_n, \zeta, z]$ is a polygonal path joining a and z that lies in Ω. Hence $z \in \Omega_1$, whence $B_r(\zeta) \subseteq \Omega_1$ and we conclude that Ω_1 is an open set.

Now let $\zeta \in \Omega_2$, so there does not exist a polygonal curve joining a and ζ. Again let $r > 0$ be such that $B_r(\zeta) \subseteq \Omega$. Fix $z \in B_r(\zeta)$. If there were a polygonal path $P \subseteq \Omega$ joining a with z, then $P \cup [z, \zeta]$ would be a polygonal path in Ω joining a and ζ, which is impossible. Thus $z \in \Omega_2$, whence $B_r(\zeta) \subseteq \Omega_2$ and we conclude that Ω_2 is open.

We now find that Ω_1 and Ω_2 are disjoint open sets with union Ω. The set Ω_1 cannot be empty, since there exists $r > 0$ such that $B_r(a) \subseteq \Omega$, and then $B_r(a) \subseteq \Omega_1$ since $B_r(a)$ is polygonally connected. Therefore, because Ω is connected by hypothesis, we must conclude that $\Omega_2 = \varnothing$, and so $\Omega_1 = \Omega$. This shows that there is a polygonal curve joining a and z for any $z \in \Omega$, and since $a \in \Omega$ is arbitrary, it follows that Ω is polygonally connected.

For the converse, suppose Ω is not connected. Then Ω is not path-connected by Proposition 2.29, and there exist $z_0, z_1 \in \Omega$ such that there is no continuous function $f : [0, 1] \to \Omega$ for which $f(0) = z_0$ and $f(1) = z_1$. This implies there can be no polygonal curve joining z_0 and z_1, since any such curve could readily be characterized as the range of a piecewise-linear continuous function $[0, 1] \to \Omega$. Therefore Ω is not polygonally connected, and we have shown that polygonal connectedness implies connectedness. ■

Definition 2.33. *A **component** of a metric space (X, d) is a nonempty connected set $C \subseteq X$ that is not a proper subset of any other connected set in X. A component of a set $S \subseteq X$ is a component of the metric space (S, d).*

It is in the sense of Definition 2.33 that a component of (X, d) is described as being a "maximal connected subset of X." We may equivalently define a component of a set $S \subseteq X$ to be a component of the *subspace* S.

Theorem 2.34. *Let C be a component of (X, d). If $S \subseteq X$ is connected and $S \cap C \neq \varnothing$, then $S \subseteq C$.*

Proof. Suppose $S \subseteq X$ is connected and $S \cap C \neq \varnothing$. Then $\mathcal{F} = \{C, S\}$ is a family of connected subsets of X with a point in common, so $C \cup S$ is connected by Theorem 2.27(2). Now, since $C \subseteq C \cup S$ and C is a maximal connected subset of X, we have $C = C \cup S$, and therefore $S \subseteq C$. ■

Proposition 2.35. *If $\Omega \subseteq \mathbb{C}$ is open, then the components of Ω are open in $\mathbb{C}$.*

Proof. Suppose $\Omega \subseteq \mathbb{C}$ is open. Let C be a component of Ω, and let $z_0 \in C$. Let $r > 0$ be such that $B_r(z_0) \subseteq \Omega$. Since $B_r(z_0)$ is connected and $B_r(z_0) \cap C \neq \varnothing$, by Theorem 2.34 it follows that $B_r(z_0) \subseteq C$. Therefore C is open in $\mathbb{C}$. ■

Two sets $A, B \subseteq X$ are **separated** if $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. If a metric space (X, d) is the union of two nonempty separated sets A and B, then we say that A and B **separate** X. Sets A and B are **separated by neighborhoods** if there are open sets $U, V \subseteq X$ with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.

Exercise 2.36. Show that a metric space (X, d) can be separated if and only if it can be disconnected.

Solution. Suppose X can be separated. Then there exist nonempty sets $A, B \subseteq X$ which separate X, which is to say $A \cap \overline{B} = \overline{A} \cap B = \varnothing$ and $X = A \cup B$. Now, $A \cap \overline{B} = \varnothing$ implies $A \subseteq \overline{B}^c$. On the other hand

$$x \in \overline{B}^c \;\Rightarrow\; x \notin \overline{B} \;\Rightarrow\; x \notin B \;\Rightarrow\; x \in A,$$

since $X = A \cup B$, so $\overline{B}^c \subseteq A$ and we have $A = \overline{B}^c$. A similar argument will show that $B = \overline{A}^c$, and thus A and B are open sets in X. Therefore A and B disconnect X.

Now suppose X can be disconnected. Then there exist nonempty sets $U, V \subseteq X$ which disconnect X, which is to say U and V are open, $U \cap V = \varnothing$, and $X = U \cup V$. Let $x \in V$. Then $x \notin U$ is immediate, and since V is open there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq V$. Thus $U \cap B_\epsilon(x) = \varnothing$, which shows that x is not a limit point of x and hence $x \notin \overline{U}$. So $\overline{U} \cap V = \varnothing$, and a similar argument will show that $U \cap \overline{V} = \varnothing$ as well. It is now clear that U and V are separated sets, and therefore U and V separate X. ■

One consequence of the exercise above is that if A and B are separated sets in a metric space (X, d), then $S = A \cup B$ is a disconnected set in (X, d). The converse of this statement is not true in general, however, since to say $S = A \cup B$ is disconnected does not necessarily imply that A and B disconnect S. Consider the sets $A = (0, 1) \cup (2, 4)$ and $B = (3, 7)$ in $\mathbb{R}$.

2.5 – Compactness

A **cover** of a set S in a metric space (X, d) is a collection $\mathcal{A}$ of subsets of X such that $S \subseteq \cup\mathcal{A}$. If all the members of $\mathcal{A}$ are open sets, then $\mathcal{A}$ is an **open cover** of S. A **subcover** of $\mathcal{A}$ is a subcollection of $\mathcal{A}$ that is also a cover of S. That is, a collection of sets $\mathcal{B}$ is a subcover of $\mathcal{A}$ if and only if $\mathcal{B} \subseteq \mathcal{A}$ and $S \subseteq \cup\mathcal{B}$. If a subcover has a finite number of members belonging to it, then it is called a **finite subcover**.

Definition 2.37. *Let (X, d) be a metric space. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover. A set $S \subseteq X$ is **precompact** if $\overline{S}$ is compact.*

It is immediate from the definition that every finite set in (X, d) is compact, and so it is only interesting to develop properties of infinite compact sets. Also, in the definition for precompactness it is understood that $\overline{S}$ must be a subset of X, since that is a condition in the definition for compactness.

Proposition 2.38. *If (X, d) is a metric space and K is a compact subset of X, then K is closed.*

Theorem 2.39. *Let (X, d) be a metric space, and let $K \subseteq X$ be an infinite set. The following statements are equivalent.*

1. *K is compact.*
2. *Every infinite subset of K has a limit point in K.*
3. *Every sequence in K has a subsequence that is convergent in K.*

Proof.
(2) → (3). Suppose that every infinite subset of K has a limit point in K. Let (x_n) be a sequence in K. If $S = \{x_n : n \in \mathbb{N}\}$ is finite, then there exists a subsequence of (x_n) that is constant and therefore trivially converges to a point in K. Assume S is infinite. Then S has a limit point $x \in K$, so that for each $k \in \mathbb{N}$ there exists some $x_{n_k} \in S$ such that $x_{n_k} \in B'_{1/k}(x)$. This allows for the creation of a subsequence (x_{n_k}) for which

$$|x_{n_k} - x| < \frac{1}{k}$$

for all k (we need only take care that $n_i < n_j$ whenever $i < j$). By construction it is clear that $x_{n_k} \to z$ as $k \to \infty$, and therefore (x_n) is seen to have a subsequence that converges to a point in K. ■

It is known from elementary analysis that a set of real numbers is compact if and only if it is closed and bounded. The same is true for sets of complex numbers.

Theorem 2.40. *A set of complex numbers is compact if and only if it is closed and bounded.*

Proof. Suppose that $K \subseteq \mathbb{C}$ is compact. If K is finite then it follows trivially that K is closed and bounded, and so assume that K is infinite.

Suppose K is unbounded. Then for any fixed $x_0 \in K$ the collection $\mathcal{A} = \{B_n(x_0) : n \in \mathbb{N}\}$ is an open cover of K with no finite subcover, implying that K is not compact. Therefore K compact implies K is bounded.

Suppose K is not closed. Then $K \neq \overline{K}$, and there exists some $x_0 \notin K$ such that x_0 is a limit point of K. For each $n \in \mathbb{N}$ let $B_n = B_{1/n}(x_0)$ and

$$A_n = B_n \setminus \overline{B}_{n+2} = \left\{ x : \frac{1}{n+2} < |x - x_0| < \frac{1}{n} \right\}.$$

Setting $A_0 = \mathbb{C} \setminus \overline{B}_2$, define $\mathcal{A} = \{A_n : n \geq 0\}$. If $x \in K$ is such that $x \notin A_0$, then $0 < |x - x_0| \leq 1/2$ and there must be some $m \geq 1$ for which

$$\frac{1}{m+2} < \frac{1}{m+1} \leq |x - x_0| < \frac{1}{m},$$

and hence $x \in A_m$. This shows that $\mathcal{A}$ is an open cover for K. However, given any finite subcollection $\mathcal{B} = \{A_{n_1}, \ldots, A_{n_k}\}$ we can find some $x \in K \cap B_{n_k+2}(x_0)$, and since $x \notin \cup\mathcal{B}$ we must conclude that there is no finite subcover and hence K is not compact. Therefore K compact implies K is closed. ■

Theorem 2.41. *Let (X, d) and (Y, ρ) be metric spaces. If $K \subseteq X$ compact and $f : K \to Y$ continuous, then $f(X)$ is compact.*

Theorem 2.42 (Extreme Value Theorem). *Let (X, d) be a metric space. If $K \subseteq X$ is compact and $f : K \to \mathbb{R}$ is continuous, then f attains both a maximum and a minimum on K.*

Proof. By Theorem 2.41 the set $f(K) \subseteq \mathbb{R}$ is compact, and thus it is closed and bounded. Since $f(K)$ is a set in $\mathbb{R}$ with an upper bound, by the Completeness Axiom it must have a least upper bound and so $\sup[f(K)] = a$ for some $a \in \mathbb{R}$.

For any $\epsilon > 0$ there is some $x \in f(K)$ with $a - \epsilon < x < a$, which is to say $B'_\epsilon(a) \cap f(K) \neq \varnothing$ and hence a is a limit point of $f(K)$. Since $f(K)$ is closed we conclude that $a \in f(K)$. That is, there is some $z \in S$ such that $f(z) = a$.

Now, for any $w \in S$ we have $f(w) \in f(K)$, and so $f(w) \leq a = f(z)$. Therefore f attains a maximum on K. A similar argument will show that f also attains a minimum on K. ■

Proposition 2.43. *If $S \subseteq \mathbb{C}$ is closed and $v \in \mathbb{C}$, then there exists some $w \in S$ such that* $\operatorname{dist}(S, v) = |w - v|$.

Proof. Suppose that $S \subseteq \mathbb{C}$ is closed and $v \in \mathbb{C}$. Let $r = 2\operatorname{dist}(S, v)$, $E = \overline{B}_r(v)$, and $S' = S \cap E$. Then S' is a (nonempty) closed and bounded set, and therefore is compact.

Define the function $f : S' \to \mathbb{R}$ by $f(z) = -|z - v|$. Since f is continuous on S' and S' is compact, by the Extreme Value Theorem f attains a maximum on S'. Thus, there is some $w \in S'$ such that $f(z) \leq f(w)$ for all $z \in S'$. That is, for all $z \in S'$ we have $|z - v| \geq |w - v|$, and so

$$\operatorname{dist}(S', v) = \inf\{|z - v| : z \in S'\} = |w - v|$$

for $w \in S$.

It remains to show that $\operatorname{dist}(S', v) = \operatorname{dist}(S, v)$. Since $S' \subseteq S$ it is clear that $\operatorname{dist}(S', v) \geq \operatorname{dist}(S, v)$. Let $0 < \epsilon < r/2$. Then there exists some $z_0 \in S$ such that

$$|z_0 - v| < \operatorname{dist}(S, v) + \epsilon,$$

and since

$$\operatorname{dist}(S, v) + \epsilon = \frac{r}{2} + \epsilon < \frac{r}{2} + \frac{r}{2} = r$$

it follows that $z_0 \in E$ and hence $z_0 \in S'$. Now,

$$\operatorname{dist}(S', v) = \inf\{|z - v| : z \in S'\} \leq |z_0 - v| < \operatorname{dist}(S, v) + \epsilon,$$

and since ϵ is arbitrarily small it must be that $\operatorname{dist}(S', v) \leq \operatorname{dist}(S, v)$, and so $\operatorname{dist}(S', v) = \operatorname{dist}(S, v)$.

Therefore $\operatorname{dist}(S, v) = |w - v|$ for $w \in S$. ■

Lemma 2.44. *For any $S, \Sigma \subseteq \mathbb{C}$, the function $f : \Sigma \to \mathbb{R}$ given by $f(z) = \operatorname{dist}(S, z)$ for each $z \in \Sigma$ is continuous.*

Proof. Fix $z_0 \in \Sigma$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon/4$, and suppose $z \in \Sigma$ is such that $|z - z_0| < \delta$. We have

$$\operatorname{dist}(S, z_0) = \inf\{|z_0 - s| : s \in S\} \in \mathbb{R}$$

since 0 is a lower bound for $\{|z_0 - s| : s \in S\}$, and so there exists some $s_0 \in S$ such that

$$\operatorname{dist}(S, z_0) \leq |z_0 - s_0| < \operatorname{dist}(S, z_0) + \epsilon/4.$$

Now,

$$|z - s_0| \leq |z - z_0| + |z_0 - s_0| < \epsilon/4 + [\operatorname{dist}(S, z_0) + \epsilon/4] = \operatorname{dist}(S, z_0) + \epsilon/2,$$

and so

$$\operatorname{dist}(S, z) = \inf\{|z - s| : s \in S\} \leq |z - s_0| \leq \operatorname{dist}(S, z_0) + \epsilon/2. \tag{2.11}$$

Next, for any fixed $s \in S$ we have

$$\operatorname{dist}(S, z_0) \leq |z_0 - s| \leq |z_0 - z| + |z - s| < |z - s| + \epsilon/4,$$

which implies that

$$\operatorname{dist}(S, z_0) - \epsilon/4 < |z - s|$$

for all $s \in S$ and so $\operatorname{dist}(S, z_0) - \epsilon/4$ is a lower bound for $\{|z - s| : s \in S\}$. Since $\operatorname{dist}(S, z)$ is the greatest lower bound for $\{|z - s| : s \in S\}$, we conclude that

$$\operatorname{dist}(S, z_0) - \epsilon/4 \leq \operatorname{dist}(S, z). \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$-\epsilon/4 \leq \operatorname{dist}(S, z) - \operatorname{dist}(S, z_0) \leq \epsilon/2,$$

whence

$$|f(z) - f(z_0)| = |\operatorname{dist}(S, z) - \operatorname{dist}(S, z_0)| \leq \epsilon/2 < \epsilon.$$

Therefore f is continuous at z_0. Since $z_0 \in \Sigma$ is arbitrary it follows that f is continuous on its domain Σ. ■

Theorem 2.45. *If $K \subseteq \mathbb{C}$ is compact and $S \subseteq \mathbb{C}$ is closed, then there exist $z_0 \in K$ and $w_0 \in S$ such that*
$$\operatorname{dist}(K, S) = |z_0 - w_0|.$$

Proof. Suppose that K is compact and S is closed. Define $f : K \to \mathbb{R}$ by $f(z) = \operatorname{dist}(S, z)$ for each $z \in K$. By Lemma 2.44 f is continuous, and so the function $-f : K \to \mathbb{R}$ is likewise continuous. By the Extreme Value Theorem $-f$ attains a maximum on K, which is to say there exists some $z_0 \in K$ such that

$$-\operatorname{dist}(S, z_0) = -f(z_0) \geq -f(z) = \operatorname{dist}(S, z)$$

for all $z \in K$, and so $\operatorname{dist}(S, z_0) \leq \operatorname{dist}(S, z)$ for all $z \in K$. By Theorem 2.43 there exists some $w_0 \in S$ such that $\operatorname{dist}(S, z_0) = |z_0 - w_0|$, and so

$$|z_0 - w_0| \leq \operatorname{dist}(S, z)$$

for all $z \in K$.

Define
$$A = \{|z - w| : z \in K, w \in S\}.$$

For any $z \in K$ and $w \in S$ we have

$$|z_0 - w_0| \leq \operatorname{dist}(S, z) = \inf\{|z - \hat{w}| : \hat{w} \in S\} \leq |z - w|,$$

and so $|z_0 - w_0|$ is a lower bound for A. Moreover $|z_0 - w_0| \in A$ implies that any lower bound β for A must be such that $\beta \leq |z_0 - w_0|$, and therefore $|z_0 - w_0|$ must be the greatest lower bound for A. That is,
$$|z_0 - w_0| = \inf(A) = \operatorname{dist}(K, S),$$
and the proof is done. ∎

Proposition 2.46. *Let (X, d) be a metric space, and let $S \subseteq X$. Then S is precompact if and only if every sequence in S has a subsequence that is convergent in X.*

Proof. Suppose that S is precompact. Let (x_n) be a sequence in S. Then (x_n) is a sequence in the compact set $\overline{S}$, and hence by Theorem 2.39 has a subsequence (x_{n_k}) that converges to some $x \in \overline{S}$. But $\overline{S} \subseteq X$, so (x_n) has a subsequence that is convergent in X.

For the converse, suppose that every sequence in S has a subsequence that is convergent in X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{S}$, and let $I = \{n \in \mathbb{N} : x_n \in S\}$. If I is an infinite set, then $(x_n)_{n\in I}$ is a subsequence that lies in S, and it in turn must have a subsequence $(x_n)_{n\in J}$, where $J \subseteq I$, that converges to some $x \in X$. In fact, since $(x_n)_{n\in J}$ lies in S it follows that x must be a limit point for S, and so $x \in \overline{S}$. Therefore $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to some $x \in \overline{S}$ if I is infinite.

Suppose that I is finite. Then all but finitely many terms in $(x_n)_{n\in\mathbb{N}}$ lie in $\overline{S} \setminus S$, the set of limit points of S that do not lie in S. By passing to a subsequence if necessary, we can assume that all terms lie in $\overline{S} \setminus S$. Since each x_n is a limit point for S, for each $n \in \mathbb{N}$ there exists some $s_n \in S$ such that $s_n \in B'_{1/n}(x_n)$. Now, $(s_n)_{n\in\mathbb{N}}$ is a sequence in S such that

$$d(x_n, s_n) < \frac{1}{n}$$

for each n, and it has a subsequence $(s_{n_j})_{j\in\mathbb{N}}$ that converges to some point $x \in X$ that must necessarily be a limit point for S, and hence $x \in \overline{S}$. Consider the sequence $(x_{n_j})_{j\in\mathbb{N}}$. Let $\epsilon > 0$ be arbitrary. Let $k \in \mathbb{N}$ be such that $1/n_k < \epsilon/2$ and

$$d(s_{n_j}, x) < \frac{\epsilon}{2}$$

for all $j \geq k$. Suppose that $j \geq k$. Then $n_j \geq n_k$ so that $1/n_j < \epsilon/2$, and we obtain

$$d(x_{n_j}, x) \leq d(x_{n_j}, s_{n_j}) + d(s_{n_j}, x) < \frac{1}{n_j} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to some $x \in \overline{S}$ if I is finite. Since every sequence in $\overline{S}$ converges to a point in $\overline{S}$, by Theorem 2.39 we conclude that $\overline{S}$ is compact, and therefore S is precompact. ■

Exercise 2.47. Show that if $\Omega \subseteq \mathbb{C}$ is open and $\overline{B}_r(z_0) \subseteq \Omega$, then there exists some $\delta > 0$ such that $B_{r+\delta}(z_0) \subseteq \Omega$.

Solution. Let $K = \overline{B}_r(z_0)$ and $S = \mathbb{C} \setminus \Omega$, so K is compact and S is closed. By Theorem 2.45 there exists $z_1 \in K$ and $w_1 \in S$ such that $\operatorname{dist}(K, S) = |z_1 - w_1|$. However, since $K \subseteq \Omega$ implies $K \cap S = \varnothing$, it follows that $z_1 \neq w_1$, and so $\operatorname{dist}(K, S) = 2\delta$ for some $\delta > 0$.

Let $z \in B_{r+\delta}(z_0) \setminus K$, so $z = z_0 + (r + \epsilon)e^{it}$ for some $t \in [0, 2\pi)$ and $0 < \epsilon < \delta$. Let $w = z_0 + re^{it}$. Then

$$|z - w| = \left|\epsilon e^{it}\right| = \epsilon < \delta < \operatorname{dist}(K, S).$$

Since $w \in K$ and $|z - w| < \operatorname{dist}(K, S)$, we conclude that $z \notin S$ and therefore $z \in \Omega$. Now we see that $B_{r+\delta}(z_0) \setminus K \subseteq \Omega$ as well as $K \subseteq \Omega$, and so $B_{r+\delta}(z_0) \subseteq \Omega$ obtains as desired. ■

2.6 – Sequences and Series of Functions

Definition 2.48. *Let $S \subseteq \mathbb{C}$, and suppose (f_n) is a sequence of functions $S \to \mathbb{C}$ such that, for each $z \in S$, the sequence $(f_n(z))$ is convergent. Defining $f : S \to \mathbb{C}$ by*

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for each $z \in S$, we say (f_n) ***converges pointwise*** *to f on S, and write $f_n \to f$ or $\lim f_n = f$.*

Definition 2.49. *Let $S \subseteq \mathbb{C}$. A sequence of functions $(f_n : S \to \mathbb{C})$ is* ***uniformly convergent*** *(or* ***converges uniformly****) on S if there exists a function $f : S \to \mathbb{C}$ for which the following holds: for every $\epsilon > 0$, there exists some $k \in \mathbb{Z}$ such that*

$$|f_n(z) - f(z)| < \epsilon$$

for all $z \in S$ and $n \geq k$. We then say (f_n) ***converges uniformly*** *to f on S, and write $f_n \xrightarrow{\mathrm{u}} f$ or $\text{u-lim}\, f_n = f$. We say (f_n) is* ***uniformly convergent*** *on $A \subseteq S$ if $(f_n|_A)$ is uniformly convergent on A.*

Given a sequence of functions $(f_n : S \to \mathbb{C})_{n\in\mathbb{N}}$, define $s_n = \sum_{k=1}^{n} f_k$ for each $n \in \mathbb{N}$. To say $\sum f_n$ **converges pointwise** to a function f on S means the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ converges pointwise to f on S, and to say $\sum f_n$ is **uniformly convergent** on S means $(s_n)_{n\in\mathbb{N}}$ is uniformly convergent on S. Finally, we say $\sum f_n$ is **absolutely convergent** on S if $\sum |f_n|$ converges pointwise on S.

Recall (2.2), the definition of the uniform metric $\|\cdot\|_S$ on $\mathcal{B}(S)$ that specifies a real-valued distance between any two bounded functions $f, g : S \to \mathbb{C}$. We may certainly apply $\|\cdot\|_S$ to the set $\mathcal{F}(S)$ of *all* functions $S \to \mathbb{C}$, bounded or otherwise, though $\|f - g\|_S$ will not necessarily always be defined as a real number (in which case $(\mathcal{F}(S), \|\cdot\|_S)$ fails to be a metric space). It should be clear that, given a sequence (f_n) in $\mathcal{F}(S)$, then (f_n) converges uniformly to f on S iff for every $\epsilon > 0$ there is some $k \in \mathbb{Z}$ such that $\|f_n - f\|_S < \epsilon$ for all $n \geq k$. Thus, given a sequence (f_n) in the metric space $(\mathcal{B}(S), \|\cdot\|)$, we conclude that (f_n) is convergent in $\mathcal{B}(S)$ iff there exists some $f \in \mathcal{B}(S)$ such that (f_n) is uniformly convergent to f on S.

Definition 2.50. *Let $S \subseteq \mathbb{C}$. A sequence of functions $(f_n : S \to \mathbb{C})$ is* ***uniformly Cauchy*** *on S if, for every $\epsilon > 0$, there exists some $k \in \mathbb{Z}$ such that*

$$|f_m(z) - f_n(z)| < \epsilon$$

for all $z \in S$ and $m, n \geq k$. We say (f_n) is ***uniformly Cauchy*** *on $A \subseteq S$ if $(f_n|_A)$ is uniformly Cauchy on A.*

It can be easily shown that a sequence $(f_n : S \to \mathbb{C})$ is uniformly Cauchy on S iff for every $\epsilon > 0$ there is some $k \in \mathbb{Z}$ such that $\|f_m - f_n\|_S < \epsilon$ for all $m, n \geq k$.

Proposition 2.51. *Suppose (f_n) and (g_n) are sequences of functions that converge uniformly to f and g on $S \subseteq \mathbb{C}$, respectively.*

1. *For any $\alpha \in \mathbb{C}$, (αf_n) converges uniformly to αf on S.*
2. *The sequence $(f_n + g_n)$ converges uniformly to $f + g$ on S.*

3. *If (f_n) and (g_n) are sequences of bounded functions on S, then $(f_n g_n)$ converges uniformly to fg on S.*
4. *If $f = u + iv$ and $f_n = u_n + iv_n$ for all n, then $u_n \xrightarrow{\text{u}} u$ and $v_n \xrightarrow{\text{u}} v$.*

Theorem 2.52. *Let (f_n) be a sequence of functions $S \to \mathbb{C}$. If (f_n) is uniformly Cauchy on S, then it is uniformly convergent on S.*

Proof. Suppose that (f_n) is uniformly Cauchy on S. For any $z \in S$ the following holds: for each $\epsilon > 0$ there exists some k such that $|f_m(z) - f_n(z)| < \epsilon$ for all $m, n \geq k$. Thus $(f_n(z))$ is a Cauchy sequence in $\mathbb{C}$, and since $(\mathbb{C}, |\cdot|)$ is a complete metric space by Theorem 2.3, we conclude that there exists some $w_z \in \mathbb{C}$ such that $f_n(z) \to w_z$. Define $f : S \to \mathbb{C}$ by $f(z) = w_z$; that is,

$$f(z) = \lim_{n\to\infty} f_n(z) \tag{2.13}$$

for all $z \in S$. We will show that (f_n) converges uniformly to f on S.

Fix $\epsilon > 0$ Choose k such that

$$|f_m(z) - f_n(z)| < \frac{\epsilon}{2}$$

for all $z \in S$ and $m, n \geq k$. Fix $z \in S$ and $n \geq k$. Since $f_m(z) \to f(z)$, there exists some $m \geq k$ such that $|f_m(z) - f(z)| < \epsilon/2$. Now,

$$|f_n(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and therefore $f_n \xrightarrow{\text{u}} f$ on S. ■

Remark. In the proof of Theorem 2.52 it can be seen that, if (f_n) is uniformly Cauchy on S, then it specifically converges uniformly on S to the function $f : S \to \mathbb{C}$ given by (2.13), which is the pointwise limit of (f_n).

Theorem 2.53 (Weierstrass M-Test). *Let (M_n) be a sequence in $[0, \infty)$ such that $\sum M_n$ converges. If (f_n) is a sequence of functions on S such that $\|f_n\|_S \leq M_n$ for all n, then $\sum f_n$ is uniformly and absolutely convergent on S.*

Theorem 2.54. *Let (f_n) be a sequence of continuous functions $S \to \mathbb{C}$. If (f_n) converges uniformly to $f : S \to \mathbb{C}$, then f is continuous on S.*

Theorem 2.55. *Suppose (f_n) converges uniformly on $S \subseteq \mathbb{C}$ and z_0 is a limit point of S. If $\lim_{z\to z_0} f_n(z)$ exists in $\mathbb{C}$ for each n, then*

$$\lim_{z\to z_0} \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \lim_{z\to z_0} f_n(z).$$

Exercise 2.56. Consider the series

$$\sum_{k=1}^{\infty} \frac{z^{k-1}}{(1 - z^k)(1 - z^{k+1})}. \tag{2.14}$$

(a) Show the series converges uniformly to $1/(1-z)^2$ on $\overline{B}_c(0)$ for any $c \in (0, 1)$.
(b) Show the series converges uniformly to $1/[z(1-z)^2)]$ on $\mathbb{C} \setminus B_d(0)$ for any $d \in (1, \infty)$.

Solution.
(a) Let $c \in (0,1)$. It must be shown that

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \forall z \in \overline{B}_c(0) \left(\left| \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{(1-z)^2} \right| < \epsilon \right).$$

Let $\epsilon > 0$ be arbitrary. Proposition 1.7 gives

$$\sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} = \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z)^2(1+z+\cdots+z^{k-1})(1+z+\cdots+z^k)},$$

and so

$$\begin{aligned} &\left| \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{(1-z)^2} \right| \\ &\qquad = \frac{1}{|1-z|^2} \left| \sum_{k=1}^{n} \frac{z^{k-1}}{(1+\cdots+z^{k-1})(1+\cdots+z^k)} - 1 \right|. \end{aligned} \tag{2.15}$$

Partial fraction decomposition of the first three terms of the summation on the right-hand side yields $1/(1+z)$,

$$\begin{aligned} \frac{z}{(1+z)(1+z+z^2)} &= -\frac{1}{1+z} + \frac{1+z}{1+z+z^2}, \\ \frac{z^2}{(1+z+z^2)(1+z+z^2+z^3)} &= -\frac{1+z}{1+z+z^2} + \frac{1+z+z^2}{1+z+z^2+z^3}. \end{aligned}$$

From this we conjecture that

$$\frac{z^{k-1}}{(1+z+\cdots+z^{k-1})(1+z+\cdots+z^k)} = -\frac{1+z+\cdots+z^{k-2}}{1+z+\cdots+z^{k-1}} + \frac{1+z+\cdots+z^{k-1}}{1+z+\cdots+z^k}. \tag{2.16}$$

Letting $w = 1+z+\cdots+z^{k-2}$, we manipulate the right-hand side of (2.16):

$$\begin{aligned} \frac{w+z^{k-1}}{w+z^{k-1}+z^k} - \frac{w}{w+z^{k-1}} &= \frac{(w+z^{k-1})^2 - w(w+z^{k-1}+z^k)}{(w+z^{k-1})(w+z^{k-1}+z^k)} \\ &= \frac{wz^{k-1} - wz^k + z^{2k-2}}{(w+z^{k-1})(w+z^{k-1}+z^k)} \\ &= \frac{(z^{k-1}+z^k+\cdots+z^{2k-3}) - (z^k+z^{k+1}+\cdots+z^{2k-2}) + z^{2k-2}}{(w+z^{k-1})(w+z^{k-1}+z^k)} \\ &= \frac{z^{k-1}}{(w+z^{k-1})(w+z^{k-1}+z^k)}. \end{aligned}$$

The last expression matches the left-hand side of (2.16), thereby confirming the conjecture and making it clear that the summation on the right-hand side of (2.15) telescopes to yield

$$\begin{aligned} \left| \sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{(1-z)^2} \right| &= \frac{1}{|1-z|^2} \left| \frac{1+z+\cdots+z^{n-1}}{1+z+\cdots+z^n} - 1 \right| \\ &= \frac{1}{|1-z|^2} \left| \frac{z^n}{1+z+\cdots+z^n} \right|. \end{aligned} \tag{2.17}$$

But,

$$\left|\frac{z^n}{1+z+\cdots+z^n}\right| = \frac{|z|^n}{|1+z+\cdots+z^n|} = \frac{|1-z||z|^n}{|1-z^{n+1}|},$$

and so from (2.17) we obtain

$$\left|\sum_{k=1}^{n}\frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{(1-z)^2}\right| = \frac{1}{|1-z|^2}\cdot\frac{|1-z||z|^n}{|1-z^{n+1}|} = \frac{|z|^n}{|1-z||1-z^{n+1}|} \tag{2.18}$$

Let

$$\alpha = \sup\left\{\frac{1}{|1-z|} : z \in \overline{B}_c(0)\right\},$$

which is a positive real number by the Extreme Value Theorem since $\overline{B}_c(0)$ is compact and $f : \overline{B}_c(0) \to \mathbb{R}$ given by $f(z) = 1/|1-z|$ is continuous. Because $0 \le c < 1$, there exists some $N \in \mathbb{N}$ such that

$$c^n < \frac{1}{\alpha/\epsilon + c}$$

for all $n > N$. Fix $n > N$ and $z \in \overline{B}_c(0)$. If $z = 0$ both the series (2.14) and $1/(1-z)^2$ equal 1 and there is nothing left to prove, so we may assume that $z \neq 0$. Since $0 < |z| \le c$ we obtain

$$|z|^n < \frac{1}{\alpha/\epsilon + |z|} \quad\Leftrightarrow\quad \frac{1}{|z|^n} > \frac{\alpha}{\epsilon} + |z| \quad\Leftrightarrow\quad \frac{1}{|z^{-n}| - |z|} < \frac{\epsilon}{\alpha} \quad\Leftrightarrow\quad \frac{1}{|z^{-n} - z|} < \frac{\epsilon}{\alpha},$$

and so

$$\frac{|z|^n}{|1-z^{n+1}|} = \left|\frac{z^n}{1-z^{n+1}}\right| = \left|\frac{1}{z^{-n}-z}\right| = \frac{1}{|z^{-n}-z|} < \frac{\epsilon}{\alpha}. \tag{2.19}$$

Now, observing that

$$\frac{1}{|1-z|} \le \alpha,$$

from (2.19) comes

$$\frac{1}{|1-z|}\cdot\frac{|z|^n}{|1-z^{n+1}|} < \epsilon,$$

and hence by (2.18) we conclude that

$$\left|\sum_{k=1}^{n}\frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{(1-z)^2}\right| < \epsilon$$

and the proof that (2.15) converges uniformly to $1/(1-z)^2$ on $\overline{B}_c(0)$ is done.

(b) Next, let $d \in (1,\infty)$. It must be shown that

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \forall z \in \mathbb{C}\setminus B_d(0)\left(\left|\sum_{k=1}^{n}\frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{z(1-z)^2}\right| < \epsilon\right).$$

Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $1/(d^{N+1}-1) < \epsilon d(d-1)$. Let $n > N$ and $z \in \mathbb{C}\setminus B_d(0)$ be arbitrary. Since $|z| \ge d > 1$,

$$\frac{1}{|z^{n+1}-1|} \le \frac{1}{|z|^{n+1}-1} \le \frac{1}{d^{n+1}-1} < \epsilon d(d-1) \le \epsilon|z|(|z|-1) \le \epsilon|z||z-1|,$$

and hence

$$\frac{1}{|z||z-1||z^{n+1}-1|} < \epsilon \tag{2.20}$$

Now, employing the same manipulations that led to (2.15), from

$$\left|\sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{z(1-z)^2}\right|$$

we obtain

$$\frac{1}{|1-z|^2}\left|\sum_{k=1}^{n} \frac{z^{k-1}}{(1+\cdots+z^{k-1})(1+\cdots+z^k)} - \frac{1}{z}\right|.$$

Again the summation telescopes, yielding

$$\frac{1}{|1-z|^2}\left|\frac{1+z+\cdots+z^{n-1}}{1+z+\cdots+z^n} - \frac{1}{z}\right|.$$

However,

$$\begin{aligned}\frac{1}{|1-z|^2}\left|\frac{1+z+\cdots+z^{n-1}}{1+z+\cdots+z^n} - \frac{1}{z}\right| &= \left|\frac{1}{z(z-1)^2(1+z+\cdots+z^n)}\right| \\ &= \left|\frac{1}{z(z-1)(z^{n+1}-1)}\right| = \frac{1}{|z||z-1||z^{n+1}-1|},\end{aligned}$$

and so recalling (2.20) we conclude that

$$\left|\sum_{k=1}^{n} \frac{z^{k-1}}{(1-z^k)(1-z^{k+1})} - \frac{1}{z(1-z)^2}\right| < \epsilon$$

and therefore (2.15) converges uniformly to $1/[z(1-z)^2)]$ on $\mathbb{C} \setminus B_d(0)$. ■

3

Differentiation and Integration

3.1 – Complex Differentiation

First we define the **derivative** of a complex-valued function of a single *real* variable, $\varphi : I \subseteq \mathbb{R} \to \mathbb{C}$, where I is an interval, to be given by

$$\varphi'(t_0) = \lim_{t \to t_0} \frac{\varphi(t) - \varphi(t_0)}{t - t_0}$$

for each $t_0 \in I$, provided the limit exists in $\mathbb{C}$. If $u, v : I \to \mathbb{R}$ are such that $\varphi(t) = u(t) + iv(t)$ for all $t \in I$, then it is easy to show that φ is differentiable at t if and only if both u and v are differentiable at t, in which case

$$\varphi'(t) = u'(t) + iv'(t). \tag{3.1}$$

Thus if u and v are continuously differentiable on I, then so too is φ.

Example 3.1. For $t \in \mathbb{R}_+$ and $z = x + iy \in \mathbb{C}$ we define

$$t^z = e^{z \ln t},$$

which can be seen to agree with the definition of t^s for $s \in \mathbb{R}$ in calculus. (A more general definition of w^z for w any nonzero complex number is given in §6.1.) Letting $\varphi : \mathbb{R}_+ \to \mathbb{C}$ be given by $\varphi(t) = t^z$, we find $\varphi'(t)$. With Definition 1.18, Proposition 1.19, and differentiation rules from calculus, we have

$$\begin{aligned}
\varphi'(t) &= \frac{d}{dt}\big(e^{z \ln t}\big) = \frac{d}{dt}\big[e^{x \ln t}\cos(y \ln t) + ie^{x \ln t}\sin(y \ln t)\big] \\
&= \frac{d}{dt}\big[t^x \cos(y \ln t)\big] + i\frac{d}{dt}\big[t^x \sin(y \ln t)\big] \\
&= xt^{x-1}\big[\cos(y \ln t) + i\sin(y \ln t)\big] + iyt^{x-1}\big[\cos(y \ln t) + i\sin(y \ln t)\big] \\
&= xt^{x-1}e^{iy \ln t} + iyt^{x-1}e^{iy \ln t} = ze^{(x-1)\ln t}e^{iy \ln t} = ze^{(x-1)\ln t + iy \ln t} \\
&= ze^{(z-1)\ln t} = zt^{z-1}.
\end{aligned}$$

That is,

$$\frac{d}{dt}(t^z) = zt^{z-1},$$

which is consonant with the power rule of differentiation from calculus. ■

Now we consider complex-valued functions of a single *complex* variable, our chief focus. Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then f is **complex-differentiable** (or **holomorphic**) at $z_0 \in \Omega$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$. If f is complex-differentiable at every point in Ω, then f is said to be **analytic on** Ω. Given an arbitrary set $S \subseteq \mathbb{C}$, we say f is **analytic on** S if f is analytic on some open $\Omega \supseteq S$. Thus f is analytic at a point z_0 if and only if it is analytic on some open neighborhood of z_0. Finally, a function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on $\mathbb{C}$ is also known as an **entire function**.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{C}$ be open. If $f : \Omega \to \mathbb{C}$ is complex-differentiable at $z_0 \in \Omega$, then f is continuous at z_0.*

Theorem 3.3 (**Rules of Differentiation**)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f, g : \Omega \to \mathbb{C}$ be complex-differentiable at $z_0 \in \Omega$.*

1. *Constant Multiple Rule: For any $\alpha \in \mathbb{C}$, αf is complex-differentiable at z_0 with $(\alpha f)'(z_0) = \alpha f'(z_0)$.*
2. *Sum Rule: $f + g$ is complex-differentiable at z_0, with*
$$(f + g)'(z_0) = f'(z_0) + g'(z_0).$$
3. *Product Rule: fg is complex-differentiable at z_0, with*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$
4. *Quotient Rule: If $g(z_0) \neq 0$, then f/g is complex-differentiable at z_0 with*
$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

Remark. In the statement of the Quotient Rule note that if $g(z_0) \neq 0$, then since g is continuous at z_0 by Theorem 3.2 it follows that $g(z) \neq 0$ for all $z \in B_r(z_0)$ for some sufficiently small $r > 0$. Thus the function f/g is defined on an open set containing z_0, namely $B_r(z_0)$, as required by the definition of complex-differentiability.

Proposition 3.4. *Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is complex-differentiable at z_0 with $f'(z_0) = \lambda$ if and only if there exists some $\epsilon : \Omega \to \mathbb{C}$ that is continuous at z_0 such that $\epsilon(z_0) = 0$ and*

$$f(z) = f(z_0) + (z - z_0)[\lambda + \epsilon(z)]$$

for all $z \in \Omega$.

Proof. Suppose $f : \Omega \to \mathbb{C}$ is complex-differentiable at z_0 with $f'(z_0) = \lambda$. Define $\epsilon : \Omega \to \mathbb{C}$ by

$$\epsilon(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - \lambda, & \text{if } z \neq z_0 \\ 0, & \text{if } z = z_0 \end{cases}$$

Fix $z \in \Omega$. If $z = z_0$, then

$$\begin{aligned} f(z) = f(z_0) &= f(z_0) + 0(\lambda + 0) = f(z_0) + (z_0 - z_0)[\lambda + \epsilon(z_0)] \\ &= f(z_0) + (z - z_0)[\lambda + \epsilon(z)]; \end{aligned}$$

and if $z \neq z_0$, then

$$\begin{aligned} f(z) &= f(z_0) + [\lambda(z - z_0) + (f(z) - f(z_0)) - \lambda(z - z_0)] \\ &= f(z_0) + (z - z_0)\left(\lambda + \frac{f(z) - f(z_0)}{z - z_0} - \lambda\right) \\ &= f(z_0) + (z - z_0)[\lambda + \epsilon(z)]. \end{aligned}$$

The converse is equally straightforward to verify, but we will have no need for it in subsequent developments. ■

Theorem 3.5 (Chain Rule). *If f is analytic on Ω and g is analytic on $f(\Omega)$, then $g \circ f$ is analytic on Ω and*

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in \Omega$.

Proof. Suppose f is analytic on Ω. Suppose g is analytic on $f(\Omega)$, so that g is analytic on some open set S containing $f(\Omega)$. Fix $z_0 \in \Omega$. Then f is differentiable (and hence continuous) at z_0, and g is differentiable at $f(z_0) \in S$.

Define the function $\rho : S \to \mathbb{C}$ by

$$\rho(y) = \begin{cases} \dfrac{g(y) - g(f(z_0))}{y - f(z_0)} - g'(f(z_0)), & \text{if } y \neq f(z_0) \\ 0, & \text{if } y = f(z_0) \end{cases}$$

By the differentiability of g at $f(z_0)$,

$$\begin{aligned} \lim_{y \to f(z_0)} \rho(y) &= \lim_{y \to f(z_0)} \left[\frac{g(y) - g(f(z_0))}{y - f(z_0)} - g'(f(z_0))\right] \\ &= \lim_{y \to f(z_0)} \frac{g(y) - g(f(z_0))}{y - f(z_0)} - \lim_{y \to f(z_0)} g'(f(z_0)) \\ &= g'(f(z_0)) - g'(f(z_0)) = 0 = \rho(f(z_0)), \end{aligned}$$

which shows that ρ is continuous at $f(z_0)$.

Since $f(z) \to f(z_0)$ as $z \to z_0$, $f(z_0)$ is in the interior of $\mathrm{Dom}(\rho)$, and ρ is continuous at $f(z_0)$, by Proposition 2.22 we obtain

$$\lim_{z \to z_0} \rho(f(z)) = \rho\left(\lim_{z \to z_0} f(z)\right) = \rho(f(z_0)) = 0.$$

Now, for any $z \in \Omega$ such that $f(z) \neq f(z_0)$, we find that

$$\rho(f(z)) = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} - g'(f(z_0))$$

and hence

$$g(f(z)) - g(f(z_0)) = [g'(f(z_0)) + \rho(f(z))][f(z) - f(z_0)]. \tag{3.2}$$

Since (3.2) also holds for $z \in \Omega$ such that $f(z) = f(z_0)$, we conclude that it holds for all $z \in \Omega$ and so

$$\begin{aligned}
(g \circ f)'(z_0) &= \lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\
&= \lim_{z \to z_0} \frac{[g'(f(z_0)) + \rho(f(z))][f(z) - f(z_0)]}{z - z_0} \\
&= \lim_{z \to z_0} [g'(f(z_0)) + \rho(f(z))] \cdot \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
&= [g'(f(z_0)) + 0] \cdot f'(z_0) = g'(f(z_0)) f'(z_0),
\end{aligned}$$

which completes the proof. ■

Theorem 3.6. *Let Ω and Ω' be open sets, with $f : \Omega \to \Omega'$ continuous and $g : \Omega' \to \Omega$ analytic. Suppose $g'(z) \neq 0$ for all $z \in \Omega'$. If $g(f(z)) = z$ for all $z \in \Omega$, then f is analytic on Ω and $f' = 1/(g' \circ f)$.*

Proof. Suppose that $g(f(z)) = z$ for all $z \in \Omega$. Then

$$f(z_1) = f(z_2) \quad \Rightarrow \quad g(f(z_1)) = g(f(z_2)) \quad \Rightarrow \quad z_1 = z_2,$$

and thus f is injective. Fix $z_0 \in \Omega$, and let $\epsilon > 0$. Since g is complex-differentiable at $f(z_0) \in \Omega'$, we have

$$g'(f(z_0)) = \lim_{z \to f(z_0)} \frac{g(z) - g(f(z_0))}{z - f(z_0)} = \lim_{z \to f(z_0)} \frac{g(z) - z_0}{z - f(z_0)} \in \mathbb{C},$$

noting that $g(f(z_0)) = z_0$. Thus, since $g'(f(z_0)) \neq 0$, by Theorem 2.15(5) we obtain

$$\lim_{z \to f(z_0)} \frac{z - f(z_0)}{g(z) - z_0} = \lim_{z \to z_0} \frac{1}{\dfrac{g(z) - z_0}{z - f(z_0)}} = \frac{1}{g'(f(z_0))},$$

and so there exists some $\delta' > 0$ such that

$$0 < |z - f(z_0)| < \delta' \quad \Rightarrow \quad \left| \frac{z - f(z_0)}{g(z) - z_0} - \frac{1}{g'(f(z_0))} \right| < \epsilon.$$

Now, since f is continuous and injective on Ω, there exists some $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad 0 < |f(z) - f(z_0)| < \delta'.$$

Hence $0 < |z - z_0| < \delta$ implies that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \frac{1}{g'(f(z_0))} \right| = \left| \frac{f(z) - f(z_0)}{g(f(z)) - z_0} - \frac{1}{g'(f(z_0))} \right| < \epsilon,$$

and so

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{g'(f(z_0))}.$$

That is, $f'(z_0)$ exists, with

$$f'(z_0) = \frac{1}{(g' \circ f)(z_0)}.$$

Since $z_0 \in \Omega$ is arbitrary, we conclude that f is analytic on Ω and $f' = 1/(g' \circ f)$. ■

3.2 – The Cauchy-Riemann Equations

Recall from calculus that if $U \subseteq \mathbb{R}^2$ is open, then the partial derivatives f_x and f_y of $f : U \to \mathbb{R}$ at $(x, y) \in U$ are given by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \tag{3.3}$$

and

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}, \tag{3.4}$$

provided the limits exist. Moreover, f is said to be differentiable at (x, y) if

$$\lim_{(h,k)\to(0,0)} \frac{f(x+h,y+k) - f(x,y) - hf_x(x,y) - kf_y(x,y)}{\sqrt{h^2+k^2}} = 0.$$

Similar notions exist for a function $u : \Omega \subseteq \mathbb{C} \to \mathbb{R}$, where Ω is open. It is natural to view u as being, like f above, a real-valued function of two independent variables $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, so that it makes sense to speak of partial derivatives of u as well as the differentiability of u. At $x + iy \in \Omega$ we define the **partial derivatives** of u by

$$u_x(x+iy) = \lim_{h\to 0} \frac{u((x+h)+iy) - u(x+iy)}{h} \tag{3.5}$$

and

$$u_y(x+iy) = \lim_{h\to 0} \frac{u(x+i(y+h)) - u(x+iy)}{h}, \tag{3.6}$$

and we say u is **real-differentiable** at $x + iy$ if

$$\lim_{(h,k)\to(0,0)} \frac{u((x+h)+i(y+k)) - u(x+iy) - hu_x(x+iy) - ku_y(x+iy)}{\sqrt{h^2+k^2}} = 0.$$

These formulas are quite cumbersome, and so henceforth it will be common practice to represent $x + iy$ by (x, y).

Remark. Analogous to the elementary analysis setting, a function $u : \Omega \subseteq \mathbb{C} \to \mathbb{R}$ is real-differentiable on Ω if u_x and u_y exist and are continuous on Ω. Also see Exercise 3.8.

Theorem 3.7. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be given by*

$$f(x+iy) = u(x,y) + iv(x,y)$$

for functions $u, v : \Omega \to \mathbb{R}$. Then f is complex-differentiable at $x + iy \in \Omega$ if and only if u, v are real-differentiable at $(x, y) = x + iy$ and

$$u_x(x,y) = v_y(x,y), \quad u_y(x,y) = -v_x(x,y) \tag{3.7}$$

both hold, in which case

$$f'(x+iy) = u_x(x,y) - iu_y(x,y).$$

Proof. Suppose that f is complex-differentiable at $z = x + iy \in \Omega$. Thus

$$f'(x+iy) = f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = a + ib$$

for some $a, b \in \mathbb{R}$. Setting $h = r + is$, we obtain

$$a + ib = \lim_{r+is\to 0} \frac{f((x+r) + i(y+s)) - f(x+iy)}{r+is},$$

and thus

$$\begin{aligned} a + ib &= \lim_{r+is\to 0} \frac{\big[u(x+r,y+s) + iv(x+r,y+s)\big] - \big[u(x,y) + iv(x,y)\big]}{r+is} \\ &= \lim_{r+is\to 0} \left[\frac{u(x+r,y+s) - u(x,y)}{r+is} + i\frac{v(x+r,y+s) - v(x,y)}{r+is}\right]. \end{aligned}$$

Hence

$$\lim_{r+is\to 0} \left[\frac{u(x+r,y+s) - u(x,y)}{r+is} + i\frac{v(x+r,y+s) - v(x,y)}{r+is} - (a+ib)\right] = 0,$$

and with a little algebra it follows that

$$\lim_{r+is\to 0} \left[\frac{\big(u(x+r,y+s) - u(x,y) - ar + bs\big) + \big(v(x+r,y+s) - v(x,y) - br - as\big)i}{r+is}\right] = 0$$

From this we conclude that for any $\epsilon > 0$ there exists some $\delta > 0$ such that $0 < |r + is| < \delta$ implies

$$\left|\frac{\big(u(x+r,y+s) - u(x,y) - ar + bs\big) + \big(v(x+r,y+s) - v(x,y) - br - as\big)i}{r+is}\right| < \epsilon.$$

That is, whenever $0 < \sqrt{r^2+s^2} < \delta$ we have

$$\frac{\big|u(x+r,y+s) - u(x,y) - ar + bs\big|}{\sqrt{r^2+s^2}},\quad \frac{\big|v(x+r,y+s) - v(x,y) - br - as\big|}{\sqrt{r^2+s^2}} < \epsilon$$

This shows that

$$\lim_{(r,s)\to(0,0)} \frac{u(x+r,y+s) - u(x,y) - ar + bs}{\sqrt{r^2+s^2}} = 0 \tag{3.8}$$

and

$$\lim_{(r,s)\to(0,0)} \frac{v(x+r,y+s) - v(x,y) - br - as}{\sqrt{r^2+s^2}} = 0, \tag{3.9}$$

and so u and v are real-differentiable at (x, y).

Now, from (3.8) it follows that

$$u_x(x,y) = a \quad \text{and} \quad u_y(x,y) = -b,$$

and from (3.9) it follows that

$$v_x(x,y) = b \quad \text{and} \quad v_y(x,y) = a,$$

and so it is clear that

$$u_x(x,y) = v_y(x,y) \quad \text{and} \quad u_y(x,y) = -v_x(x,y)$$

as claimed, and also

$$f'(x+iy) = a+ib = u_x(x,y) - iu_y(x,y).$$

For the converse, suppose that u and v are real-differentiable at $(x,y) \in \Omega$ such that (3.7) hold. We have

$$\lim_{(r,s)\to(0,0)} \frac{u(x+r,y+s) - u(x,y) - u_x(x,y)r - u_y(x,y)s}{\sqrt{r^2+s^2}} = 0 \tag{3.10}$$

and

$$\lim_{(r,s)\to(0,0)} \frac{v(x+r,y+s) - v(x,y) - v_x(x,y)r - v_y(x,y)s}{\sqrt{r^2+s^2}} = 0. \tag{3.11}$$

Let $\epsilon > 0$. By (3.10) there exists some $\delta_1 > 0$ such that $0 < \sqrt{r^2+s^2} < \delta_1$ implies

$$\left| \frac{u(x+r,y+s) - u(x,y) - u_x(x,y)r - u_y(x,y)s}{\sqrt{r^2+s^2}} \right| < \frac{\epsilon}{2},$$

and by (3.11) there exists some $\delta_2 > 0$ such that $0 < \sqrt{r^2+s^2} < \delta_2$ implies

$$\left| \frac{v(x+r,y+s) - v(x,y) - v_x(x,y)r - v_y(x,y)s}{\sqrt{r^2+s^2}} \right| < \frac{\epsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, and suppose $h = r+is \in \mathbb{C}$ is such that $0 < |h| < \delta$. Then

$$0 < |h| = |r+is| = \sqrt{r^2+s^2} < \delta \leq \delta_1, \delta_2,$$

and so

$$\begin{aligned}&\frac{\left|u(x+r,y+s) - u(x,y) - u_x(x,y)r - u_y(x,y)s\right|}{|r+is|} \\ &\qquad\qquad + \frac{\left|v(x+r,y+s) - v(x,y) - v_x(x,y)r - v_y(x,y)s\right|}{|r+is|} < \epsilon.\end{aligned}$$

Suppressing (x,y), we obtain

$$\left| \frac{u(x+r,y+s) - u - u_x r - u_y s}{r+is} + i\frac{v(x+r,y+s) - v - v_x r - v_y s}{r+is} \right| < \epsilon$$

by the Triangle Inequality. Replacing $-v_x(x,y)$ and $v_y(x,y)$ with $u_y(x,y)$ and $u_x(x,y)$, respectively, then yields

$$\left| \frac{u(x+r,y+s) - u - u_x r - u_y s}{r+is} + i\frac{v(x+r,y+s) - v + u_y r - u_x s}{r+is} \right| < \epsilon,$$

and thus

$$\left| \left(\frac{u(x+r,y+s) - u}{r+is} + i\frac{v(x+r,y+s) - v}{r+is} \right) - \left(\frac{u_x r + u_y s}{r+is} + i\frac{u_x s - u_y r}{r+is} \right) \right| < \epsilon. \tag{3.12}$$

Now, since

$$\frac{u_x r + u_y s}{r+is} + i\frac{u_x s - u_y r}{r+is} = \frac{(u_x - iu_y)(r+is)}{r+is} = u_x - iu_y,$$

from (3.12) we arrive at

$$\left| \left[\frac{u(x+r,y+s)-u(x,y)}{r+is} + i\frac{v(x+r,y+s)-v(x,y)}{r+is} \right] - \big[u_x(x,y) - iu_y(x,y)\big] \right| < \epsilon,$$

or equivalently

$$\left| \frac{f((x+iy)+h)-f(x+iy)}{h} - \big[u_x(x,y) - iu_y(x,y)\big] \right| < \epsilon.$$

That is,

$$f'(x+iy) = \lim_{h\to 0} \frac{f((x+iy)+h)-f(x+iy)}{h} = u_x(x,y) - iu_y(x,y),$$

which shows that f is complex-differentiable at $x+iy \in \Omega$. ■

In general we see that if $f(x+iy) = u(x,y) + iv(x,y)$ is complex-differentiable on open set $\Omega \subseteq \mathbb{C}$, then

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

on Ω. These are known as the **Cauchy-Riemann equations**.

Exercise 3.8 (AN1.4). Let $U \subseteq \mathbb{R}^2$ be open, and let $g : U \to \mathbb{R}$ be such that g_x and g_y exist at $(x_0, y_0) \in U$. Suppose that g_x exists in an open neighborhood V of (x_0, y_0) and is continuous at (x_0, y_0). Show that g is real-differentiable at (x_0, y_0).

Solution. We can assume $V \subseteq U$. Since V is open and $(x_0, y_0) \in V$, there exists some $\epsilon > 0$ sufficiently small such that $(x_0 + h, y_0 + k) \in V$ for all $h, k \in [-\epsilon, \epsilon]$.

For each $-\epsilon \le h, k \le \epsilon$ let I_{hk} be the line segment with endpoints $(x_0, y_0 + k)$ and $(x_0 + h, y_0 + k)$, so $I_{hk} \subseteq V \subseteq U$. Also let J_{hk} be the closed interval in $\mathbb{R}$ with endpoints x_0 and $x_0 + h$. The existence of g_x on I_{hk} implies that $g(\,\cdot\,, y_0 + k) : J_{hk} \to \mathbb{R}$ is continuous on J_{hk} and differentiable (with respect to x) on $\mathrm{Int}(J_{hk})$. Hence by the Mean Value Theorem there exists some c_{hk} between x_0 and $x_0 + h$ such that

$$g_x(c_{hk}, y_0 + k) = \frac{g(x_0 + h, y_0 + k) - g(x_0, y_0 + k)}{h}.$$

Letting

$$S = \{(x_0 + h, y_0 + k) : -\epsilon \le h, k \le \epsilon\},$$

by the Axiom of Choice we can choose some c_{hk} value for each $h, k \in [-\epsilon, \epsilon]$ so as construct a function $c_{hk} : S \to \mathbb{R}$. Clearly $c_{hk} \to x_0$ as $h \to 0$.

Now,

$$\begin{aligned}
L &= \lim_{(h,k)\to(0,0)} \frac{g(x_0+h, y_0+k) - g(x_0, y_0) - g_x(x_0, y_0)h - g_y(x_0, y_0)k}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{g_x(c_{hk}, y_0+k)h + g(x_0, y_0+k) - g(x_0, y_0) - g_x(x_0, y_0)h - g_y(x_0, y_0)k}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \left(\frac{[g_x(c_{hk}, y_0+k) - g_x(x_0, y_0)]h}{\sqrt{h^2+k^2}} + \frac{g(x_0, y_0+k) - g(x_0, y_0) - g_y(x_0, y_0)k}{\sqrt{h^2+k^2}} \right)
\end{aligned}$$

The existence of g_y at (x_0, y_0) implies that

$$\lim_{k\to 0} \frac{g(x_0, y_0+k) - g(x_0, y_0) - g_y(x_0, y_0)k}{k} = 0,$$

which immediately implies that

$$\lim_{(h,k)\to(0,0)} \frac{g(x_0, y_0+k) - g(x_0, y_0) - g_y(x_0, y_0)k}{\sqrt{h^2+k^2}} = 0. \tag{3.13}$$

Since $c_{hk} \to x_0$ and $y_0 + k \to y_0$ as $(h,k) \to (0,0)$, the continuity of g_x at (x_0, y_0) implies that $g_x(c_{hk}, y_0+k) \to g_x(x_0, y_0)$ as $(h,k) \to (0,0)$, and thus

$$\lim_{(h,k)\to(0,0)} \frac{[g_x(c_{hk}, y_0+k) - g_x(x_0, y_0)]h}{\sqrt{h^2+k^2}} = 0. \tag{3.14}$$

The limits (3.13) and (3.14) taken together lead us to conclude that $L = 0$ and therefore g is real-differentiable at (x_0, y_0). ■

Exercise 3.9 (AN1.5). Let $f(z) = \bar{z}$ for all $z \in \mathbb{C}$. Show that f is everywhere continuous but nowhere differentiable.

Solution. We have

$$f(x+iy) = u(x,y) + iv(x,y) = x - iy$$

for all $x + iy \in \mathbb{C}$, so $u(x,y) = x$ and $v(x,y) = -y$ on $\mathbb{C}$. Let $x, y \in \mathbb{R}$ be arbitrary. Certainly u and v are continuous at (x,y), so that f is continuous at $x + iy$ by Theorem 2.23. On the other hand

$$u_x(x,y) = 1 \neq -1 = v_y(x,y),$$

and therefore f is not complex-differentiable at $x + iy$ by Theorem 3.6. ■

Exercise 3.10 (AN1.7). Let $u : \mathbb{C} \to \mathbb{R}$ be given by $u(x,y) = \sqrt{|xy|}$. Show that $u_x(0,0)$ and $u_y(0,0)$ exist, but u is not real-differentiable at $(0,0)$.

Solution. We have

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

and

$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

so $u_x(0,0)$ and $u_y(0,0)$ exist. Now,

$$\lim_{(h,k)\to(0,0)} \frac{u(h,k) - u(0,0) - u_x(0,0)h - u_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{\frac{|hk|}{h^2+k^2}}. \tag{3.15}$$

For any $(h,k) \neq (0,0)$ such that $h = k$ we have

$$\sqrt{\frac{|hk|}{h^2+k^2}} = \sqrt{\frac{h^2}{h^2+h^2}} = \frac{|h|}{\sqrt{2}|h|} = \frac{1}{\sqrt{2}}.$$

This makes clear that the limit (3.15) cannot equal zero, and therefore u is not real-differentiable at $(0,0)$. ■

Exercise 3.11 (AN1.16). Show that $f(z) = e^{\mathrm{Re}(z)}$ is nowhere complex-differentiable.

Solution. For any $x + iy \in \mathbb{C}$ we have $f(x + iy) = e^x$, which is to say

$$f(x + iy) = u(x, y) + iv(x, y)$$

with $u(x, y) = e^x$ and $v(x, y) = 0$. Now, since

$$u_x(x, y) = e^x \neq 0 = v_y(x, y),$$

for any $(x, y) \in \mathbb{R}^2$, it follows by Theorem 3.6 that f is not complex-differentiable at any $x + iy \in \mathbb{C}$. ∎

Proposition 3.12. *The exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ *is analytic everywhere, with*

$$\exp'(z) = \exp(z)$$

for all $z \in \mathbb{C}$*.*

Proof. By definition we have

$$\exp(x + iy) = u(x, y) + iv(x, y)$$

with

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Let $z = x+iy$ be arbitrary. It is clear that u_x, u_y, v_x, and v_y exist and are continuous everywhere, and thus by Exercise 3.8 both u and v are real-differentiable at (x, y). Now, since

$$u_x(x, y) = e^x \cos y = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -e^x \sin y = -v_x(x, y),$$

it follows by Theorem 3.6 that exp is complex-differentiable at $x + iy$, and

$$\begin{aligned} \exp'(x + iy) &= u_x(x, y) - iu_y(x, y) = e^x \cos y - i(-e^x \sin y) \\ &= e^x \cos y + ie^x \sin y = \exp(x + iy). \end{aligned}$$

Therefore exp is analytic on $\mathbb{C}$, and $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$. ∎

Example 3.13. We determine here the derivative of the function $z \mapsto e^{\alpha z}$, where $\alpha \in \mathbb{C}$ is a constant. Let $f(z) = \alpha z$, which is clearly an entire function with $f'(z) = \alpha$; and also let $g(z) = e^z$, which is entire with $g'(z) = e^z$ by Proposition 3.12. By the Chain Rule $g \circ f$ is entire, with

$$(g \circ f)'(z) = g'(f(z))f'(z) = \alpha e^{f(z)} = \alpha e^{\alpha z}$$

for any $z \in \mathbb{C}$. Since $(g \circ f)(z) = e^{\alpha z}$, we see that $(e^{\alpha z})' = \alpha e^{\alpha z}$. ∎

Let $\Omega \subseteq \mathbb{C}$ be an open set. A function $u : \Omega \to \mathbb{R}$ is **harmonic** on Ω if the first- and second-order partial derivatives of u are continuous on Ω, and

$$u_{xx}(x + iy) + u_{yy}(x + iy) = 0$$

for all $x + iy \in \Omega$. The equation $u_{xx} + u_{yy} = 0$ is known as **Laplace's Equation**.

We need now the following classical result, proven in §10.9 of the *Elementary Analysis Notes*.

Proposition 3.14 (Leibniz's Rule). *Let $u : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuous, and define $\varphi : [a,b] \to \mathbb{R}$ by*

$$\varphi(x) = \int_c^d u(x,y)\,dy.$$

Then φ is continuous. If in addition u_x exists and is continuous on $[a,b] \times [c,d]$, then φ is continuously differentiable with

$$\varphi'(x) = \int_c^d u_x(x,y)\,dy.$$

The rectangle $[a,b] \times [c,d]$ may be taken to be a subset of $\mathbb{C}$ rather than $\mathbb{R}^2$, so that

$$[a,b] \times [c,d] = \big\{x + iy : x \in [a,b] \text{ and } y \in [c,d]\big\}.$$

Only slight changes to the proof are needed, such as substituting (3.5) for the usual definition of u_x given in calculus. Of course there is another "version" of the rule that interchanges the roles of the two variables x and y. If $\psi : [c,d] \to \mathbb{R}$ is given by

$$\psi(y) = \int_a^b u(x,y)\,dx$$

and u_y is continuous on $[a,b] \times [c,d]$, then ψ is continuously differentiable with

$$\psi'(y) = \int_a^b u_y(x,y)\,dx.$$

Theorem 3.15. *Let $\Omega = B_r(z_0)$ for some $r \in (0,\infty]$. If $u : \Omega \to \mathbb{R}$ is a harmonic function, then there is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + iv$ is analytic on Ω.*

Proof. Suppose $u : \Omega \to \mathbb{R}$ is a harmonic function. Setting $z_0 = x_0 + iy_0$, define $v : \Omega \to \mathbb{R}$ by

$$v(x,y) = \int_{y_0}^y u_x(x,t)\,dt - \int_{x_0}^x u_y(s,y_0)\,ds \tag{3.16}$$

for all $(x,y) \in \Omega$. Note that both integrals on the right-hand side of (3.16) are defined for all $(x,y) \in \Omega$ on account of Ω being a disc. Fix $(x,y) \in \Omega$. Since u_x is continuous on Ω, by the Fundamental Theorem of Calculus (FTC) we have $v_y(x,y) = u_x(x,y)$, and hence $u_x = v_y$.

To show that $u_y = -v_x$, first assume that $x \neq x_0$ and $y \neq y_0$, so

$$R = [x_0 \wedge x, x_0 \vee x] \times [y_0 \wedge y, y_0 \vee y]$$

is a closed rectangle in Ω. Now, u_{xx} is continuous on Ω, and so Leibniz's Rule, the FTC, and the fact that $u_{xx} = -u_{yy}$ yields

$$\begin{aligned} v_x(x,y) &= \frac{\partial}{\partial x}\int_{y_0}^y u_x(x,t)\,dt - \frac{\partial}{\partial x}\int_{x_0}^x u_y(s,y_0)\,ds \\ &= \int_{y_0}^y u_{xx}(x,t)\,dt - u_y(x,y_0) = -\int_{y_0}^y u_{yy}(x,t)\,dt - u_y(x,y_0) \\ &= -\big[u_y(x,y) - u_y(x,y_0)\big] - u_y(x,y_0) = -u_y(x,y) \end{aligned}$$

as desired.

If we now let $y = y_0$, we have

$$v(x, y_0) = -\int_{x_0}^{x} u_y(s, y_0)\, ds$$

for all x such that $(x, y_0) \in \Omega$ (including $x = x_0$), whence

$$v_x(x, y_0) = -u_y(x, y_0)$$

by the FTC. In particular $v_x(x_0, y_0) = -u_y(x_0, y_0)$.

Finally, fix (x_0, y) for $y \neq y_0$. For definiteness we can assume $y > y_0$. Let $\delta > 0$ be such that

$$[x_0 - \delta, x_0 + \delta] \times [y_0, y] \subseteq \Omega.$$

Applying Leibniz's Rule,

$$\frac{d}{dx}\int_{y_0}^{y} u_x(x, t)\, dt = \int_{y_0}^{y} u_{xx}(x, t)\, dt = -\int_{y_0}^{y} u_{yy}(x, t)\, dt = u_y(x, y_0) - u_y(x, y)$$

for any $x \in [x_0 - \delta, x_0 + \delta]$, and thus $v_x(x, y) = -u_y(x, y)$ for $x \in [x_0 - \delta, x_0 + \delta]$. In particular $v_x(x_0, y) = -u_y(x_0, y)$, and since the argument is much the same if $y < y_0$, we conclude that $u_y = -v_x$ on Ω.

Next, since $v_x = -u_y$, $v_y = u_x$, and the first- and second-order partial derivatives of u exist and are continuous, we obtain

$$v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$

using Clairaut's Theorem in §10.8 of the *Elementary Analysis Notes*, and so $v_{xx} + v_{yy} = 0$. This shows that v is harmonic on Ω, and the analyticity of $f = u + iv$ on Ω follows immediately from Theorem 3.7 together with the remark preceding it. ■

The function v in the theorem is called the **harmonic conjugate** of u on Ω. Thus the theorem states that every harmonic function on an open disc has a harmonic conjugate on the disc.

3.3 – Complex Riemann Integration

The **Riemann integral** of $\varphi : [a, b] \to \mathbb{C}$ is defined to be

$$\int_a^b \varphi = \int_a^b \operatorname{Re}\varphi + i\int_a^b \operatorname{Im}\varphi, \tag{3.17}$$

provided the integrals on the right-hand side exist in $\mathbb{R}$, in which case we say φ is **Riemann integrable** on $[a, b]$. The set of all Riemann integrable complex-valued functions on $[a, b]$, which includes real-valued functions, is denoted by $\mathcal{R}[a, b]$. It is immediate that $\varphi \in \mathcal{R}[a, b]$ if and only if $\operatorname{Re}\varphi, \operatorname{Im}\varphi \in \mathcal{R}[a, b]$. Nearly all properties of the integral (3.17) derive directly from established properties of the Riemann integral of real-valued functions.

Theorem 3.16. *Let $\varphi, \psi : [a, b] \to \mathbb{C}$ be such that $\varphi, \psi \in \mathcal{R}[a, b]$.*

1. *We have*
$$\int_b^a \varphi = -\int_a^b \varphi \quad \text{and} \quad \int_a^a \varphi = 0. \tag{3.18}$$
2. *$\alpha\varphi + \beta\psi \in \mathcal{R}[a, b]$ for all $\alpha, \beta \in \mathbb{C}$, with*
$$\int_a^b (\alpha\varphi + \beta\psi) = \alpha\int_a^b \varphi + \beta\int_a^b \psi. \tag{3.19}$$
3. *$\varphi\psi \in \mathcal{R}[a, b]$ and $\varphi^n \in \mathcal{R}[a, b]$ for $n \in \mathbb{N}$.*
4. *If $c \in (a, b)$, then*
$$\int_a^b \varphi = \int_a^c \varphi + \int_c^b \varphi.$$
5. *$|\varphi| \in \mathcal{R}[a, b]$ with*
$$\left|\int_a^b \varphi\right| \le \int_a^b |\varphi|. \tag{3.20}$$

Proof.
Proof of Part (1). By definition $\int_b^a f = -\int_a^b f$ and $\int_a^a f = 0$ if f is a real-valued integrable function on $[a, b]$, from which the properties in (3.18) easily follow.

Proof of Part (2). Fix $\alpha = r + is \in \mathbb{C}$. Since $r, s \in \mathbb{R}$ and $\operatorname{Re}\varphi, \operatorname{Im}\varphi \in \mathcal{R}[a, b]$ are real-valued functions, from calculus it follows that $r\operatorname{Re}\varphi - s\operatorname{Re}\varphi$ and $r\operatorname{Im}\varphi + s\operatorname{Re}\varphi$ are in $\mathcal{R}[a, b]$, and thus

$$\alpha\varphi = (r\operatorname{Re}\varphi - s\operatorname{Re}\varphi) + i(r\operatorname{Im}\varphi + s\operatorname{Im}\varphi) \in \mathcal{R}[a, b].$$

Now,

$$\begin{aligned}
\int_a^b \alpha\varphi &= \int_a^b \big[(r\operatorname{Re}\varphi - s\operatorname{Im}\varphi) + i(r\operatorname{Im}\varphi + s\operatorname{Re}\varphi)\big] \\
&= \int_a^b (r\operatorname{Re}\varphi - s\operatorname{Im}\varphi) + i\int_a^b (r\operatorname{Im}\varphi + s\operatorname{Re}\varphi) \\
&= r\int_a^b \operatorname{Re}\varphi - s\int_a^b \operatorname{Im}\varphi + ir\int_a^b \operatorname{Im}\varphi + is\int_a^b \operatorname{Re}\varphi
\end{aligned}$$

$$\begin{aligned}
&= r\left(\int_a^b \operatorname{Re}\varphi + i\int_a^b \operatorname{Im}\varphi\right) + is\left(\int_a^b \operatorname{Re}\varphi + i\int_a^b \operatorname{Im}\varphi\right) \\
&= r\int_a^b \varphi + is\int_a^b \varphi = (r+is)\int_a^b \varphi = \alpha\int_a^b \varphi.
\end{aligned}$$

Next, since $\operatorname{Re}\psi, \operatorname{Im}\psi \in \mathcal{R}[a,b]$, from calculus we have $\operatorname{Re}\varphi + \operatorname{Re}\psi, \operatorname{Im}\varphi + \operatorname{Im}\psi \in \mathcal{R}[a,b]$, and so

$$\varphi + \psi = (\operatorname{Re}\varphi + \operatorname{Re}\psi) + i(\operatorname{Im}\varphi + \operatorname{Im}\psi) \in \mathcal{R}[a,b].$$

Now,

$$\begin{aligned}
\int_a^b (\varphi+\psi) &= \int_a^b (\operatorname{Re}\varphi + \operatorname{Re}\psi) + i\int_a^b (\operatorname{Im}\varphi + \operatorname{Im}\psi) \\
&= \int_a^b \operatorname{Re}\varphi + \int_a^b \operatorname{Re}\psi + i\int_a^b \operatorname{Im}\varphi + i\int_a^b \operatorname{Im}\psi \\
&= \int_a^b \varphi + \int_a^b \psi.
\end{aligned}$$

The two linearity results now established combine to give (3.19) for any $\alpha, \beta \in \mathbb{C}$.

Proof of Part (3). Let $\mathcal{F} = \{\operatorname{Re}\varphi, \operatorname{Im}\varphi, \operatorname{Re}\psi, \operatorname{Im}\psi\}$, so $\mathcal{F} \subseteq \mathcal{R}[a,b]$, and from calculus $fg \in \mathcal{R}[a,b]$ for any $f, g \in \mathcal{F}$. By Part (2) it follows that $\sum_{k=1}^m \alpha_k f_k g_k \in \mathcal{R}[a,b]$ for any $\alpha_k \in \mathbb{C}$ and $f_k, g_k \in \mathcal{F}$, where $m \in \mathbb{N}$. Hence

$$\varphi\psi = (\operatorname{Re}\varphi)(\operatorname{Re}\psi) - (\operatorname{Im}\varphi)(\operatorname{Re}\psi) + i(\operatorname{Re}\varphi)(\operatorname{Im}\psi) + i(\operatorname{Im}\varphi)(\operatorname{Im}\psi) \in \mathcal{R}[a,b],$$

from which $\varphi^n \in \mathcal{R}[a,b]$ follows by induction.

Proof of Part (4). For any $c \in (a,b)$ we have, using the corresponding property from calculus,

$$\begin{aligned}
\int_a^b \varphi = \int_a^b \operatorname{Re}\varphi + i\int_a^b \operatorname{Im}\varphi &= \left(\int_a^c \operatorname{Re}\varphi + \int_c^b \operatorname{Re}\varphi\right) + i\left(\int_a^c \operatorname{Im}\varphi + \int_c^b \operatorname{Im}\varphi\right) \\
&= \left(\int_a^c \operatorname{Re}\varphi + i\int_a^c \operatorname{Im}\varphi\right) + \left(\int_c^b \operatorname{Re}\varphi + i\int_c^b \operatorname{Im}\varphi\right),
\end{aligned}$$

and therefore

$$\int_a^b \varphi = \int_a^c \varphi + \int_c^b \varphi.$$

Proof of Part (5). Let $u = \operatorname{Re}\varphi$ and $v = \operatorname{Im}\varphi$. Since u and v are real-valued functions such that $u, v \in \mathcal{R}[a,b]$, from elementary analysis it follows that $u^2, v^2 \in \mathcal{R}[a,b]$, whence $|\varphi|^2 = u^2 + v^2 \in \mathcal{R}[a,b]$, and finally $|\varphi| \in \mathcal{R}[a,b]$ since the square root function is continuous.

Now, if $\int_a^b \varphi = 0$, then (3.20) follows trivially. Suppose $\int_a^b \varphi \neq 0$, and define

$$\lambda = \left|\int_a^b \varphi\right| \bigg/ \int_a^b \varphi,$$

so $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then

$$\int_a^b \lambda\varphi = \lambda \int_a^b \varphi = \left|\int_a^b \varphi\right| \in \mathbb{R},$$

and so by (3.17),

$$\left|\int_a^b \varphi\right| = \mathrm{Re}\left(\int_a^b \lambda\varphi\right) = \int_a^b \mathrm{Re}(\lambda\varphi) \leq \int_a^b |\lambda\varphi| = \int_a^b |\varphi| \tag{3.21}$$

as desired. The inequality in (3.21) obtains from the general property $\mathrm{Re}(z) \leq |z|$, and the result in elementary analysis that states that if $f, g \in \mathcal{R}[a,b]$ are real-valued functions such that $f \leq g$ on $[a,b]$, then $\int_a^b f \leq \int_a^b g$. ■

Theorem 3.17 (**Fundamental Theorem of Calculus**)**.** *If $\varphi : [a,b] \to \mathbb{C}$ is continuous, then $\Phi : [a,b] \to \mathbb{C}$ given by*

$$\Phi(t) = \int_a^t \varphi$$

is differentiable with $\Phi'(t) = \varphi(t)$. Moreover if F is any antiderivative of φ on $[a,b]$, then

$$\int_a^b \varphi = F(b) - F(a).$$

Proof. Suppose $\varphi : [a,b] \to \mathbb{C}$ is continuous, so that $u = \mathrm{Re}\,\varphi$ and $v = \mathrm{Im}\,\varphi$ are likewise continuous on $[a,b]$ by Theorem 2.23. The Fundamental Theorem of Calculus for real-valued functions implies that

$$U(t) = \int_a^t u \quad \text{and} \quad V(t) = \int_a^t v$$

are differentiable with $U'(t) = u(t)$ and $V'(t) = v(t)$. Now,

$$\Phi(t) = \int_a^t \varphi = \int_a^t u + i\int_a^t v = U(t) + iV(t),$$

and so we see that Φ is differentiable with $\Phi'(t) = \varphi(t)$.

The proof of the second statement follows just as easily using the corresponding result from calculus. ■

If $f : [a,b] \to \mathbb{R}$ is differentiable, $f' \in \mathcal{R}[a,b]$, and g is a continuous real-valued function on $f([a,b])$, then we have the following change of variable formula from elementary analysis:

$$\int_a^b g(f(t))f'(t)\,dt = \int_{f(a)}^{f(b)} g(x)\,dx.$$

There is a similar result for Riemann integrals of complex-valued functions.

Theorem 3.18 (**Change of Variable**)**.** *If $f : [a,b] \to \mathbb{R}$ is differentiable, $f' \in \mathcal{R}[a,b]$, and φ is a continuous complex-valued function on $f([a,b])$, then*

$$\int_a^b \varphi(f(t))f'(t)\,dt = \int_{f(a)}^{f(b)} \varphi(x)\,dx.$$

Proof. Let $u = \operatorname{Re}\varphi$ and $v = \operatorname{Im}\varphi$. By Theorem 2.23, the continuity of $\varphi : f([a,b]) \to \mathbb{C}$ implies the continuity of $u, v : f([a,b]) \to \mathbb{R}$. Applying the change of variable formula from elementary analysis, we immediately obtain

$$\begin{aligned}\int_a^b \varphi(f(t))f'(t)\,dt &= \int_a^b u(f(t))f'(t)\,dt + i\int_a^b v(f(t))f'(t)\,dt \\ &= \int_{f(a)}^{f(b)} u(x)\,dx + i\int_{f(a)}^{f(b)} v(x)\,dx \\ &= \int_{f(a)}^{f(b)} \varphi(x)\,dx,\end{aligned}$$

as desired. ■

3.4 – Parametrizations and Path Integrals

Definition 3.19. *Let $I \subseteq \mathbb{R}$ be an interval. A* ***curve*** *in $\mathbb{C}$ is a continuous mapping $\gamma : I \to \mathbb{C}$. The* ***trace*** *of γ is $\gamma^* = \gamma(I)$. For $S \subseteq \mathbb{C}$, a* ***curve in*** *S is a curve γ such that $\gamma^* \subseteq S$.*

Any set $\Gamma \subseteq \mathbb{C}$ for which there exists a continuous $\gamma : I \to \mathbb{C}$ such that $\gamma^* = \Gamma$ is also called a ***curve****, and then we say γ is a* ***parametrization*** *of Γ.*

A curve $\gamma : I \to \mathbb{C}$ is **smooth** if the derivative γ' exists and is continuous on I, and also $\gamma'(t) \neq 0$ for all $t \in I$. In particular, if $I = [a, b]$, this means the one-sided derivatives

$$\gamma'_+(a) := \lim_{h\to 0^+} \frac{\gamma(a+h) - \gamma(a)}{h} \quad \text{and} \quad \gamma'_-(b) := \lim_{h\to 0^-} \frac{\gamma(b+h) - \gamma(b)}{h}$$

exist in $\mathbb{C}$. A curve $\gamma : I \to \mathbb{C}$ is **piecewise-smooth** if there exists a partition of I such that γ is smooth on each subinterval of the partition. Again considering the case when $I = [a, b]$, which means there exist

$$a := t_0 < t_1 < \cdots < t_{n-1} < t_n := b$$

such that $\gamma : [t_{k-1}, t_k] \to \mathbb{C}$ is smooth for $1 \leq k \leq n$. A **path** is a piecewise-smooth curve.

Two smooth curves $\gamma : I \to \mathbb{C}$ and $\xi : J \to \mathbb{C}$ are **equivalent**, written $\gamma \sim \xi$, if there exists a continuously differentiable increasing bijection $s : J \to I$ such that $\xi = \gamma \circ s$. Two paths $\gamma : [a, b] \to \mathbb{C}$ and $\xi : [c, d] \to \mathbb{C}$ are **equivalent** if there exist

$$a := \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n := b \quad \text{and} \quad c := t_0 < t_1 < \cdots < t_{n-1} < t_n := d$$

such that $\gamma|_{[\tau_{k-1},\tau_k]}$ and $\xi|_{[t_{k-1},t_k]}$ are smooth with $\gamma|_{[\tau_{k-1},\tau_k]} \sim \xi|_{[t_{k-1},t_k]}$ for each $1 \leq k \leq n$. It's straightforward to show that $\sim$ is an equivalence relation, and so we may define the equivalence class of all paths equivalent to a given path γ_0:

$$[\gamma_0] = \{\gamma : \gamma \sim \gamma_0\}.$$

Any set $\Gamma \subseteq \mathbb{C}$ for which there exists some smooth (resp. piecewise-smooth) curve γ such that $\gamma^* = \Gamma$ is called a **smooth (resp. piecewise-smooth) curve**. It is natural to identify the set of points Γ with the equivalence class $[\gamma]$.

If $\gamma : [a, b] \to \mathbb{C}$ is a path, we define the path $\overline{\gamma} : [a, b] \to \mathbb{C}$ by

$$\overline{\gamma}(t) = \gamma(a + b - t).$$

Thus $\overline{\gamma}$ generates the same curve (viewed as a point set) in $\mathbb{C}$ as γ, but with the opposite orientation: we proceed from $\gamma(b)$ to $\gamma(a)$ instead of from $\gamma(a)$ to $\gamma(b)$.

Definition 3.20. *Let $\gamma : [a, b] \to \mathbb{C}$ be a path that is smooth on $[t_0, t_1], \ldots, [t_{n-1}, t_n]$ for $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, and let f be continuous on γ^*. The* ***path integral of f on γ*** *is*

$$\int_\gamma f = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)\, dt. \tag{3.22}$$

In particular if γ is smooth on $[a,b]$ then

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{3.23}$$

In the definition, if we let $\gamma_k = \gamma|_{[t_{k-1},t_k]}$ for each $1 \le k \le n$, then we straightaway obtain

$$\int_\gamma f = \sum_{k=1}^n \int_{\gamma_k} f$$

in the case when γ is smooth on $[t_0,t_1],\dots,[t_{n-1},t_n]$.

Letting (3.23) be the definition for the path integral of f on γ in the case when γ is piecewise-smooth—but not smooth—on $[a,b]$ would lead to a snag: namely, how to handle $\gamma'(t_k)$, seeing as each t_k would lie in the interior of the interval of integration. After all, $\gamma'(t_k)$ may not exist as a two-sided derivative. There would be workarounds, such as setting $\gamma'(t_k)=0$ for each k, or venturing into the badlands of improper integrals. On the other hand, for each integral

$$\int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)\,dt$$

in (3.22) the resolution is clear and natural: we take $\gamma'(t_{k-1}) = \gamma'_+(t_{k-1})$ and $\gamma'(t_k) = \gamma'_-(t_k)$. This is the motivation for the wording of Definition 3.20.

A curve $\gamma : [a,b] \to \mathbb{C}$ is **closed** if $\gamma(a) = \gamma(b)$. If γ is a closed path it is customary to use the symbol $\oint_\gamma$ instead of $\int_\gamma$. The symbol $\oint\!\!\!\!\square_\gamma$ is reserved for when γ is a closed rectangular path with sides parallel to the real and imaginary axes of $\mathbb{C}$. (Note: a path can have rectangular trace and yet not be closed!) Other notations for $\int_\gamma f$ for any path γ are

$$\int_\gamma f = \int_\gamma f\,dz = \int_\gamma f(z)\,dz,$$

and we define

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt$$

if $\gamma : [a,b] \to \mathbb{C}$ is smooth (passing to a sum as in Definition 3.20 if γ is piecewise-smooth), which is integration along a path with respect to "arc length."

Theorem 3.21. *Let $\gamma : [a,b] \to \mathbb{C}$ be a path, and let f be continuous on γ^*. If $\xi \sim \gamma$, then*

$$\int_\xi f = \int_\gamma f.$$

Proof. Suppose $\xi \sim \gamma$, so there exist

$$a := \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n := b \quad \text{and} \quad c := t_0 < t_1 < \cdots < t_{n-1} < t_n := d$$

such that $\gamma_k = \gamma|_{[\tau_{k-1},\tau_k]}$ and $\xi_k = \xi|_{[t_{k-1},t_k]}$ are smooth with $\gamma_k \sim \xi_k$ for each k. Thus, for each k, we have $\xi_k = \gamma_k \circ s_k$ for some continuously differentiable increasing bijection $s_k : [t_{k-1},t_k] \to [\tau_{k-1},\tau_k]$. By Definition 3.20, Theorem 3.16(4), and Theorem 3.19,

$$\int_\xi f = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f(\xi_k(t))\xi_k'(t)dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f\big((\gamma_k \circ s_k)(t)\big)(\gamma_k \circ s_k)'(t)dt$$

$$= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma_k(s_k(t)))\gamma_k'(s_k(t))s_k'(t)dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \big((f \circ \gamma_k)\gamma_k'\big)(s_k(t))s_k'(t)dt$$

$$= \sum_{k=1}^{n} \int_{s_k(t_{k-1})}^{s(t_k)} \big((f \circ \gamma_k)\gamma_k'\big)(\tau)d\tau = \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} f(\gamma_k(\tau))\gamma_k'(\tau)d\tau = \int_\gamma f,$$

as desired. ■

Definition 3.22. *The **length** of a path $\gamma : [a, b] \to \mathbb{C}$ is defined to be*

$$\mathcal{L}(\gamma) = \int_a^b |\gamma'(t)|dt.$$

If $\gamma \sim \xi$, then by an argument similar to the proof of Theorem 3.21 it can be shown that $\mathcal{L}(\gamma) = \mathcal{L}(\xi)$, and thus we may define the length of a smooth curve C to be the length of any path γ such that $\gamma^* = C$.

Theorem 3.23. *If $\gamma : [a, b] \to \mathbb{C}$ is a path and f is continuous on γ^*, then*

$$\left| \int_\gamma f \right| \le \mathcal{L}(\gamma) \sup_{z \in \gamma^*} |f(z)|.$$

Theorem 3.24 (**Fundamental Theorem of Path Integrals**). *Suppose that $f : \Omega \to \mathbb{C}$ is continuous, and there exist a function F such that $F' = f$ on Ω. If $\gamma : [a, b] \to \Omega$ is a path, then*

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a path. Assume that γ is continuously differentiable on $[a, b]$. Now, applying first the Chain Rule and then the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} \int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (F \circ \gamma)'(t)dt = (F \circ \gamma)(b) - (F \circ \gamma)(a) \\ &= F(\gamma(b)) - F(\gamma(a)), \end{aligned}$$

as desired.

If γ is piecewise continuously differentiable, then we need only apply the above argument to each subinterval of $[a, b]$ on which γ is continuously differentiable and add the results. ■

Example 3.25. In Example 3.13 it was found that $(e^{\alpha z})' = \alpha e^{\alpha z}$ for any $\alpha \in \mathbb{C}$. Thus for any $\alpha \in \mathbb{C}_*$ a primitive for $f(z) = e^{\alpha z}$ is $F(z) = \alpha^{-1}e^{\alpha z}$. For any $r > 0$ let $\gamma : [0, r] \to \mathbb{C}$ be given by $\gamma(t) = t$. Then

$$\int_0^r e^{\alpha t}dt = \int_0^r f(\gamma(t))\gamma'(t)dt = \int_\gamma f = F(\gamma(r)) - F(\gamma(0)) = F(r) - F(0) = \frac{e^{\alpha r} - 1}{\alpha}$$

by the Fundamental Theorem of Path Integrals. ■

Theorem 3.26. *Let $\Omega \subseteq \mathbb{C}$ be a connected open set, and suppose f is analytic on Ω. If $f' \equiv 0$ on Ω, then f is constant on Ω.*

Proof. Fix $z_0 \in \Omega$, and let $z \in \Omega$ be arbitrary. Since Ω is open and connected, it is also polygonally connected, and hence there exists a polygonal path $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = z_0$ and $\gamma(b) = z$. Clearly f' is continuous and has primitive f on Ω, so by the Fundamental Theorem of Path Integrals

$$0 = \int_\gamma 0 = \int_\gamma f' = f(\gamma(b)) - f(\gamma(a)) = f(z) - f(z_0),$$

which implies that $f(z) = f(z_0)$. Therefore $f \equiv f(z_0)$ on Ω. ∎

Proposition 3.27. *If $\gamma : [a, b] \to \mathbb{C}$ is a path and f is continuous on γ^*, then*

$$\int_{\overline{\gamma}} f = -\int_\gamma f.$$

Proof. By the Chain Rule

$$\overline{\gamma}'(t) = \gamma'(a + b - t) \cdot (a + b - t)' = -\gamma'(a + b - t),$$

and so

$$\int_{\overline{\gamma}} f = \int_a^b f(\overline{\gamma}(t))\overline{\gamma}'(t)dt = -\int_a^b f(\gamma(a + b - t))\gamma'(a + b - t)dt.$$

If we let $\varphi(t) = a + b - t$, then $\varphi'(t) = -1$ and we may write

$$\int_{\overline{\gamma}} f = \int_a^b \big((f \circ \gamma)\gamma'\big)(\varphi(t))\varphi'(t)dt.$$

Since $\varphi : [a, b] \to [a, b]$ is a strictly decreasing bijection, and $(f \circ \gamma)\gamma'$ and φ' are integrable on $[a, b]$ owing to both being piecewise continuous on $[a, b]$, by Theorem 3.19 it follows that

$$\int_{\overline{\gamma}} f = \int_{\varphi(a)}^{\varphi(b)} \big((f \circ \gamma)\gamma'\big)(\tau)d\tau = \int_b^a f(\gamma(\tau))\gamma'(\tau)d\tau = -\int_a^b f(\gamma(\tau))\gamma'(\tau)d\tau = -\int_\gamma f$$

as was to be shown. ∎

Suppose that $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are paths such that $\gamma_1(b_1) = \gamma_2(a_2)$. The **concatenation** of γ_1 and γ_2 is the path $\gamma_1 * \gamma_2 : [0, 1] \to \mathbb{C}$ given by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1\big((1 - 2t)a_1 + 2tb_1\big), & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2\big((2 - 2t)a_2 + (2t - 1)b_2\big), & \text{if } t \in [1, \frac{1}{2}] \end{cases}$$

Thus

$$(\gamma_1 * \gamma_2)|_{[0,1/2]} = \gamma_1 \circ h_1 \;\text{ with }\; h_1(t) = (1 - 2t)a_1 + 2tb_1$$

and

$$(\gamma_1 * \gamma_2)|_{[1/2,1]} = \gamma_2 \circ h_2 \;\text{ with }\; h_2(t) = (2 - 2t)a_2 + (2t - 1)b_2.$$

By Definition 3.20, if f is continuous on $(\gamma_1 * \gamma_2)^*$ then

$$\int_{\gamma_1 * \gamma_2} f = \int_{\gamma_1 \circ h_1} f + \int_{\gamma_2 \circ h_2} f.$$

Now, since f is continuous on γ_1^* and γ_2^*, h_1 and h_2 are continuously differentiable increasing bijections, and $h_1(0) = a_1$, $h_1(1/2) = b_1$, $h_2(1/2) = a_2$, and $h_2(1) = b_2$, by Theorem 3.21 we have

$$\int_{\gamma_1 \circ h_1} f = \int_{\gamma_1} f \quad \text{and} \quad \int_{\gamma_2 \circ h_2} f = \int_{\gamma_2} f.$$

This proves the following.

Proposition 3.28. *If $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are paths such that $\gamma_1(b_1) = \gamma_2(a_2)$, and f is continuous on $(\gamma_1 * \gamma_2)^*$, then*

$$\int_{\gamma_1 * \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Proposition 3.29. *Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two paths such that $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$. If f is continuous on $\gamma_1^* \cup \gamma_2^*$, then*

$$\int_{\gamma_1 * \overline{\gamma}_2} f = 0 \quad \textit{iff} \quad \int_{\gamma_1} f = \int_{\gamma_2} f.$$

Proof. Suppose that

$$\int_{\gamma_1 * \overline{\gamma}_2} f = 0.$$

Since $\overline{\gamma}_2 : [a_2, b_2] \to \mathbb{C}$ is a path such that $\gamma_1(b_1) = \overline{\gamma}_2(a_2)$, by Propositions 3.27 and 3.28 we have

$$\int_{\gamma_1 * \overline{\gamma}_2} f = \int_{\gamma_1} f + \int_{\overline{\gamma}_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f,$$

and therefore

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

The converse is proven merely by reversing the order of the foregoing manipulations, and so we are done. ∎

In §2.4 we defined the closed line segment joining $z, w \in \mathbb{C}$ to be the set of complex numbers

$$[z, w] = \big\{(1 - t)z + tw : t \in [0, 1]\big\}.$$

The same symbol $[z, w]$ will also be used to denote a path: specifically, the path given by

$$t \mapsto (1 - t)z + tw \tag{3.24}$$

for $0 \le t \le 1$. Context will make clear which interpretation of $[z, w]$—that of a set or a path—is intended. In the path integral symbol

$$\int_{[z,w]} f$$

the path interpretation is understood.

More generally for $z_1, \ldots, z_n \in \mathbb{C}$ we define

$$\int_{[z_1,\ldots,z_n]} f = \sum_{k=1}^{n-1} \int_{[z_k, z_{k+1}]} f.$$

Thus, given a triangle $\Delta = [z_1, z_2, z_3, z_1]$ we have

$$\oint_\Delta f = \int_{[z_1,z_2]} f + \int_{[z_2,z_3]} f + \int_{[z_3,z_1]} f.$$

Proposition 3.30. *Let $z, w \in \mathbb{C}$. If f is continuous on $[z, w]$, then*

$$\int_{[z,w]} f = -\int_{[w,z]} f.$$

Proof. Suppose f is continuous on $[z, w]$. Let $\gamma_1 : [0, 1] \to \mathbb{C}$ represent the path $[z, w]$ as defined by (3.24), and let $\gamma_2 : [0, 1] \to \mathbb{C}$ represent the path $[w, z]$, so

$$\gamma_2(t) = (1 - t)w + tz$$

for all $0 \le t \le 1$. Since

$$\overline{\gamma}_1(t) = \gamma_1(1 - t) = [1 - (1 - t)]z + (1 - t)w = (1 - t)w + tz = \gamma_2(t)$$

for all $t \in [0, 1]$, we see that $\gamma_2 = \overline{\gamma}_1$, and thus

$$\int_{[z,w]} f = \int_{\gamma_1} f = -\int_{\overline{\gamma}_1} f = -\int_{\gamma_2} f = -\int_{[w,z]} f$$

by Proposition 3.27. ∎

Theorem 3.31 (Path Integral Change of Variable). *For any $\alpha \in \mathbb{C}_*$,*

$$\int_{\gamma/\alpha} f(\alpha z)\, dz = \frac{1}{\alpha} \int_\gamma f(w)\, dw. \tag{3.25}$$

Proof. Fix $\alpha \in \mathbb{C}_*$. For $\gamma : [a, b] \to \mathbb{C}$ define $\hat{\gamma} = \gamma/\alpha$, and also let $\hat{f}(z) = f(\alpha z)$. Now,

$$\begin{aligned}\int_\gamma f(w)\, dw &= \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b f(\alpha\hat{\gamma}(t))\alpha\hat{\gamma}'(t)\, dt \\ &= \alpha \int_a^b \hat{f}(\hat{\gamma}(t))\hat{\gamma}'(t)\, dt = \alpha \int_{\hat{\gamma}} \hat{f}(z)\, dz = \alpha \int_{\gamma/\alpha} f(\alpha z)\, dz,\end{aligned}$$

since $\gamma = \alpha\hat{\gamma}$. ∎

Remark. For the integral at left in (3.25), the substitution in practice is carried by setting $w = \alpha z$, so that (formally) we have $dz = (1/\alpha)dw$. This nearly gives the integral at right in (3.25), with the only adjustment left to be made is to scale the parametrization γ/α up by a factor of α to offset the loss of α in the argument of f.

Example 3.32. Assuming $\alpha \in \mathbb{C}_*$, for the integral

$$\int_{C_r(0)} f(\alpha z)\, dz$$

we may make the substitution $w = \alpha z$, whereupon Theorem 3.31 implies that

$$\int_{C_r(0)} f(\alpha z)\,dz = \frac{1}{\alpha}\int_{\alpha C_r(0)} f(w)\,dw.$$

If $C_r(0)$ in the integral at left represents parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = re^{it}$, then $\alpha C_r(0)$ represents the parametrization $\hat{\gamma} : [0, 2\pi] \to \mathbb{C}$ given by $\hat{\gamma}(t) = \alpha\gamma(t) = r\alpha e^{it}$ for $t \in [0, 2\pi]$. In polar form we have $\alpha = |\alpha|e^{it_0}$ for some $t_0 \in \mathbb{R}$, and so

$$\hat{\gamma}(t) = |\alpha|re^{i(t+t_0)}.$$

Letting $s(t) = t - t_0$, define $\xi : [0, 2\pi] \to \mathbb{C}$ by $\xi = \hat{\gamma} \circ s$, so that

$$\xi(t) = \hat{\gamma}(s(t)) = \hat{\gamma}(t - t_0) = |\alpha|re^{it}$$

for each $t \in [0, 2\pi]$. Since $\xi \sim \hat{\gamma}$, Theorem 3.21 implies that

$$\int_{\alpha C_r(0)} f(w)\,dw = \int_{\hat{\gamma}} f(w)\,dw = \int_{\xi} f(w)\,dw = \int_{C_{|\alpha|r}(0)} f(w)\,dw,$$

and therefore

$$\int_{C_r(0)} f(\alpha z)\,dz = \frac{1}{\alpha}\int_{C_{|\alpha|r}(0)} f(w)\,dw. \tag{3.26}$$

■

3.5 – Preservation Properties

Under certain circumstances a limit or derivative operation may pass to the inside of, say, an integral without change, and so is "preserved." In this section we develop several such results for integrals of complex-valued functions, including path integrals. The starting point is always some analogous result for integrals of a real-valued function that has been proven in elementary analysis.

First, if $U \subseteq \mathbb{R}^2$ is open, we define the **partial derivatives** of a complex-valued function $f : U \to \mathbb{C}$ at $(x, y) \in U$, $f_x(x, y)$ and $f_y(x, y)$, to be given by the same limits in (3.3) and (3.4). Other symbols for f_x are $\partial_x f$ and $\partial f/\partial x$.

We now come to our first preservation property of this section, which states that the partial differentiation operator ∂_x is preserved by the functions $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$. There is of course a corresponding result for ∂_y.

Proposition 3.33. *Let $U \subseteq \mathbb{R}^2$ be open. If $f : U \to \mathbb{C}$ is such that $f_x(x, y)$ exists for some $(x, y) \in U$, then $(\mathrm{Re}\, f)_x(x, y)$ and $(\mathrm{Im}\, f)_x(x, y)$ also exist, and*

$$(\mathrm{Re}\, f)_x(x, y) = \mathrm{Re}(f_x(x, y)) \quad \textit{and} \quad (\mathrm{Im}\, f)_x(x, y) = \mathrm{Im}(f_x(x, y)).$$

Therefore $(\mathrm{Re}\, f)_x = \mathrm{Re}(f_x)$ and $(\mathrm{Im}\, f)_x = \mathrm{Im}(f_x)$.

Proof. Suppose $f_x(x, y)$ exists for some $(x, y) \in U$. For $\mathrm{Re}\, f : U \to \mathbb{R}$ we have, by (3.3) and (2.9),

$$\begin{aligned}(\mathrm{Re}\, f)_x(x, y) &= \lim_{h\to 0} \frac{(\mathrm{Re}\, f)(x + h, y) - (\mathrm{Re}\, f)(x, y)}{h} = \lim_{h\to 0} \mathrm{Re}\left(\frac{f(x + h, y) - f(x, y)}{h}\right) \\ &= \mathrm{Re}\left(\lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}\right) = \mathrm{Re}(f_x(x, y)).\end{aligned}$$

A similar argument is carried out for $(\mathrm{Im}\, f)_x(x, y)$. ■

With the foregoing definitions and proposition in place, we now present a complex-analytic version of Leibniz's Rule, recalling the real-analytic version that is Proposition 3.14.

Theorem 3.34 (Leibniz's Rule). *Let $f : [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{C}$ be continuous, and define $\varphi : [a, b] \to \mathbb{C}$ by*

$$\varphi(x) = \int_c^d f(x, y)\, dy.$$

Then φ is continuous. If in addition f_x exists and is continuous on $[a, b] \times [c, d]$, then φ is continuously differentiable with

$$\varphi'(x) = \int_c^d f_x(x, y)\, dy.$$

Proof. Set $R = [a, b] \times [c, d]$. By definition,

$$\varphi(x) = \int_c^d (\mathrm{Re}\, f)(x, y)\, dy + i \int_c^d (\mathrm{Im}\, f)(x, y)\, dy$$

for each $x \in [a,b]$. Define $u, v : [a,b] \to \mathbb{R}$ by

$$u(x) = \int_c^d (\operatorname{Re} f)(x,y)\,dy \quad \text{and} \quad v(x) = \int_c^d (\operatorname{Im} f)(x,y)\,dy,$$

so that $\varphi = u + iv$. Both $\operatorname{Re} f : R \to \mathbb{R}$ and $\operatorname{Im} f : R \to \mathbb{R}$ are continuous by Theorem 2.23, whereupon Proposition 3.14 implies that u and v are continuous, and thus so too is φ.

Now suppose f_x exists and is continuous on R. By Theorem 2.23 both $\operatorname{Re}(f_x)$ and $\operatorname{Im}(f_x)$ are continuous, and thus so are $(\operatorname{Re} f)_x$ and $(\operatorname{Im} f)_x$ by Proposition 3.33. It follows by Proposition 3.14 that u and v are continuously differentiable with

$$u'(x) = \int_c^d (\operatorname{Re} f)_x(x,y)\,dy \quad \text{and} \quad v'(x) = \int_c^d (\operatorname{Im} f)_x(x,y)\,dy.$$

Since $\varphi' = u' + iv'$ on $[a,b]$, we find that φ is continuously differentiable as well, and for each $x \in [a,b]$ we have

$$\begin{aligned}
\varphi'(x) &= \int_c^d (\operatorname{Re} f)_x(x,y)\,dy + i\int_c^d (\operatorname{Im} f)_x(x,y)\,dy \\
&= \int_c^d \big[(\operatorname{Re} f)_x(x,y) + i(\operatorname{Im} f)_x(x,y)\big]\,dy \\
&= \int_c^d \big[\operatorname{Re}(f_x)(x,y) + i\operatorname{Im}(f_x)(x,y)\big]\,dy \\
&= \int_c^d f_x(x,y)\,dy,
\end{aligned}$$

making use of Proposition 3.33 once more. ■

Theorem 3.35. *Let (φ_n) be a sequence of functions $[a,b] \to \mathbb{C}$ such that $\varphi_n \in \mathcal{R}[a,b]$ for all n. If (φ_n) converges uniformly to φ on $[a,b]$, then $\varphi \in \mathcal{R}[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b \varphi_n = \int_a^b \lim_{n\to\infty} \varphi_n = \int_a^b \varphi.$$

Proof. We take as given the corresponding theorem for real-valued functions. Suppose that (φ_n) converges uniformly to φ on $[a,b]$. Since each φ_n is a complex-valued function on $[a,b]$, there exist functions $u_n, v_n : [a,b] \to \mathbb{R}$ such that $\varphi_n = u_n + iv_n$. Recalling equation (3.17), $\varphi_n \in \mathcal{R}[a,b]$ implies that $u_n, v_n \in \mathcal{R}[a,b]$, and by Proposition 2.51(4) the sequences (u_n) and (v_n) converge uniformly to $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ on $[a,b]$, respectively. Therefore

$$\lim_{n\to\infty} \int_a^b u_n = \int_a^b \operatorname{Re}\varphi \quad \text{and} \quad \lim_{n\to\infty} \int_a^b v_n = \int_a^b \operatorname{Im}\varphi,$$

and thus

$$\begin{aligned}
\lim_{n\to\infty} \int_a^b \varphi_n &= \lim_{n\to\infty} \left(\int_a^b u_n + i\int_a^b v_n\right) = \lim_{n\to\infty} \int_a^b u_n + i\lim_{n\to\infty} \int_a^b v_n \\
&= \int_a^b \operatorname{Re}\varphi + i\int_a^b \operatorname{Im}\varphi = \int_a^b \varphi,
\end{aligned}$$

as desired. ■

Corollary 3.36. *Let $(\varphi_n)_{n\geq n_0}$ be a sequence of functions $[a,b]\to\mathbb{C}$ such that $\varphi_n\in\mathcal{R}[a,b]$ for all $n\geq n_0$. If the series $\sum\varphi_n$ is uniformly convergent on $[a,b]$, then $\sum\varphi_n\in\mathcal{R}[a,b]$ and*

$$\int_a^b\sum_{n=n_0}^\infty\varphi_n=\sum_{n=n_0}^\infty\int_a^b\varphi_n.$$

Proof. Suppose the series $\sum\varphi_n$ is uniformly convergent on $[a,b]$. This means that the sequence of partial sums (σ_n) converges uniformly to a function σ on $[a,b]$, and since

$$\sigma_n=\sum_{k=n_0}^n\varphi_k\in\mathcal{R}[a,b]$$

for all $n\geq n_0$ by Theorem 3.16, Theorem 3.35 implies that

$$\lim_{n\to\infty}\int_a^b\sigma_n=\int_a^b\lim_{n\to\infty}\sigma_n=\int_a^b\sigma. \tag{3.27}$$

Of course $\sigma(t)=\sum\varphi_n(t)$ for all $a\leq t\leq b$, which is to say $\sigma=\sum\varphi_n$ and so

$$\sum_{n=n_0}^\infty\int_a^b\varphi_n=\lim_{n\to\infty}\left(\sum_{k=n_0}^n\int_a^b\varphi_k\right)=\lim_{n\to\infty}\left(\int_a^b\sum_{k=n_0}^n\varphi_k\right)=\int_a^b\sum_{n=n_0}^\infty\varphi_n,$$

where the last equality follows from (3.27) to complete the proof. ■

Proposition 3.37. *Let $\gamma:[a,b]\to\mathbb{C}$ be a path, and let (f_n) be a sequence of functions $\gamma^*\to\mathbb{C}$ such that $f_n\circ\gamma\in\mathcal{R}[a,b]$ for all n. If (f_n) converges uniformly to f on γ^*, then $f\circ\gamma\in\mathcal{R}[a,b]$ and*

$$\lim_{n\to\infty}\int_\gamma f_n=\int_\gamma\lim_{n\to\infty}f_n=\int_\gamma f.$$

Proof. Assume that γ is continuously differentiable on $[a,b]$. Suppose (f_n) converges uniformly to f on γ^*. Since γ is continuous on $[a,b]$, it is easy to show that $(f_n\circ\gamma)$ is a sequence of functions $[a,b]\to\mathbb{C}$ that converges uniformly to $f\circ\gamma$ on $[a,b]$, where each $f_n\circ\gamma$ is bounded on account of being integrable. Now, the continuity of γ' on $[a,b]$ implies that γ' is bounded and integrable on $[a,b]$, so that $(f_n\circ\gamma)\gamma'\in\mathcal{R}[a,b]$ for all n, and

$$(f_n\circ\gamma)\gamma'\xrightarrow{\text{u}}(f\circ\gamma)\gamma',$$

by Proposition 2.51(3). Therefore $(f\circ\gamma)\gamma'\in\mathcal{R}[a,b]$ by Theorem 3.35, and

$$\lim_{n\to\infty}\int_\gamma f_n=\lim_{n\to\infty}\int_a^b(f_n\circ\gamma)\gamma'=\int_a^b\lim_{n\to\infty}(f_n\circ\gamma)\gamma'=\int_a^b(f\circ\gamma)\gamma'=\int_\gamma f$$

as was to be shown.

If γ is piecewise continuously differentiable on $[a,b]$, then we need only apply the above argument to each closed subinterval of $[a,b]$ on which γ is continuously differentiable, and then invoke Theorem 3.16 to obtain the desired result. ■

Corollary 3.38. *Let $\gamma : [a,b] \to \mathbb{C}$ be a path, and let $(f_n)_{n \geq n_0}$ be a sequence of functions $\gamma^* \to \mathbb{C}$ such that $f_n \circ \gamma \in \mathcal{R}[a,b]$ for all n. If the series $\sigma = \sum f_n$ is uniformly convergent on γ^*, then $\sigma \circ \gamma \in \mathcal{R}[a,b]$ and*

$$\int_\gamma \sum_{n=n_0}^{\infty} f_n = \sum_{n=n_0}^{\infty} \int_\gamma f_n.$$

Proof. Assume γ is continuously differentiable on $[a,b]$. Suppose the series $\sum f_n$ is uniformly convergent on $[a,b]$, meaning the sequence of partial sums (σ_n) converges uniformly to a function σ on $[a,b]$, and since

$$\sigma_n = \sum_{k=n_0}^{n} f_k \in \mathcal{R}[a,b]$$

for all $n \geq n_0$, Proposition 3.37 implies that

$$\lim_{n\to\infty} \int_\gamma \sigma_n = \int_\gamma \lim_{n\to\infty} \sigma_n = \int_\gamma \sigma. \tag{3.28}$$

Since $\sigma(t) = \sum f_n(t)$ for all $a \leq t \leq b$, we obtain

$$\sum_{n=n_0}^{\infty} \int_\gamma f_n = \lim_{n\to\infty} \left(\sum_{k=n_0}^{n} \int_\gamma f_k \right) = \lim_{n\to\infty} \left(\int_\gamma \sum_{k=n_0}^{n} f_k \right) = \int_\gamma \sum_{n=n_0}^{\infty} f_n,$$

where the last equality follows from (3.28). ∎

3.6 – Triangles and Primitives

We define the **convex hull** of a set $S \subseteq \mathbb{C}$ to be the set.[1]

$$\operatorname{Conv}(A) = \bigcap \{C : A \subseteq C \text{ and } C \text{ is convex}\}.$$

Theorem 3.39 (**Goursat's Theorem**). *Suppose $f : \Omega \to \mathbb{C}$ is analytic and Δ is a triangle. If $\operatorname{Conv}(\Delta) \subseteq \Omega$, then*

$$\oint_{\Delta} f = 0.$$

Proof. For $\Delta = [z_1, z_2, z_3, z_1]$, let a, b, and c be the midpoints of $[z_1, z_2]$, $[z_2, z_3]$, and $[z_3, z_1]$, respectively. As shown in Figure 5 we obtain four triangles,

$$T_1 = [z_1, a, c, z_1], \quad T_2 = [z_2, b, a, z_2], \quad T_3 = [z_3, c, b, z_3], \quad M = [a, b, c, a],$$

and using Proposition 3.27 we find that

$$\begin{aligned}
\oint_{\Delta} f &= \left(\int_{[z_1,a]} f + \int_{[a,z_2]} f + \int_{[z_2,b]} f + \int_{[b,z_3]} f + \int_{[z_3,c]} f + \int_{[c,z_1]} f \right) \\
&\quad + \left(\int_{[a,b]} f + \int_{[b,a]} f + \int_{[b,c]} f + \int_{[c,b]} f + \int_{[c,a]} f + \int_{[a,c]} f \right) \\
&= \left(\int_{[z_1,a]} f + \int_{[a,c]} f + \int_{[c,z_1]} f \right) + \left(\int_{[z_2,b]} f + \int_{[b,a]} f + \int_{[a,z_2]} f \right) \\
&\quad + \left(\int_{[z_3,c]} f + \int_{[c,b]} f + \int_{[b,z_3]} f \right) + \left(\int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f \right) \\
&= \oint_{T_1} f + \oint_{T_2} f + \oint_{T_3} f + \oint_{M} f
\end{aligned}$$

Letting $\Delta_1 \in \{T_1, T_2, T_3, M\}$ be such that

$$\left| \oint_{\Delta_1} f \right| = \max \left\{ \left| \oint_{T_1} f \right|, \left| \oint_{T_2} f \right|, \left| \oint_{T_3} f \right|, \left| \oint_{M} f \right| \right\},$$

[1]See §10.1 of the *Linear Algebra Notes* for a more thorough discussion of convex sets in general.

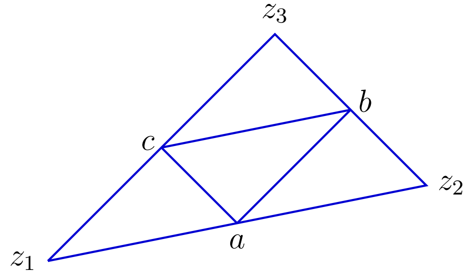

Figure 5.

we have

$$\operatorname{Conv}(\Delta_1) \subseteq \operatorname{Conv}(\Delta), \quad \left|\oint_{\Delta} f\right| \leq 4\left|\oint_{\Delta_1} f\right|, \quad \text{and} \quad \mathcal{L}(\Delta_1) = 2^{-1}\mathcal{L}(\Delta),$$

with the last observation hailing from elementary geometry.

Let $\Delta_0 = \Delta$. The procedure we applied to Δ_0 we may next apply to Δ_1. Proceeding inductively, if we let $n \in \mathbb{N}$ be arbitrary and suppose that

$$\operatorname{Conv}(\Delta_n) \subseteq \operatorname{Conv}(\Delta_{n-1}), \quad \left|\oint_{\Delta_{n-1}} f\right| \leq 4\left|\oint_{\Delta_n} f\right|, \quad \text{and} \quad \mathcal{L}(\Delta_n) = 2^{-n}\mathcal{L}(\Delta_{n-1}), \tag{3.29}$$

it is routine to show—by the same procedure as above—that (3.29) holds if we substitute $n+1$ for n, and therefore (3.29) must hold for all $\mathbb{N}$. Hence

$$\left|\oint_{\Delta_0} f\right| \leq 4^n\left|\oint_{\Delta_n} f\right| \tag{3.30}$$

for all $n \in \mathbb{N}$.

Since each $\operatorname{Conv}(\Delta_n)$ is a compact set and $\operatorname{Conv}(\Delta_{n+1}) \subseteq \operatorname{Conv}(\Delta_n)$ for all $n \in \mathbb{N}$, there exists some $z_0 \in \Omega$ such that

$$z_0 \in \bigcap_{n=1}^{\infty} \operatorname{Conv}(\Delta_n).$$

Now, f is analytic at z_0, so there exists some $\lambda \in \mathbb{C}$ and $\epsilon : \Omega \to \mathbb{C}$ that is continuous at z_0 such that $\epsilon(z_0) = 0$ and

$$f(z) = f(z_0) + (z - z_0)[\lambda + \epsilon(z)]$$

for all $z \in \Omega$. Fix $n \in \mathbb{N}$. If we designate $\Delta_0 = [w_0, w_1, w_2, w_0]$, then

$$\begin{aligned}
\oint_{\Delta_n} f &= \oint_{\Delta_n} \big[f(z_0) + (z - z_0)(\lambda + \epsilon(z))\big]\,dz \\
&= \oint_{\Delta_n} [f(z_0) - z_0\lambda]dz + \lambda\oint_{\Delta_n} z\,dz + \oint_{\Delta_n} (z - z_0)\epsilon(z)dz \\
&= [f(z_0) - z_0\lambda](w_0 - w_0) + \lambda\big(\tfrac{1}{2}w_0^2 - \tfrac{1}{2}w_0^2\big) + \oint_{\Delta_n} (z - z_0)\epsilon(z)dz \\
&= \oint_{\Delta_n} (z - z_0)\epsilon(z)dz,
\end{aligned}$$

and so by Theorem 3.23

$$\left|\oint_{\Delta_n} f\right| = \left|\oint_{\Delta_n} (z - z_0)\epsilon(z)dz\right| \leq \mathcal{L}(\Delta_n)\sup_{z\in\Delta_n}\big(|z - z_0||\epsilon(z)|\big).$$

Given $z_0 \in \operatorname{Conv}(\Delta_n)$, for any $z \in \Delta_n$ we have

$$|z - z_0| < \mathcal{L}(\Delta_n)$$

since the distance between any two points in a triangular region is less than the length of the region's boundary. Also

$$\mathcal{L}(\Delta_n) \leq 2^{-n}\mathcal{L}(\Delta_0)$$

follows from the equality in (3.29). Thus

$$\left|\oint_{\Delta_n} f\right| \le \big[\mathcal{L}(\Delta_n)\big]^2 \sup_{z\in\Delta_n} |\epsilon(z)| \le 4^{-n}\big[\mathcal{L}(\Delta_0)\big]^2 \sup_{z\in\Delta_n} |\epsilon(z)|,$$

and recalling (3.30) we obtain

$$0 \le \left|\oint_{\Delta_0} f\right| \le 4^n \left|\oint_{\Delta_n} f\right| \le \big[\mathcal{L}(\Delta_0)\big]^2 \sup_{z\in\Delta_n} |\epsilon(z)|.$$

for all $n \in \mathbb{N}$.

From

$$\Delta_n \subseteq B_{\mathcal{L}(\Delta_n)}(z_0) := B_n$$

we have

$$\sup_{z\in\Delta_n} |\epsilon(z)| \le \sup_{z\in B_n} |\epsilon(z)|,$$

and since $\mathcal{L}(\Delta_n) \to 0$ as $n \to \infty$ it is clear that

$$\lim_{n\to\infty} \sup_{z\in B_n} |\epsilon(z)| = |\epsilon(z_0)| = 0$$

and hence

$$\lim_{n\to\infty} \sup_{z\in\Delta_n} |\epsilon(z)| = 0$$

by the Squeeze Theorem. Therefore

$$\left|\oint_{\Delta_0} f\right| = 0$$

by another application of the Squeeze Theorem. ■

Theorem 3.40 (**Cauchy's Theorem for Starlike Regions**). *If Ω is a starlike region and f is analytic on Ω, then f has a primitive on Ω.*

Proof. Suppose that Ω is a starlike region with star center z_0, and let f be analytic on Ω. For every $z \in \Omega$ we have $[z_0, z] \subseteq \Omega$, and since f is analytic on Ω it follows that f is continuous on $[z_0, z]$ and therefore

$$\int_{[z_0,z]} f = \int_0^1 f\big((1-t)z_0 + z\big)(z - z_0)dt$$

exists in $\mathbb{C}$. Define $F : \Omega \to \mathbb{C}$ by

$$F(z) = \int_{[z_0,z]} f.$$

Fix $w \in \Omega$, and let $r > 0$ be sufficiently small such that $B_r(w) \subseteq \Omega$. For all $z \in B_r'(w)$ we have

$$\frac{F(z) - F(w)}{z - w} = \frac{1}{z - w}\left(\int_{[z_0,z]} f - \int_{[z_0,w]} f\right).$$

By Goursat's Theorem

$$\int_{[z,z_0]} f + \int_{[z_0,w]} f + \int_{[w,z]} f = \oint_{[z,z_0,w,z]} f = 0, \tag{3.31}$$

so

$$\int_{[z_0,z]} f - \int_{[z_0,w]} f = \int_{[w,z]} f$$

by Proposition 3.27, whence

$$\frac{F(z)-F(w)}{z-w} = \frac{1}{z-w}\int_{[w,z]} f$$

and therefore

$$\begin{aligned} F'(w) &= \lim_{z\to w}\frac{F(z)-F(w)}{z-w} = \lim_{z\to w}\frac{1}{z-w}\int_{[w,z]} f \\ &= \lim_{z\to w}\frac{1}{z-w}\int_0^1 f\big((1-t)w+tz\big)(z-w)dt = \lim_{z\to w}\int_0^1 f\big((1-t)w+tz\big)dt \end{aligned}$$

Let $0<\epsilon<r$. Since f is continuous at w there exists some $\delta>0$ such that $0<|z-w|<\delta$ implies

$$\big|f\big((1-t)w+tz\big)-f(w)\big|<\epsilon$$

for all $0\le t\le 1$, and then we obtain

$$\begin{aligned} \left|\int_0^1 f\big((1-t)w+tz\big)dt - f(w)\right| &= \left|\int_0^1 \big[f\big((1-t)w+tz\big)-f(w)\big]dt\right| \\ &\le \int_0^1 \big|f\big((1-t)w+tz\big)-f(w)\big|dt \le \int_0^1 \epsilon\, dt = \epsilon. \end{aligned}$$

We have now shown that

$$F'(w) = \lim_{z\to w}\int_0^1 f\big((1-t)w+tz\big)dt = f(w),$$

and since $w\in\Omega$ is arbitrary it follows that $F'=f$ on Ω. That is, F is a primitive for f on Ω. ■

Theorem 3.41. *If $f:\Omega\to\mathbb{C}$ is continuous and $\int_\gamma f = 0$ for every closed path γ in Ω, then f has a primitive on Ω.*

Exercise 3.42 (AN2.1.1). Evaluate

$$\int_{[-i,1+2i]} \mathrm{Im}(z)dz$$

Solution. The relevant path is $\gamma:[0,1]\to\mathbb{C}$ given by

$$\gamma(t) = (1-t)(-i)+t(1+2i) = t+(3t-1)i.$$

Now

$$\begin{aligned} \int_{[-i,1+2i]} \mathrm{Im}(z)dz &= \int_0^1 \mathrm{Im}(\gamma(t))\gamma'(t)dt = \int_0^1 \mathrm{Im}\big(t+(3t-1)t\big)(1+3i)dt \\ &= \int_0^1 (3t-1)(1+3i)dt = (1+3i)\left[\frac{3}{2}t^2-t\right]_0^1 = \frac{1}{2}+\frac{3}{2}i. \end{aligned}$$

■

Exercise 3.43 (AN2.1.2). Evaluate
$$\int_\gamma \bar{z}dz,$$
where γ is the path on the parabola $\{x + ix^2 : x \in \mathbb{R}\}$ from $1 + i$ to $2 + 4i$.

Solution. The relevant path is $\gamma : [1, 2] \to \mathbb{C}$ given by
$$\gamma(t) = t + it^2.$$
Now
$$\int_\gamma \bar{z}dz = \int_1^2 \overline{\gamma(t)}\gamma'(t)dt = \int_1^2 (t - it^2)(1 + 2it)dt$$
$$= \int_1^2 (2t^3 + t + it^2)dt = \left[\frac{1}{2}t^4 + \frac{1}{2}t^2 + \frac{i}{3}t^3\right]_1^2 = 9 + \frac{7}{3}i$$

■

Exercise 3.44 (AN2.1.3). Evaluate
$$\int_{[z_1,z_2,z_3]} f(z)dz,$$
where $z_1 = -i$, $2 + 5i$, $z_3 = 5i$, and $f(x + iy) = x^2 + iy$.

Solution. We have
$$\int_{[z_1,z_2,z_3]} f = \int_{[z_1,z_2]} f + \int_{[z_2,z_3]} f = \int_{[-i,2+5i]} f + \int_{[2+5i,5i]} f.$$
Defining $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ by
$$\gamma_1(t) = (1 - t)(-i) + t(2 + 5i) = 2t + (6t - 1)i$$
and
$$\gamma_2(t) = (1 - t)(2 + 5i) + t(5i) = (2 - 2t) + 5i,$$
we calculate
$$\int_{[-i,2+5i]} f = \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = \int_0^1 [(2t)^2 + i(6t - 1)](2 + 6i)dt$$
$$= (2 + 6i)\int_0^1 (4t^2 + 6it - i)dt = (2 + 6i)\left[\frac{4}{3}t^3 + 3it^2 - it\right]_0^1 = -\frac{28}{3} + 12i.$$
and
$$\int_{[2+5i,5i]} f = \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt = \int_0^1 [(2 - 2t)^2 + 5i](-2)dt$$
$$= -2\int_0^1 (4 - 8t + 4t^2 + 5i)dt = -2\left[4t - 4t^2 + \frac{4}{3}t^3 + 5it\right]_0^1 = -\frac{8}{3} - 10i.$$
Therefore
$$\int_{[z_1,z_2,z_3]} f(z)dz = \left(-\frac{28}{3} + 12i\right) + \left(-\frac{8}{3} - 10i\right) = -12 + 2i$$

is the value of the integral. ∎

Example 3.45. Let Γ be a rectangle with center at the origin and vertex z_k in the kth quadrant for $1 \le k \le 4$. Evaluate

$$\oint_\Gamma \frac{1}{z} dz$$

for both possible orientations of Γ, assuming the standard parameterization in each case.

Solution. First suppose Γ has the positive (i.e. counterclockwise) orientation. The standard parameterization is $[z_1, z_2, z_3, z_4, z_1]$, and it's convenient to write $\Gamma = [z_1, z_2, z_3, z_4, z_1]$. Letting $\xi_1 = [z_1, z_2]$, $\xi_2 = [z_2, z_3]$, $\xi_3 = [z_3, z_4]$, and $\xi_4 = [z_4, z_1]$, by definition

$$\oint_\Gamma \frac{1}{z} dz = \int_{\xi_1} \frac{1}{z} dz + \int_{\xi_2} \frac{1}{z} dz + \int_{\xi_3} \frac{1}{z} dz + \int_{\xi_4} \frac{1}{z} dz. \tag{3.32}$$

For each k we have $z_k = re^{i\tau_k}$ for some $r > 0$ and $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < 2\pi$, with $\tau_1 \in (0, \pi/2)$ in particular. Let C be the circle of radius r that circumscribes Γ. We may parameterize C with the function $\gamma : [0, 2\pi] \to \mathbb{C}$ given by

$$\gamma(t) = re^{i(t+\tau_1)},$$

which is a closed path with positive orientation that begins and ends at z_1. Now, define

$$\gamma_1 = \gamma|_{[0,\tau_2-\tau_1]}, \quad \gamma_2 = \gamma|_{[\tau_2-\tau_1,\tau_3-\tau_1]}, \quad \gamma_3 = \gamma|_{[\tau_3-\tau_1,\tau_4-\tau_1]}, \quad \text{and} \quad \gamma_4 = \gamma|_{[\tau_4-\tau_1,2\pi]}$$

(see Figure 6). Then by Definition 3.20

$$\oint_\gamma \frac{1}{z} dz = \int_{\gamma_1} \frac{1}{z} dz + \int_{\gamma_2} \frac{1}{z} dz + \int_{\gamma_3} \frac{1}{z} dz + \int_{\gamma_4} \frac{1}{z} dz \tag{3.33}$$

since $z \mapsto 1/z$ is continuous on $C = \gamma^*$.

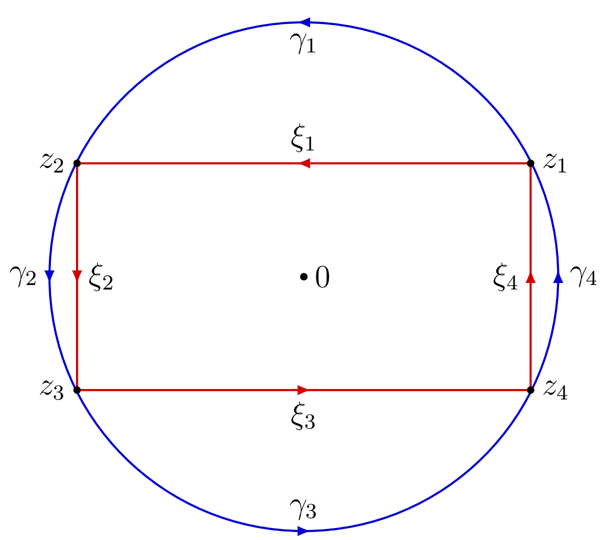

Figure 6.

Define $\Omega = \{z : \mathrm{Im}(z) > 0\}$. Since Ω is a starlike region in $\mathbb{C}$ and $1/z$ is analytic on Ω, by Theorem 3.40 $1/z$ has a primitive on Ω. Now, $\xi_1 * \overline{\gamma}_1$ is a closed path in Ω, so

$$\oint_{\xi_1 * \overline{\gamma}_1} \frac{1}{z} dz = 0$$

by the Fundamental Theorem of Path Integrals. Similar arguments will show that

$$\oint_{\xi_k * \overline{\gamma}_k} \frac{1}{z} dz = 0$$

for all $1 \leq k \leq 4$. Since ξ_k and γ_k have the same initial and terminal points, and $1/z$ is continuous on $\xi_k^* \cup \gamma_k^*$, by Proposition 3.29

$$\int_{\xi_k} \frac{1}{z} dz = \int_{\gamma_k} \frac{1}{z} dz \tag{3.34}$$

for each k. Combining equations (3.32), (3.33), and (3.34) yields

$$\oint_{\Gamma} \frac{1}{z} dz = \sum_{k=1}^{4} \int_{\xi_k} \frac{1}{z} dz = \sum_{k=1}^{4} \int_{\gamma_k} \frac{1}{z} dz = \oint_{\gamma} \frac{1}{z} dz.$$

Therefore

$$\oint_{\Gamma} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{\gamma(t)} \gamma'(t) dt = \int_0^{2\pi} \frac{\left[re^{i(t+\tau_1)}\right]'}{re^{i(t+\tau_1)}} dt = \int_0^{2\pi} \frac{ire^{i(t+\tau_1)}}{re^{i(t+\tau_1)}} dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

If Γ has the negative (clockwise) orientation, then $[z_4, z_3, z_2, z_1, z_4]$ is the standard parameterization. If we let $\overline{\Gamma} = [z_4, z_3, z_2, z_1, z_4]$, then by Proposition 3.27,

$$\begin{aligned}
\oint_{\overline{\Gamma}} \frac{1}{z} dz &= \int_{[z_4, z_3]} \frac{1}{z} dz + \int_{[z_3, z_2]} \frac{1}{z} dz + \int_{[z_2, z_1]} \frac{1}{z} dz + \int_{[z_1, z_4]} \frac{1}{z} dz \\
&= -\int_{[z_3, z_4]} \frac{1}{z} dz - \int_{[z_2, z_3]} \frac{1}{z} dz - \int_{[z_1, z_2]} \frac{1}{z} dz - \int_{[z_4, z_1]} \frac{1}{z} dz \\
&= -\left(\int_{\xi_1} \frac{1}{z} dz + \int_{\xi_2} \frac{1}{z} dz + \int_{\xi_3} \frac{1}{z} dz + \int_{\xi_4} \frac{1}{z} dz \right) \\
&= -\oint_{\Gamma} \frac{1}{z} dz = -2\pi i.
\end{aligned}$$

■

Exercise 3.46 (AN2.1.5a). Let f be analytic on a convex open set Ω. Show that if $\mathrm{Re}\, f' > 0$, $\mathrm{Re}\, f' < 0$, $\mathrm{Im}\, f' > 0$, or $\mathrm{Im}\, f' < 0$ on Ω, then f is injective on Ω.

Solution. Suppose that $\mathrm{Re}\, f' > 0$ on Ω. As will be established in the next section, the analyticity of f on Ω implies that f' is also analytic—and in particular continuous—on Ω. Fix $z_1, z_2 \in \Omega$. Since f is a primitive for f' on Ω, by the Fundamental Theorem for Path Integrals we have

$$\int_{\gamma} f' = f(\gamma(b)) - f(\gamma(a))$$

for any path $\gamma : [a,b] \to \Omega$, so if we define $\gamma : [0,1] \to \Omega$ by $\gamma(t) = (1-t)z_1 + tz_2$, then since $\gamma^* = [z_1, z_2]$ and Ω is convex, it follows that γ is a path in Ω and hence

$$\int_\gamma f' = f(\gamma(1)) - f(\gamma(0)) = f(z_2) - f(z_1).$$

On the other hand

$$\int_\gamma f' = \int_0^1 f'(\gamma(t))\gamma'(t)dt = \int_0^1 f'\big((1-t)z_1 + tz_2\big)(z_2 - z_1)dt,$$

and so

$$\int_0^1 f'\big((1-t)z_1 + tz_2\big)dt = \frac{f(z_2) - f(z_1)}{z_2 - z_1}.$$

Applying the definition in (3.17) next yields

$$\int_0^1 \operatorname{Re} f'\big((1-t)z_1 + tz_2\big)dt + i\int_0^1 \operatorname{Im} f'\big((1-t)z_1 + tz_2\big)dt = \frac{f(z_2) - f(z_1)}{z_2 - z_1}, \tag{3.35}$$

whence we obtain

$$\operatorname{Re}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) = \int_0^1 \operatorname{Re} f'\big((1-t)z_1 + tz_2\big)dt = \int_0^1 \big((\operatorname{Re} f') \circ \gamma\big)(t)dt.$$

Now, $((\operatorname{Re} f') \circ \gamma)(t) > 0$ for all $t \in [0,1]$, which implies that the rightmost integral is greater than zero and thus

$$\operatorname{Re}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) > 0.$$

This immediately implies that $f(z_2) \neq f(z_1)$, and therefore f is injective on Ω.

If $\operatorname{Re} f' < 0$ on Ω, then a nearly identical argument leads to

$$\operatorname{Re}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) < 0$$

for any $z_1, z_2 \in \Omega$, which again shows that f is injective on Ω.

If $\operatorname{Im} f' > 0$ on Ω, the same argument as above leads to equation (3.35), whereupon we obtain

$$\operatorname{Im}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) = \int_0^1 \operatorname{Im} f'\big((1-t)z_1 + tz_2\big)dt = \int_0^1 \big((\operatorname{Im} f') \circ \gamma\big)(t)dt.$$

Now, since $(\operatorname{Im} f') \circ \gamma > 0$ on $[0,1]$, the rightmost integral must be positive and thus

$$\operatorname{Im}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) > 0.$$

Clearly $f(z_2) \neq f(z_1)$, and so f is injective on Ω. The argument is nearly identical if $\operatorname{Im} f' < 0$ on Ω. ■

Exercise 3.47 (AN2.1.5b). Show that the conclusion of the proposition in Exercise 3.46 does not necessarily hold in the case when Ω is a starlike region.

Solution. Consider the function $f(z) = z + 1/z$, which is analytic on $\mathbb{C}_*$. For all $z \neq 0$ we have

$$f'(z) = 1 - \frac{1}{z^2},$$

which in polar form gives

$$\begin{aligned} f'(re^{i\theta}) &= 1 - \frac{1}{(re^{i\theta})^2} = 1 - \frac{e^{-2i\theta}}{r^2} = 1 - \frac{1}{r^2}\big[\cos(-2\theta) + i\sin(-2\theta)\big] \\ &= \left(1 - \frac{\cos 2\theta}{r^2}\right) + \left(\frac{\sin 2\theta}{r^2}\right)i. \end{aligned}$$

Hence we have

$$\operatorname{Re} f'(re^{i\theta}) = 1 - \frac{\cos 2\theta}{r^2}.$$

Now,

$$\operatorname{Re} f'(re^{i\theta}) > 0 \quad \Leftrightarrow \quad 1 - \frac{\cos 2\theta}{r^2} > 0 \quad \Leftrightarrow \quad r^2 > \cos 2\theta,$$

the solution set for which is the shaded region shown at left in Figure 7. Let

$$z_1 = 2, \quad z_2 = -2 + \frac{3}{2}i, \quad z_3 = -2 + 2i, \quad z_4 = 3, \quad z_5 = -2 - 2i, \quad z_6 = -2 - \frac{3}{2}i.$$

Let Ω be the open region enclosed by the polygonal path $[z_1, z_2, z_3, z_4, z_5, z_6, z_1]$, shown at right in Figure 7. It can be seen that Ω is a starlike open set, f is analytic on Ω, and $\operatorname{Re} f' > 0$ on Ω. However f is not injective on Ω, for we have $-i, i \in \Omega$, and

$$f(i) = i + \frac{1}{i} = i - i = 0 = -i + i = -i - \frac{1}{-i} = f(-i).$$

Similar analyses can be carried out in the cases when $\operatorname{Re} f' < 0$, $\operatorname{Im} f' > 0$ and $\operatorname{Im} f' < 0$ on a starlike region. ■

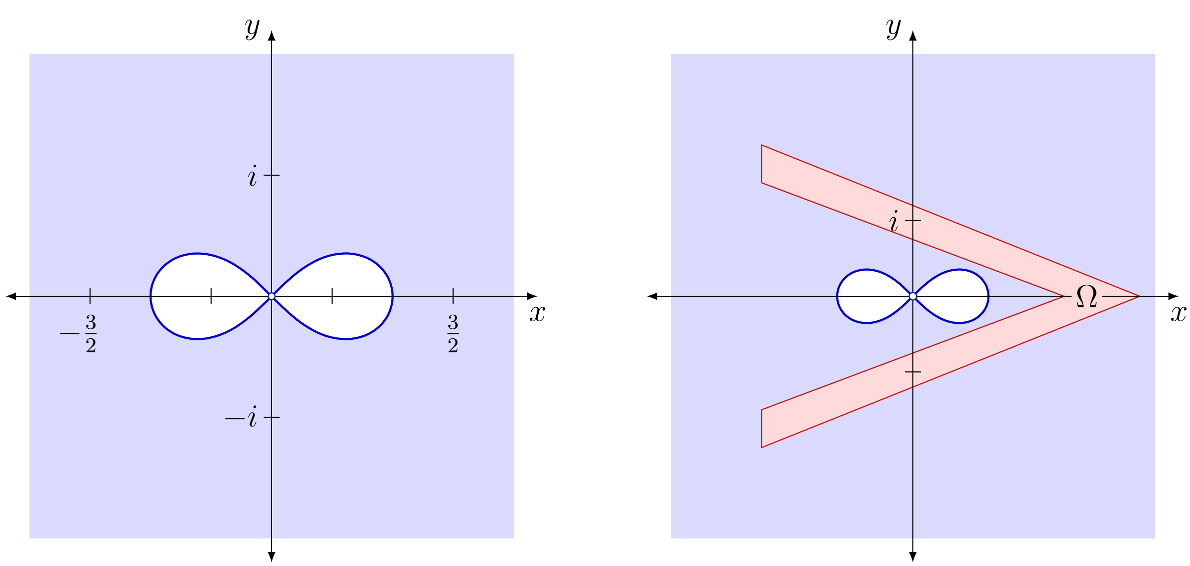

FIGURE 7.

Exercise 3.48 (AN2.1.5c). Let f be analytic on Ω, and let $z_0 \in \Omega$. Show that if $f'(z_0) \neq 0$, then there exists some $r > 0$ such that f is injective on $B_r(z_0)$.

Solution. The analyticity of f on Ω implies the analyticity, and hence continuity, of f' on Ω, and then by Theorem 2.23 it follows that $\operatorname{Re} f'$ and $\operatorname{Im} f'$ are also continuous on Ω.

Suppose that $f'(z_0) \neq 0$. Then either $\operatorname{Re} f'(z_0) \neq 0$ or $\operatorname{Im} f'(z_0) \neq 0$. Assume first that $\operatorname{Re} f'(z_0) \neq 0$. Since $\operatorname{Re} f'$ is continuous at z_0 there must exist some $r > 0$ such that either $\operatorname{Re} f' > 0$ or $\operatorname{Re} f' < 0$ on $B_r(z_0)$, depending on whether $\operatorname{Re} f'(z_0)$ is positive or negative. In either event, since $B_r(z_0)$ is convex we conclude by Exercise 3.46 that f is injective on $B_r(z_0)$.

Now assume that $\operatorname{Im} f'(z_0) \neq 0$. Since $\operatorname{Im} f'$ is continuous at z_0 there must exist some $r > 0$ such that either $\operatorname{Im} f' > 0$ or $\operatorname{Im} f' < 0$ on $B_r(z_0)$, depending on whether $\operatorname{Im} f'(z_0)$ is positive or negative. Either way, we conclude by Exercise 3.46 that f is injective on $B_r(z_0)$. ■

The following theorem extends Theorems 3.39 and 3.40 to the case when a function $f : \Omega \to \mathbb{C}$ is analytic except at a single point $z_0 \in \Omega$. Continuity on all of Ω is still required. A further extension to the case when a continuous function $f : \Omega \to \mathbb{C}$ is analyic on Ω except at a *finite* number of points $z_1, \ldots, z_n \in \Omega$ is possible.

Theorem 3.49 (Extended Cauchy Theorem). *Let $f : \Omega \to \mathbb{C}$ be continuous on Ω and analytic on $\Omega \setminus \{z_0\}$.*

1. *If Δ is a triangle such that $\operatorname{Conv}(\Delta) \subseteq \Omega$, then $\int_\Delta f = 0$.*
2. *If Ω is a starlike region, then f has a primitive on Ω.*

4

Analytic Functions

4.1 – Complex Power Series

Definition 4.1. *Given a power series* $\sum a_n(z-z_0)^n$*, let* $\rho = \limsup \sqrt[n]{|a_n|}$*. Define the* ***radius of convergence*** r *of* $\sum a_n(z-z_0)^n$ *to be given by*

$$r = \begin{cases} 1/\rho, & \text{if } 0 < \rho < \infty \\ 0, & \text{if } \rho = \infty \\ \infty, & \text{if } \rho = 0 \end{cases}$$

It should be emphasized that, given any sequence (a_n) in $\mathbb{C}$, the limit superior ρ will always exist in $[0, \infty]$. The following theorem will make clear the reason why $r \in \overline{\mathbb{R}}$ in the definition is called the radius of convergence.

Theorem 4.2. *Suppose the series* $\sum a_n(z-z_0)^n$ *has radius of convergence* r*.*

1. *If* $0 < r < \infty$*, the series converges absolutely if* $|z-z_0| < r$ *and diverges if* $|z-z_0| > r$*.*
2. *If* $r = 0$*, the series converges absolutely if* $z = z_0$ *and diverges if* $z \neq z_0$*.*
3. *If* $r = \infty$ *the series converges absolutely if* $|z-z_0| < \infty$*.*
4. *The series converges uniformly on compact subsets of* $B_r(z_0)$*.*

Proof.
Proof of Part (1). Let $\rho = \limsup \sqrt[n]{|a_n|}$. Suppose $0 < r < \infty$. Then $r = 1/\rho$ for some $0 < \rho < \infty$, which is to say

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{r}.$$

By the Root Test $\sum a_n(z-z_0)^n$ converges absolutely if

$$\limsup \sqrt[n]{|a_n(z-z_0)^n|} < 1,$$

and diverges if

$$\limsup \sqrt[n]{|a_n(z-z_0)^n|} > 1.$$

Since by Proposition 2.5(3)

$$\limsup \sqrt[n]{|a_n(z-z_0)^n|} = \limsup \left(|z-z_0| \sqrt[n]{|a_n|}\right) = |z-z_0| \limsup \sqrt[n]{|a_n|} = \frac{|z-z_0|}{r}$$

it follows that $\sum a_n(z-z_0)^n$ converges absolutely if $|z-z_0| < r$ and diverges if $|z-z_0| > r$.

Proof of Part (2). Suppose that $r = 0$. Then $\rho = \infty$, so for any $z \neq z_0$ we have

$$\limsup \sqrt[n]{|a_n(z-z_0)^n|} = |z-z_0| \limsup \sqrt[n]{|a_n|} = |z-z_0| \cdot \infty = \infty,$$

and thus $\sum a_n(z-z_0)^n$ diverges by the Root Test. It is clear that $\sum a_n(z-z_0)^n$ converges absolutely if $z = z_0$.

Proof of Part (3). Finally, suppose $r = \infty$. Then $\rho = 0$, so for any $z \in \mathbb{C}$ (i.e. $|z-z_0| < \infty$) we have

$$\limsup \sqrt[n]{|a_n(z-z_0)^n|} = |z-z_0| \limsup \sqrt[n]{|a_n|} = |z-z_0| \cdot 0 = 0,$$

and thus $\sum a_n(z-z_0)^n$ converges absolutely by the Root Test.

Proof of Part (4). Let $K \subseteq B_r(z_0)$ be a compact set. Choose $s \in (0,r)$ such that $K \subseteq \overline{B}_s(z_0)$. Part (1) implies that

$$\sum |a_n||z-z_0|^n$$

converges for any $z \in \mathbb{C}$ such that $|z-z_0| < r$. Choosing $z = z_0 + s$ and noting that

$$|z-z_0| = |(z_0+s)-z_0| = |s| = s < r,$$

it follows that $\sum |a_n|s^n$ converges. Now, for each n define $f_n : B_r(z_0) \to \mathbb{C}$ by

$$f_n(z) = a_n(z-z_0)^n.$$

Since

$$\|f_n\|_K \leq \|f_n\|_{\overline{B}_s(z_0)} = \sup_{z \in \overline{B}_s(z_0)} \left(|a_n||z-z_0|^n\right) \leq |a_n|s^n,$$

and $\sum |a_n|s^n$ converges, we conclude by the Weierstrass M-Test that $\sum f_n$ converges uniformly on K. Therefore $\sum a_n(z-z_0)^n$ converges uniformly on compact subsets of $B_r(z_0)$. ■

Proposition 4.3. *If the series $\sum a_n(z-z_0)^n$ converges on $B_r(z_0)$, then it converges absolutely on $B_r(z_0)$.*

Proof. Suppose that $\sum a_n(z-z_0)^n$ converges on $B_r(z_0)$, and let s be the radius of convergence of the series. Assume there exists some $z \in B_r(z_0)$ such that $\sum a_n(z-z_0)^n$ does not converge absolutely. (Clearly this implies that $z \neq z_0$.) Then $s \leq |z-z_0| < r$, since by Theorem 4.2(1) the series must be absolutely convergent on $B_s(z_0)$. Now, let $\hat{z}$ be such that

$$|z-z_0| < |\hat{z}-z_0| < r.$$

Then $\hat{z} \in B_r(z_0)$, so by hypothesis $\sum a_n(\hat{z}-z_0)^n$ converges. However we also have $|\hat{z}-z_0| > s$, so that by Theorem 4.2(1) it follows that $\sum a_n(\hat{z}-z_0)^n$ diverges—a contradiction. We conclude that $\sum a_n(z-z_0)^n$ must converge absolutely, and therefore $\sum a_n(z-z_0)^n$ converges absolutely on $B_r(z_0)$. ■

Example 4.4. Consider the series

$$\sigma(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Clearly the series converges absolutely if $z = 0$. If $z \neq 0$, we find that

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0,$$

and so

$$\limsup_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = 0$$

as well, and by the Ratio Test we conclude that the series converges absolutely at z. Hence the series converges absolutely on $\mathbb{C}$, and the radius of converges of the series must be $r = \infty$. Indeed, if $r < \infty$, then by Theorem 4.2 the series must diverge for any z such that $|z| > r$, which is a contradiction. We conclude by Theorem 4.2 that the series converges uniformly on any compact subset of $\mathbb{C}$. This is to say that if (σ_k) is the sequence of functions $\mathbb{C} \to \mathbb{C}$ given by

$$\sigma_k(z) = \sum_{n=0}^{k} \frac{1}{n!} z^n$$

for each $k \geq 0$, then on any compact set K the sequence (σ_k) converges uniformly to some function $K \to \mathbb{C}$. It is easy to see that, in fact, $\sigma_k \xrightarrow{\text{u}} \sigma$ on K. ■

Proposition 4.5. *Suppose that $\sum a_n(z-z_0)^n$ has radius of convergence $r > 0$. Then for any $A > 1/r$ there exists some $C > 0$ such that $|a_n| \leq CA^n$ for all $n \geq 0$.*

Proof. We have

$$\limsup \sqrt[n]{|a_n|} = \begin{cases} 1/r, & \text{if } 0 < r < \infty \\ 0, & \text{if } r = \infty \end{cases}$$

Assume to start that $0 < r < \infty$, so

$$\sup_{k\geq n} \sqrt[k]{|a_k|} \downarrow \frac{1}{r} > 0$$

as $n \to \infty$. Let $A > 1/r$, which is to say $A = 1/r + \epsilon$ for some $\epsilon > 0$. Then there exists some $N \in \mathbb{Z}$ such that

$$\frac{1}{r} \leq \sup_{k\geq n} \sqrt[k]{|a_k|} < \frac{1}{r} + \epsilon = A$$

for all $n \geq N$. In particular this implies that

$$0 \leq \sqrt[k]{|a_k|} \leq A$$

for all $k \geq N$, and hence

$$|a_k| \leq A^k$$

for all $k \geq N$. Choose $C > 0$ sufficiently large that $|a_k| \leq CA^k$ for $k = 0, \ldots, N-1$. Then we have

$$|a_n| \leq CA^n$$

for all $n \geq 0$ as desired.

If $r = \infty$ the argument above is unchanged, and the conclusion reached the same, except that $1/r$ is 0. ■

Corollary 4.6. *Suppose that $\sum a_n(z - z_0)^n$ has radius of convergence $r > 0$. Then there exists some $B > 1/r$ such that $|a_n| \leq B^n$ for all $n \geq 0$.*

Proof. The proof is identical to the proof of Proposition 4.5, except that when it comes time to choose some $C > 0$ large enough that $|a_k| \leq CA^k$ for $k = 0, \ldots, N-1$, we instead choose some $B \geq A$ such that $|a_k| \leq B^k$ for $k = 0, \ldots, N-1$. The desired result follows. ■

Definition 4.7. *The **Cauchy product** of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is*

$$\left(\sum_{n=0}^{\infty} a_n\right) * \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right).$$

In Definition 4.7 there is no assumption that either $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=0}^{\infty} b_n$ is convergent, so a Cauchy product (a kind of "discrete convolution") is a formal construct. However we do have the following result.

Theorem 4.8 (Mertens' Theorem). *If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and $\sum_{n=0}^{\infty} b_n$ is convergent, then*

$$\left(\sum_{n=0}^{\infty} a_n\right) * \left(\sum_{n=0}^{\infty} b_n\right) = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Exercise 4.9 (L2.2.1). Show that

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$$

if $|z| < 1$.

Solution. Suppose $|z| < 1$. From exercise La1.2.11 we have

$$\sum_{n=0}^{k} z^n = \frac{z^{k+1} - 1}{z - 1},$$

for all $k \geq 0$, so that

$$\sum_{n=0}^{\infty} z^n = \lim_{k\to\infty} \sum_{n=0}^{k} z^n = \lim_{k\to\infty} \frac{z^{k+1} - 1}{z - 1} = \frac{0-1}{z-1} = \frac{1}{1-z}.$$

Now,

$$\sum_{n=1}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n+1} = z\sum_{n=0}^{\infty} z^n = z \cdot \frac{1}{1-z} = \frac{z}{1-z}$$

as was to be shown. ■

Exercise 4.10 (L2.2.2). Show that if $r \in [0,1)$ and $\theta \in \mathbb{R}$, then

$$\sum_{n=0}^{\infty} r^n e^{in\theta} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}$$

converge, and find a formula for the value of each series.

Before proceeding it must be acknowledged that Lang has thus far neglected to give a definition for what it means for a series of the form $\sum_{n\in\mathbb{Z}} z_n$ to converge. Such a series converges if and only if

$$\sum_{n\geq 0} z_n \quad \text{and} \quad \sum_{n<0} z_n$$

both converge, in which case we define

$$\sum_{n\in\mathbb{Z}} z_n = \sum_{n\geq 0} z_n + \sum_{n<0} z_n.$$

Solution. Fix $r \in [0,1)$ and $\theta \in \mathbb{R}$. Since $|re^{i\theta}| = |r||e^{i\theta}| = |r| < 1$,

$$\sum_{n\geq 0} r^n e^{in\theta} = \sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} (re^{i\theta})^n = \frac{1}{1-re^{i\theta}}$$

(see the previous exercise).

Next, since $|re^{-i\theta}| < 1$ we have

$$\begin{aligned}\sum_{n<0} r^{|n|} e^{in\theta} &= \lim_{k\to\infty} \sum_{n=-1}^{-k} r^{|n|} e^{in\theta} = \lim_{k\to\infty} \sum_{n=-1}^{-k} r^{-n} e^{in\theta} = \lim_{k\to\infty} \sum_{n=1}^{k} r^n e^{-in\theta} \\ &= \sum_{n=1}^{\infty} (re^{-i\theta})^n = \frac{re^{-i\theta}}{1-re^{-i\theta}} = \frac{r}{e^{i\theta}-r}\end{aligned}$$

by Exercise 4.9.

Since the series $\sum_{n\geq 0}$ and $\sum_{n<0}$ both converge, we conclude that $\sum_{n\in\mathbb{Z}}$ also converges, and

$$\sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = \sum_{n\geq 0} r^{|n|} e^{in\theta} + \sum_{n<0} r^{|n|} e^{in\theta} = \frac{1}{1-re^{i\theta}} + \frac{r}{e^{i\theta}-r}$$

by definition. ■

Exercise 4.11 (L2.2.4h). Determine the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^n.$$

Solution. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| &= \lim_{n\to\infty}\left|\frac{[(n+1)!]^3 z^{n+1}}{[3(n+1)]!}\cdot\frac{(3n)!}{(n!)^3 z^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^3 z}{(3n+1)(3n+2)(3n+3)}\right| = \frac{1}{27}|z|,\end{aligned}$$

by Proposition 2.5(4) we have

$$\rho = \limsup_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \frac{1}{27}|z|$$

and so by the Ratio Test the series converges absolutely if $0 \le |z|/27 < 1$ (i.e. $0 \le |z| < 27$) and diverges if $|z|/27 > 1$ (i.e. $|z| > 27$). Therefore the radius of convergence of the series is 27. ■

Exercise 4.12 (L2.2.5c). Let $\sum a_n z^n$ have radius of convergence $r > 0$, where $a_n \neq 0$ for all but finitely many n. Show that $\sum n^d a_n z^n$ also has radius of convergence r for any $d \in \mathbb{N}$.

Solution. Let

$$\rho_1 = \limsup\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \limsup\left|\frac{a_{n+1}}{a_n}\right||z|.$$

By the Ratio Test $\sum a_n z^n$ converges absolutely if $0 \le \rho_1 < 1$ and diverges if $\rho_1 > 1$, and thus we see that $r = \rho_1$.

Next, let

$$\rho_2 = \limsup\left|\frac{(n+1)^d a_{n+1} z^{n+1}}{n^d a_n z^n}\right| = \limsup\left(\frac{(n+1)^d}{n^d}\left|\frac{a_{n+1}}{a_n}\right||z|\right).$$

The Ratio Test indicates that $\sum n^d a_n z^n$ converges absolutely if $0 \le \rho_2 < 1$ and diverges if $\rho_2 > 1$, and so the radius of convergence of $\sum n^d a_n z^n$ is ρ_2. We must show that $\rho_2 = \rho_1$.

By Proposition 2.5(2),

$$\begin{aligned}\rho_2 &= \limsup\left(\frac{(n+1)^d}{n^d}\cdot\left|\frac{a_{n+1}}{a_n}\right||z|\right) \\ &\le \limsup\frac{(n+1)^d}{n^d}\cdot\limsup\left|\frac{a_{n+1}}{a_n}\right||z| = \limsup\left|\frac{a_{n+1}}{a_n}\right||z| = \rho_1,\end{aligned}$$

and also

$$\begin{aligned}\rho_1 &= \limsup\left[\frac{n^d}{(n+1)^d}\cdot\frac{(n+1)^d}{n^d}\left|\frac{a_{n+1}}{a_n}\right||z|\right] \\ &\le \limsup\frac{n^d}{(n+1)^d}\cdot\limsup\left(\frac{(n+1)^d}{n^d}\left|\frac{a_{n+1}}{a_n}\right||z|\right) \\ &= \limsup\left(\frac{(n+1)^d}{n^d}\left|\frac{a_{n+1}}{a_n}\right||z|\right) = \rho_2.\end{aligned}$$

Hence $\rho_1 \le \rho_2$ and $\rho_2 \le \rho_1$, and so $\rho_2 = \rho_1 = r$. ■

Exercise 4.13 (L2.2.8). Let $(a_n)_{n=0}^{\infty}$ be a decreasing sequence in $\mathbb{R}_+$ that converges to 0. Prove that for all $\delta > 0$ the series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on

$$S_\delta = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } |z-1| \geq \delta\},$$

provided that $S_\delta \neq \varnothing$.

Solution. Let $\delta > 0$ such that $S_\delta \neq \varnothing$. Consider the sequence $(c_n)_{n=0}^{\infty}$ for which $c_n = a_n - a_{n+1}$. Since $a_{n+1} \leq a_n$ for all $n \geq 0$ we have

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} |a_n - a_{n+1}| = \lim_{n\to\infty} \sum_{k=0}^{n} (a_k - a_{k+1}) = \lim_{n\to\infty} (a_0 - a_{n+1}) = a_0,$$

so $\sum |c_n|$ converges and it follows that $\sum |2c_n/\delta|$ also converges.

Define the sequence of functions (f_n) on S_δ by

$$f_n(z) = \frac{z^{n+1}-1}{z-1} c_n$$

for all $n \geq 0$. We have

$$\begin{aligned} \|f_n\|_{S_\delta} = \sup_{z\in S_\delta} |f_n(z)| &= \sup_{z\in S_\delta} \left(\frac{|z^{n+1}-1|}{|z-1|} c_n \right) \leq \sup_{z\in S_\delta} \left(\frac{|z^{n+1}-1|}{\delta} c_n \right) \\ &\leq \sup_{z\in S_\delta} \left(\frac{|z|^{n+1}+1}{\delta} c_n \right) \leq \sup_{z\in S_\delta} \left| \frac{2c_n}{\delta} \right|, \end{aligned}$$

and since $\sum |2c_n/\delta|$ converges the Weierstrass M-Test implies that $\sum f_n$ converges uniformly on S_δ. That is, the sequence of functions (s_n), where

$$s_n = \sum_{k=0}^{n} f_k,$$

converges uniformly on S_δ.

Next, define functions $\sigma_n : S_\delta \to \mathbb{C}$ by

$$\sigma_n(z) = \frac{z^{n+1}-1}{z-1} a_n.$$

Let $\epsilon > 0$. Choose N such that $a_n < \delta\epsilon/4$ for all $n \geq N$. Let $n \geq N$. Then

$$\sup_{z\in S_\delta} |\sigma_n(z)| = \sup_{z\in S_\delta} \frac{|z^{n+1}-1|}{|z-1|} a_n \leq \sup_{z\in S_\delta} \frac{|z|^{n+1}+1}{\delta} a_n \leq \sup_{z\in S_\delta} \left(\frac{2}{\delta} a_n \right) \leq \frac{2}{\delta} \cdot \frac{\delta\epsilon}{4} < \epsilon,$$

which shows that (σ_n) converges uniformly on S_δ.

For each $n \geq 0$ define functions $\varphi_n, \tau_n : S_\delta \to \mathbb{C}$ by

$$\varphi_n(z) = \sum_{k=0}^{n} a_k z^k$$

and $\tau_n = s_{n-1}$ (with $\tau_0 \equiv 0$). Using summation by parts, we obtain

$$\varphi_n(z) = a_n \sum_{k=0}^{n} z^k + \sum_{k=0}^{n-1} \left((a_k - a_{k+1}) \sum_{j=0}^{k} z^j \right)$$

$$= \frac{z^{n+1}-1}{z-1}a_n + \sum_{k=0}^{n-1}\left(\frac{z^{k+1}-1}{z-1}c_k\right) = \sigma_n(z) + \tau_n(z)$$

for all $n \geq 0$ and $z \in S_\delta$. Thus, $\varphi_n = \sigma_n + \tau_n$, and since (σ_n) and (τ_n) converge uniformly on S_δ is follows by Proposition 2.51(2) that φ_n also converges uniformly on S_δ. Therefore $\sum a_n z^n$ converges uniformly on S_δ. ■

Exercise 4.14 (L2.2.9). Let $\sum_{n=0}^\infty a_n z^n$ have radius of convergence $r \geq 1$. If $\sum_{n=0}^\infty a_n$ converges, then

$$\lim_{x\to 1^-} \sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty a_n.$$

Solution. Suppose that $\sum a_n$ converges. It can be assumed that $r = 1$, so that $\sum a_n z^n$ converges absolutely for all $|z| < 1$, and in particular $\sum a_n x^n$ converges for all $x \in [0,1]$. Define functions $s_n : [0,1] \to \mathbb{C}$ by

$$s_n(x) = \sum_{k=0}^n a_k x^k$$

for $n \geq 0$. Also, for convenience, let

$$\sigma_n = s_n(1) = \sum_{k=0}^n a_k$$

for $n \geq 0$. We will use summation by parts to show the sequence (s_n) is Cauchy on $[0,1]$.

Let $\epsilon > 0$. We have

$$\lim_{n\to\infty} \sigma_n = \sum_{n=0}^\infty a_n = a$$

for some $a \in \mathbb{C}$, so there exists some N such that $|\sigma_n - a| < \epsilon/2$ for all $n \geq N$, and then for any $m, n > N$ we have

$$|\sigma_n - \sigma_m| \leq |\sigma_n - a| + |a - \sigma_m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Let $m, n \geq N$ with $n > m$. Then for any $x \in [0,1]$,

$$\begin{aligned}
s_n(x) - s_m(x) &= \sum_{k=m+1}^{n} a_k x^k = \sum_{k=0}^{n-m-1} a_{k+m+1}x^{k+m+1} \\
&= x^n \sum_{k=0}^{n-m-1} a_{k+m+1} - \sum_{k=0}^{n-m-2}\left[\left(x^{k+m+2} - x^{k+m+1}\right)\sum_{j=0}^{k} a_{j+m+1}\right] \\
&= x^n \sum_{k=m+1}^{n} a_k - \sum_{k=m+1}^{n-1}\left[\left(x^{k+1} - x^k\right)\sum_{j=m+1}^{k} a_j\right] \\
&= x^n(\sigma_n - \sigma_m) + \underbrace{\sum_{k=m+1}^{n-1}(x^k - x^{k+1})(\sigma_k - \sigma_m)}_{\text{Zero if } n = m+1}.
\end{aligned}$$

Now,

$$\begin{aligned}
\|s_n - s_m\|_{[0,1]} &= \sup_{x\in[0,1]} |s_n(x) - s_m(x)| \\
&\le \sup_{x\in[0,1]} \left(x^n|\sigma_n - \sigma_m| + \sum_{k=m+1}^{n-1} (x^k - x^{k+1})|\sigma_k - \sigma_m| \right) \\
&\le \sup_{x\in[0,1]} \left(x^n\epsilon + \sum_{k=m+1}^{n-1} \epsilon(x^k - x^{k+1}) \right) \\
&= \sup_{x\in[0,1]} \left[x^n\epsilon + (x^{m+1} - x^n)\epsilon\right] = \sup_{x\in[0,1]} (x^{m+1}\epsilon) = \epsilon,
\end{aligned}$$

and we conclude that (s_n) is uniformly Cauchy on $[0,1]$. By Theorem 2.52 it follows that (s_n) converges uniformly to some function $s : [0,1] \to \mathbb{C}$, and the proof of Theorem 2.52 establishes that s is given by

$$s(x) = \lim_{n\to\infty} s_n(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.1}$$

for each $x \in [0,1]$. Since (s_n) is clearly a sequence of continuous functions on $[0,1]$, by Theorem 2.54 we conclude that s is continuous on $[0,1]$, and so we have in particular

$$\lim_{x\to 1^-} s(x) = s(1).$$

Therefore

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \lim_{x\to 1^-} s(x) = s(1) = \sum_{n=0}^{\infty} a_n (1)^n = \sum_{n=0}^{\infty} a_n,$$

and the proof is finished. ■

Solution (**Alternate**). We can show directly that

$$\forall\epsilon > 0\, \exists\delta > 0\, \forall x \in [0,1) \left(1 - \delta < x < 1 \longrightarrow \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n \right| < \epsilon \right).$$

We start by establishing (as before) that (s_n) converges uniformly on $[0,1]$ to the function s given by (4.1), so that we have

$$\forall\epsilon > 0\, \exists N\, \forall n \ge N\, \forall x \in [0,1]\big(|s_n(x) - s(x)| < \epsilon\big).$$

Let $\epsilon > 0$ be arbitrary. There exists $N \in \mathbb{N}$ such that

$$\forall n \ge N\, \forall x \in [0,1]\big(|s_n(x) - s(x)| < \epsilon/3\big).$$

Choose $\delta > 0$ sufficiently small so that $1 - \delta < x < 1$ implies

$$|s_N(1) - s_N(x)| = \left| \sum_{n=0}^{N} a_n(1 - x^n) \right| < \epsilon/3.$$

Let $1-\delta < x < 1$ be arbitrary. Then

$$\begin{aligned}\left|\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n\right| &= |s(x) - s(1)| \\ &\leq |s(x) - s_N(x)| + |s_N(x) - s_N(1)| + |s_N(1) - s(1)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\end{aligned}$$

as desired. ■

Exercise 4.15 (L2.2.12). Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ defined recursively as follows:

$$a_n = a_{n-1} + a_{n-2}, \quad a_0 = 1,\ a_1 = 2. \tag{4.2}$$

Determine the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

Solution. From Exercise A.14 in the Appendix it is known that the difference equation in (4.2) has general solution $a_n = A\alpha^n + B\beta^n$ (A and B arbitrary constants) if α and β are distinct roots to the auxiliary equation

$$T^2 - T - 1 = 0.$$

Solving the auxiliary equation yields

$$T = \frac{1 \pm \sqrt{5}}{2},$$

and so

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We find a particular solution to the initial value problem (4.2) by applying the initial conditions $a_0 = 1$ and $a_1 = 2$ to obtain the system

$$\begin{cases} A + \phantom{\tfrac{1+\sqrt{5}}{2}} B = 1 \\ \tfrac{1+\sqrt{5}}{2}A + \tfrac{1+\sqrt{5}}{2}B = 2 \end{cases}$$

Solving the system yields

$$A = \frac{5+3\sqrt{5}}{10} \quad \text{and} \quad B = \frac{5-3\sqrt{5}}{10},$$

and so

$$a_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

To find the radius of convergence of $\sum a_n z^n$ we may employ the Ratio Test, since it is clear from (4.2) that $a_n \neq 0$ for all n. To start, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| &= \lim_{n\to\infty}\left|\frac{\left[(5+3\sqrt{5}\,)(1+\sqrt{5}\,)^{n+1} + (5-3\sqrt{5}\,)(1-\sqrt{5}\,)^{n+1}\right] z^{n+1}\cdot 10(2^n)}{10(2^{n+1})\cdot\left[(5+3\sqrt{5}\,)(1+\sqrt{5}\,)^n + (5-3\sqrt{5}\,)(1-\sqrt{5}\,)^n\right] z^n}\right| \\ &= \lim_{n\to\infty}\left(\left|\frac{(5+3\sqrt{5}\,)(1+\sqrt{5}\,)^{n+1} + (5-3\sqrt{5}\,)(1-\sqrt{5}\,)^{n+1}}{(5+3\sqrt{5}\,)(1+\sqrt{5}\,)^n + (5-3\sqrt{5}\,)(1-\sqrt{5}\,)^n}\right|\cdot\frac{|z|}{2}\right)\end{aligned}$$

$$= \lim_{n\to\infty}\left(\left|\frac{(20+8\sqrt{5}\,)+(20-8\sqrt{5}\,)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{(5+3\sqrt{5}\,)+(5-3\sqrt{5}\,)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}\right|\cdot\frac{|z|}{2}\right).$$

Observing that

$$\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = 0$$

since $|(1-\sqrt{5})/(1+\sqrt{5})| < 1$, we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \left|\frac{20+8\sqrt{5}}{5+3\sqrt{5}}\right|\cdot\frac{|z|}{2} = \frac{10+4\sqrt{5}}{5+3\sqrt{5}}|z|,$$

so by Proposition 2.5(4)

$$\limsup\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \frac{10+4\sqrt{5}}{5+3\sqrt{5}}|z|.$$

By the Ratio Test the series $\sum a_n z^n$ converges absolutely if

$$\frac{10+4\sqrt{5}}{5+3\sqrt{5}}|z| < 1,$$

or equivalently

$$|z| < \frac{\sqrt{5}-1}{2}.$$

The series diverges if $|z| > \frac{1}{2}(\sqrt{5}-1)$. Therefore the radius of convergence is $\frac{1}{2}(\sqrt{5}-1)$. ■

Exercise 4.16 (L2.2.13). Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ defined recursively as follows:

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}, \quad a_0 = a,\ a_1 = b. \tag{4.3}$$

Suppose that the difference equation's auxiliary equation,

$$T^2 - u_1 T - u_2 = 0,$$

has two distinct roots $\alpha, \beta \neq 0$, such that $|\alpha| < |\beta|$. Determine the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

Solution. From Exercise A.14 it is known that the difference equation in (4.3) has general solution $a_n = A\alpha^n + B\beta^n$, where A and B are arbitrary constants. The initial conditions $a_0 = a$ and $a_1 = b$ can be used to determine A and B and thereby obtain a particular solution to the initial value problem (refer to Equation (9) in the Appendix for the relevant formulas). To start we assume that $B \neq 0$. We have

$$\lim_{n\to\infty}\sqrt[n]{|a_n z^n|} = \lim_{n\to\infty}\sqrt[n]{\left|(A\alpha^n + B\beta^n)z^n\right|} = \lim_{n\to\infty}|\beta z|\left|A(\alpha/\beta)^n + B\right|^{1/n}. \tag{4.4}$$

Since $|\alpha| < |\beta|$ implies that $(\alpha/\beta)^n \to 0$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that $A(\alpha/\beta)^n + B \neq 0$ for all $n \geq N$, and thus

$$\left|A(\alpha/\beta)^n + B\right|^{1/n} = \exp\Big(\ln\left(\left|A(\alpha/\beta)^n + B\right|^{1/n}\right)\Big) = \exp\left(\frac{\ln\left|A(\alpha/\beta)^n + B\right|}{n}\right)$$

for all $n \geq N$. Now,

$$\lim_{n\to\infty} \ln \left|A(\alpha/\beta)^n + B\right| = \ln |B|,$$

whence

$$\lim_{n\to\infty} \frac{\ln \left|A(\alpha/\beta)^n + B\right|}{n} = 0$$

obtains, and so

$$\lim_{n\to\infty} \left|A(\alpha/\beta)^n + B\right|^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\ln \left|A(\alpha/\beta)^n + B\right|}{n}\right) = \exp(0) = 1.$$

Returning to (4.4), we finally have

$$\lim_{n\to\infty} \sqrt[n]{|a_n z^n|} = |\beta z| \lim_{n\to\infty} \left|A(\alpha/\beta)^n + B\right|^{1/n} = |\beta||z|,$$

and so by Proposition 2.5(4)

$$\limsup \sqrt[n]{|a_n z^n|} = |\beta||z|.$$

From this we conclude by the Root Test that the series $\sum a_n z^n$ converges absolutely if $|z| < 1/|\beta|$ and diverges if $|z| > 1/|\beta|$. Therefore the radius of convergence is $1/|\beta|$ if $B \neq 0$, regardless of whether $A = 0$ or $A \neq 0$.

If $B = 0$ and $A \neq 0$, then

$$\limsup \sqrt[n]{|a_n z^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n z^n|} = \lim_{n\to\infty} \sqrt[n]{\left|A\alpha^n z^n\right|} = \lim_{n\to\infty} |\alpha||z||A|^{1/n} = |\alpha||z|$$

and we conclude that the radius of convergence is $1/|\alpha|$.

The case $A = B = 0$ can only occur if $a_0 = a_1 = 0$, in which case $a_n = 0$ for all $n \geq 0$ and the series has radius of convergence ∞. ■

4.2 – Integral Formulas

Given $z_0 \in \mathbb{C}$ and $r > 0$, let $C_r(z_0)$ denote the circle with center z_0 and radius r; that is,

$$C_r(z_0) = \overline{B}_r(z_0) \setminus B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The circle $C_r(z_0)$ will be parameterized by the function

$$\gamma(t) = z_0 + re^{it}, \quad t \in [0, 2\pi], \tag{4.5}$$

unless otherwise specified. We define

$$\oint_{C_r(z_0)} f = \oint_\gamma f,$$

with γ the path given by (4.5).

Theorem 4.17 (**Cauchy's Integral Formula for a Circle**). *Let f be analytic on Ω, $z_0 \in \Omega$, and $r > 0$ such that $\overline{B}_r(z_0) \subseteq \Omega$. Then*

$$f(z) = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{w - z}\, dw$$

for all $z \in B_r(z_0)$.

Proof. The proof in [AN] is clear, in the main, save for the step

$$\oint_{C_r(z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw = \sum_{n=0}^{\infty} \left(\oint_{C_r(z_0)} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw \right). \tag{4.6}$$

Fix $z \in B_r(z_0)$, so that $|z - z_0| = \hat{r}$ for some $\hat{r} < r$, and for each $n \geq 0$ define $f_n : C_r(z_0) \to \mathbb{C}$ by

$$f_n(w) = \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Also define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = z_0 + re^{it}$, so that $\gamma^* = C_r(z_0)$ and $f_n \circ \gamma : [0, 2\pi] \to \mathbb{C}$ is given by

$$(f_n \circ \gamma)(t) = \frac{(z - z_0)^n}{r^{n+1}} e^{-it(n+1)}.$$

We see that $f_n \circ \gamma \in \mathcal{R}[0, 2\pi]$ for all $n \geq 0$, since $(z - z_0)^n / r^{n+1}$ (a constant) and $e^{-it(n+1)}$ are each integrable on $[0, 2\pi]$. Moreover, since

$$\sum_{n=0}^{\infty} \frac{1}{r} \left(\frac{\hat{r}}{r} \right)^n$$

is a convergent geometric series, and

$$\|f_n\| = \sup_{w \in C_r(z_0)} \left| \frac{(z - z_0)^n}{(w - z_0)^{n+1}} \right| = \sup_{w \in C_r(z_0)} \frac{|z - z_0|^n}{|w - z_0|^{n+1}} = \frac{1}{r} \left(\frac{\hat{r}}{r} \right)^n$$

for all $n \geq 0$, the Weierstrass M-Test implies that the series

$$\sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}$$

converges uniformly on $C_r(z_0)$. Equation (4.6) now follows by Corollary 3.38. ∎

Corollary 4.18. *If f is analytic on Ω and $\overline{B}_r(z_0) \subseteq \Omega$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt.$$

Proof. By Theorem 4.17, letting $\gamma(t) = z_0 + re^{it}$,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{w - z_0} dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0} \gamma'(t)dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{ire^{it} f(z_0 + re^{it})}{re^{it}} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt, \end{aligned}$$

as desired. ∎

Theorem 4.19. *Let γ be a path, $g : \gamma^* \to \mathbb{C}$ a continuous function, and $\Omega = \mathbb{C} \setminus \gamma^*$. Define $F : \Omega \to \mathbb{C}$ by*

$$F(z) = \int_\gamma \frac{g(w)}{w - z} dw.$$

Then F has derivatives of all orders on Ω,

$$F^{(n)}(z) = n! \int_\gamma \frac{g(w)}{(w-z)^{n+1}} dw$$

for all $z \in \Omega$ and $n \in \mathbb{N}$, and $F^{(n)}(z) \to 0$ as $|z| \to \infty$.

Corollary 4.20. *If f is analytic on Ω, then f has derivatives of all orders on Ω, and for any $z_0 \in \Omega$ and $r > 0$ such that $\overline{B}_r(z_0) \subseteq \Omega$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}$.

Proof. Suppose f is analytic on Ω. Fix $z_0 \in \Omega$, and let $r > 0$ be such that $\overline{B}_r(z_0) \subseteq \Omega$. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by (4.5), so that $\gamma^* = C_r(z_0) \subseteq \Omega$, and define $g : \gamma^* \to \mathbb{C}$ by $g(w) = f(z)/2\pi i$. Since f is analytic—and hence continuous—on Ω, we see that g is continuous on γ^*. If we define $F : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ by

$$F(z) = \oint_\gamma \frac{g(w)}{w - z} dw,$$

then F has derivatives of all orders on $\mathbb{C} \setminus \gamma^*$ by Theorem 4.19. Now, since $B_r(z_0) \subseteq \mathbb{C} \setminus \gamma^*$, by Theorem 4.17

$$f(z) = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{w - z} dw = \oint_{C_r(z_0)} \frac{f(w)/2\pi i}{w - z} dw = \oint_\gamma \frac{g(w)}{w - z} dw = F(z)$$

for all $z \in B_r(z_0)$, and so f has derivatives of all orders on $B_r(z_0)$ such that

$$f^{(n)}(z) = F^{(n)}(z) = n! \oint_\gamma \frac{g(w)}{(w-z)^{n+1}}\,dw = \frac{n!}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}$.

Finally, we observe that f has derivatives of all orders at z_0 in particular, and since $z_0 \in \Omega$ is arbitrary, it follows that f has derivatives of all orders on Ω. ∎

Corollary 4.21. *If f has a primitive on Ω, then f is analytic on Ω.*

Proof. Suppose f has a primitive on Ω. Thus there exists $F : \Omega \to \mathbb{C}$ with $F' = f$. Since F is analytic on Ω, by Corollary 4.20 it has derivatives of all orders, and so $f' = F''$ exists on Ω. Therefore f is analytic on Ω. ∎

Corollary 4.22. *If f is continuous on Ω and analytic on $\Omega \setminus \{z_0\}$, then f is analytic on Ω.*

Proof. Suppose f is continuous on Ω and analytic on $\Omega \setminus \{z_0\}$. Fix $z \in \Omega$, and let $\epsilon > 0$ be such that $B_\epsilon(z) \subseteq \Omega$. If $z_0 \notin B_\epsilon(z)$, then the analyticity of f on $B_\epsilon(z)$ is immediate. Suppose $z_0 \in B_\epsilon(z)$, then since $B_\epsilon(z)$ is a starlike region, f is continuous on $B_\epsilon(z)$, and f is analytic on $B_\epsilon(z) \setminus \{z_0\}$, Theorem 3.49(2) implies that f has a primitive on $B_\epsilon(z)$. Now Corollary 4.21 implies that f is analytic on $B_\epsilon(z)$, and in particular analytic at z. Since $z \in \Omega$ is arbitrary, we conclude that f is analytic on Ω. ∎

Theorem 4.23 (**Morera's Theorem**)**.** *If f is continuous on Ω and $\oint_\Delta f = 0$ for every triangle Δ such that* $\mathrm{Conv}(\Delta) \subseteq \Omega$, *then f is analytic on Ω.*

Proof. Suppose f is continuous on Ω and $\oint_\Delta f = 0$ for every triangle Δ such that $\mathrm{Conv}(\Delta) \subseteq \Omega$. Fix $z_0 \in \Omega$, and choose $\epsilon > 0$ so that $\Omega' := B_\epsilon(z_0) \subseteq \Omega$. Then Ω' is starlike with star center z_0. Since f is continuous on Ω', we may define $F : \Omega' \to \mathbb{C}$ by

$$F(z) = \int_{[z_0,z]} f.$$

Fix $w \in \Omega'$, and let $r > 0$ be such that $B_r(w) \subseteq \Omega'$. For all $z \in B'_r(w)$ we have

$$\frac{F(z) - F(w)}{z-w} = \frac{1}{z-w}\left(\int_{[z_0,z]} f - \int_{[z_0,w]} f\right).$$

Since $\mathrm{Conv}[z, z_0, w, z] \subseteq \Omega$, the equation (3.31) in the proof of Cauchy's Theorem for Starlike Regions holds,[2] and as in that proof we come to conclude that F is a primitive for f on Ω'. Hence f is analytic on $B_\epsilon(z_0)$ by Corollary 4.21, implying that f is analytic at $z_0 \in \Omega$, and therefore f is analytic on Ω. ∎

[2] In the proof of Theorem 3.40 the hypothesized analyticity of f enabled the use of Goursat's Theorem to obtain (3.31).

We define $\mathbb{H}$ to be the **upper half-plane** of $\mathbb{C}$, and $\mathbb{C} \setminus \overline{\mathbb{H}}$ the **lower half-plane**; that is,

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \quad \text{and} \quad \mathbb{C} \setminus \overline{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im} z < 0\},$$

where of course $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$ is the closure of the upper half-plane.

Theorem 4.24 (Schwarz Reflection Principle). *Suppose $f : \overline{\mathbb{H}} \to \mathbb{C}$ is continuous on $\overline{\mathbb{H}}$, analytic on $\mathbb{H}$, and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Then f has an analytic extension to all of $\mathbb{C}$.*

A full proof of the Schwarz Reflection Principle is furnished by Exercises 4.25 and 4.26 below, after which we apply results of this section to a matter concerning harmonic functions.

Exercise 4.25 (AN2.2.10). Suppose $f : \overline{\mathbb{H}} \to \mathbb{C}$ is continuous on $\overline{\mathbb{H}}$, analytic on $\mathbb{H}$, and $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Define $f^* : \mathbb{C} \to \mathbb{C}$ by

$$f^*(z) = \begin{cases} f(z), & z \in \overline{\mathbb{H}} \\ \overline{f(\overline{z})}, & z \in \mathbb{C} \setminus \overline{\mathbb{H}} \end{cases}$$

Show that f^* is analytic on $\mathbb{C} \setminus \mathbb{R}$ and continuous on $\mathbb{C}$.

Solution. Clearly f^* is continuous on $\mathbb{H}$. Fix $z \in \mathbb{C} \setminus \overline{\mathbb{H}}$, and let $\epsilon > 0$ be arbitrary. Since $\overline{z}$ is in the open set $\mathbb{H}$, and f is continuous on $\mathbb{H}$, there exists some $\delta > 0$ such that $B_\delta(\overline{z}) \subseteq \mathbb{H}$, and $w \in B_\delta(\overline{z})$ implies $|f(w) - f(\overline{z})| < \epsilon$. Suppose that $w \in B_\delta(z)$, where $B_\delta(z) \subseteq \mathbb{C} \setminus \overline{\mathbb{H}}$. Then $\overline{w} \in B_\delta(\overline{z})$, and so $|f(\overline{w}) - f(\overline{z})| < \epsilon$. Now,

$$|f^*(w) - f^*(z)| = \left|\overline{f(\overline{w})} - \overline{f(\overline{z})}\right| = \left|\overline{f(\overline{w}) - f(\overline{z})}\right| = |f(\overline{w}) - f(\overline{z})| < \epsilon$$

and we conclude that f^* is continuous on $\mathbb{C} \setminus \overline{\mathbb{H}}$.

Fix $x \in \mathbb{R}$, and again let $\epsilon > 0$. Since f is continuous on $\overline{\mathbb{H}}$, there exists some $\delta > 0$ such that $w \in B_\delta(x) \cap \overline{\mathbb{H}}$ implies $|f(w) - f(x)| < \epsilon$. Suppose that $w \in B_\delta(x)$. If $w \in \overline{\mathbb{H}}$, then $|f(w) - f(x)| < \epsilon$, and hence $|f^*(w) - f^*(x)| < \epsilon$. Suppose that $w \notin \overline{\mathbb{H}}$. Then $w \in \mathbb{C} \setminus \overline{\mathbb{H}}$, so that $\overline{w} \in \mathbb{H}$. Since

$$|\overline{w} - x| = \left|\overline{\overline{w} - x}\right| = |w - \overline{x}| = |w - x| < \delta,$$

we see that $\overline{w} \in B_\delta(x) \cap \overline{\mathbb{H}}$ and therefore $|f(\overline{w}) - f(x)| < \epsilon$. Now, recalling that $x, f(x) \in \mathbb{R}$, we obtain

$$|f^*(w) - f^*(x)| = \left|\overline{f(\overline{w})} - \overline{f(x)}\right| = \left|\overline{f(\overline{w}) - f(x)}\right| = |f(\overline{w}) - f(x)| < \epsilon$$

once again, and we conclude that f^* is continuous on $\mathbb{R}$. Therefore f^* is continuous on $\mathbb{C}$.

Clearly f^* is analytic on $\mathbb{H}$. Fix $z \in \mathbb{C} \setminus \overline{\mathbb{H}}$. Then $\overline{z} \in \mathbb{H}$, and so

$$f'(\overline{z}) = \lim_{h \to 0} \frac{f(\overline{z} + h) - f(\overline{z})}{h} \in \mathbb{C}.$$

Fix $\epsilon > 0$. Then there exists some $\delta > 0$ such that

$$0 < |h| < \delta \;\; \Rightarrow \;\; \left|\frac{f(\overline{z} + h) - f(\overline{z})}{h} - f'(\overline{z})\right| < \epsilon.$$

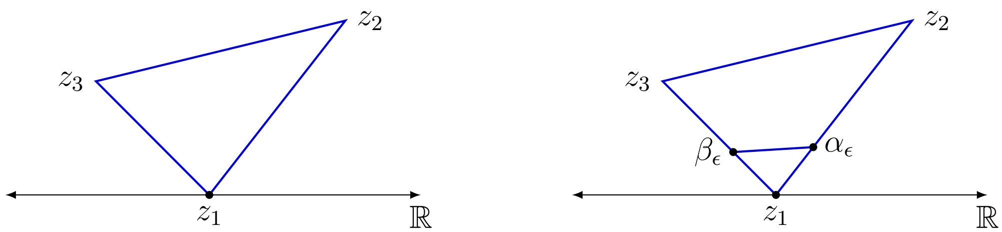

FIGURE 8.

Now, $0 < |h| < \delta$ if and only if $0 < |\overline{h}| < \delta$, and so we obtain

$$\left| \frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}} - f'(\overline{z}) \right| < \epsilon,$$

which yields

$$\left| \frac{\overline{f\left(\overline{z+h}\right)} - \overline{f(\overline{z})}}{h} - \overline{f'(\overline{z})} \right| < \epsilon$$

by taking the conjugate of the number inside the absolute value. Assuming δ to be sufficiently small so that $z + h \in \mathbb{C} \setminus \overline{\mathbb{H}}$, we then have

$$\left| \frac{f^*(z+h) - f^*(z)}{h} - \overline{f'(\overline{z})} \right| < \epsilon.$$

Therefore

$$(f^*)'(z) = \lim_{h \to 0} \frac{f^*(z+h) - f^*(z)}{h} = \overline{f'(\overline{z})} \in \mathbb{C},$$

which shows that f^* is differentiable at z, and therefore f^* is analytic on $\mathbb{C} \setminus \overline{\mathbb{H}}$. ■

Exercise 4.26 (AN2.2.11). Prove that the function f^* defined in the previous exercise is analytic on $\mathbb{C}$.

Solution. It has already been shown that f^* is continuous on $\mathbb{C}$. If we can show that $\int_\Delta f^* = 0$ for every triangle $\Delta = [z_1, z_2, z_3, z_1]$, then the desired result will follow from Morera's Theorem. Since f^* is already known to be analytic on $\mathbb{H}$ and $\mathbb{C} \setminus \overline{\mathbb{H}}$, we have $\int_\Delta f^* = 0$ for any Δ in $\mathbb{H}$ or $\mathbb{C} \setminus \overline{\mathbb{H}}$ by Goursat's Theorem. The degenerate case when z_1, z_2, and z_3 are collinear is easily dispatched with Proposition 3.30, and so it remains to analyze triangles Δ for which $\Delta \cap \mathbb{R} \neq \varnothing$. There are three cases: (I) $\Delta \cap \mathbb{R}$ contains one point, (II) two points, or (III) a line segment.

Case (I) occurs when all but a single vertex z_1 of Δ is in $\mathbb{H}$ or $\mathbb{C} \setminus \overline{\mathbb{H}}$. Suppose $\Delta \setminus \{z_1\} \subseteq \mathbb{H}$, as shown at left in Figure 8. Let $\epsilon > 0$ and choose points $\alpha_\epsilon \in [z_1, z_2]$ and $\beta_\epsilon \in [z_1, z_3]$ such that $d(\alpha_\epsilon, z_1) = d(\beta_\epsilon, z_1) = \epsilon$, as at right in Figure 8, so that $d(\alpha_\epsilon, \beta_\epsilon) < 2\epsilon$. Also let

$$M = \sup_{z \in \operatorname{Conv} \Delta} |f^*(z)|,$$

which is a real number since f^* is continuous on the compact set $\operatorname{Conv}(\Delta)$. Let γ be the closed path $[\alpha_\epsilon, \beta_\epsilon, z_3, z_2, \alpha_\epsilon]$. Since f^* is analytic on the starlike region $\mathbb{H}$ and $\gamma^* \subseteq \mathbb{H}$, by Cauchy's

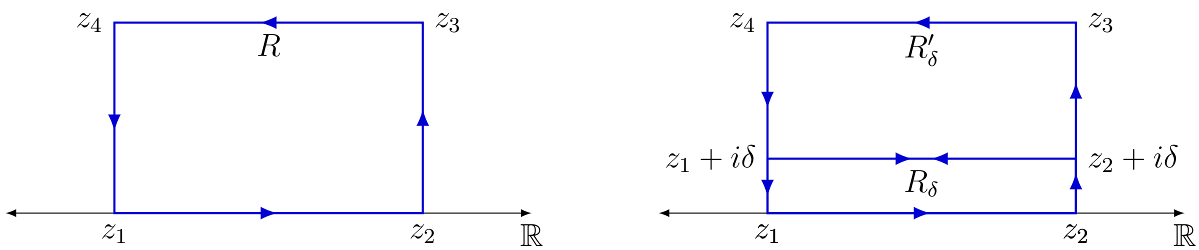

FIGURE 9.

Theorem for Starlike Regions we obtain $\int_\gamma f^* = 0$. This, along with Proposition 3.30, implies that

$$\int_{[z_2,z_3]} f^* = \int_{[\alpha_\epsilon,\beta_\epsilon]} f^* - \int_{[z_3\beta_\epsilon]} f^* - \int_{[\alpha_\epsilon,z_2]} f^*.$$

Substituting this into

$$\oint_\Delta f^* = \int_{[z_1,\alpha_\epsilon]} f^* + \int_{[\alpha_\epsilon,z_2]} f^* + \int_{[z_2,z_3]} f^* + \int_{[z_3,\beta_\epsilon]} f^* + \int_{[\beta_\epsilon,z_1]} f^*$$

yields

$$\oint_\Delta f^* = \int_{[z_1,\alpha_\epsilon]} f^* + \int_{[\alpha_\epsilon,\beta_\epsilon]} f^* + \int_{[\beta_\epsilon,z_1]} f^* = \oint_{\Delta_\epsilon} f^*,$$

where we define $\Delta_\epsilon = [z_1, \alpha_\epsilon, \beta_\epsilon, z_1]$. By Theorem 3.23,

$$\left|\oint_\Delta f^*\right| = \left|\oint_{\Delta_\epsilon} f^*\right| \le \mathcal{L}(\Delta_\epsilon) \sup_{z\in\Delta_\epsilon} |f^*(z)| \le 4M\epsilon,$$

and since $\epsilon > 0$ is arbitrary we conclude that $\int_\Delta f^* = 0$. By symmetry the same holds if $\Delta \setminus \{z_1\} \subseteq \mathbb{C} \setminus \overline{\mathbb{H}}$.

Before addressing Case (II), we first consider a rectangular path $R = [z_1, z_2, z_3, z_4, z_1]$ in $\overline{\mathbb{H}}$ such that $[z_1, z_2] \subseteq \mathbb{R}$, as shown at left in Figure 9. For small $\delta > 0$ define the rectangular paths

$$R_\delta = [z_1, z_2, z_2 + i\delta, z_1 + i\delta, z_1] \quad \text{and} \quad R'_\delta = [z_1 + i\delta, z_2 + i\delta, z_3, z_4, z_1 + i\delta],$$

shown at right in Figure 9. It's straightforward to verify that

$$\oint_R f^* = \oint_{R_\delta} f^* + \oint_{R'_\delta} f^* = \oint_{R_\delta} f^*,$$

where

$$\oint_{R'_\delta} f^* = 0$$

by Cauchy's Theorem for Starlike Regions since the closed path R'_δ lies in $\mathbb{H}$ and f^* is analytic on $\mathbb{H}$. Fix $\epsilon > 0$. Let

$$M = \sup_{z\in\operatorname{Conv} R} |f^*(z)|.$$

Since f^* is uniformly continuous on $\operatorname{Conv}(R)$, there exists some $0 < \delta < \epsilon$ such that $z, w \in R$ with $|z - w| < 2\delta$ implies that

$$|f^*(z) - f^*(w)| < \frac{\epsilon}{2|z_2 - z_1|}.$$

Defining $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ by

$$\gamma_1(t) = (1-t)z_1 + tz_2 \quad \text{and} \quad \gamma_2(t) = (1-t)(z_1 + i\delta) + t(z_2 + i\delta),$$

and observing that $\gamma_1(t), \gamma_2(t) \in R$ are such that

$$|\gamma_2(t) - \gamma_1(t)| = |i\delta| = \delta$$

for all $0 \leq t \leq 1$, we obtain

$$\begin{aligned}\left|\int_{[z_1,z_2]} f^* - \int_{[z_1+i\delta, z_2+i\delta]} f^*\right| &= \left|\int_0^1 f^*(\gamma_1(t))(z_2 - z_1)dt - \int_0^1 f^*(\gamma_2(t))(z_2 - z_1)dt\right| \\ &= |z_2 - z_1| \left|\int_0^1 \big[f^*(\gamma_1(t)) - f^*(\gamma_2(t))\big] dt\right| \\ &\leq |z_2 - z_1| \int_0^1 \big|f^*(\gamma_1(t)) - f^*(\gamma_2(t))\big| dt \\ &\leq |z_2 - z_1| \cdot \frac{\epsilon}{2|z_2 - z_1|} < \epsilon.\end{aligned}$$

Applying Theorem 3.23,

$$\begin{aligned}\left|\oint_R f^*\right| = \left|\oint_{R_\delta} f^*\right| &= \left|\int_{[z_1,z_2]} f^* + \int_{[z_2,z_2+i\delta]} f^* + \int_{[z_2+i\delta, z_1+i\delta]} f^* + \int_{[z_1+i\delta, z_1]} f^*\right| \\ &\leq \left|\int_{[z_1,z_2]} f^* - \int_{[z_1+i\delta,z_2+i\delta]} f^*\right| + \left|\int_{[z_1,z_1+i\delta]} f^*\right| + \left|\int_{[z_2,z_2+i\delta]} f^*\right| \\ &< \epsilon + M|i\delta| + M|i\delta| = \epsilon + 2M\delta \leq \epsilon + 2M\epsilon = (1+2M)\epsilon.\end{aligned}$$

Since

$$\left|\oint_R f^*\right| < (1+2M)\epsilon$$

for arbitrary $\epsilon > 0$, we conclude that $\int_R f^* = 0$. By symmetry the same holds if $R \subseteq \overline{\mathbb{C} \setminus \overline{\mathbb{H}}}$.

Next, let $\Delta = [z_1, z_2, z_3, z_1]$ be a right triangle in $\overline{\mathbb{H}}$ such that $[z_1, z_2] \subseteq \mathbb{R}$ and $\operatorname{Re} z_2 = \operatorname{Re} z_3$. Set $z_4 = z_1 + \operatorname{Im} z_3$, so that $R = [z_1, z_2, z_3, z_4, z_1]$ is a rectangle with side $[z_1, z_2]$ in $\mathbb{R}$. Defining $\Delta' = [z_1, z_3, z_4, z_1]$, we obtain

$$\oint_R f^* = \oint_\Delta f^* + \oint_{\Delta'} f^*,$$

but since

$$\oint_R f^* = 0 \quad \text{and} \quad \oint_{\Delta'} f^* = 0$$

(Δ' is a Case (I) triangle), it follows that

$$\oint_\Delta f^* = 0. \tag{4.7}$$

We now know that every right triangle $\Delta \subseteq \overline{\mathbb{H}}$, including any with a vertex or a side in $\mathbb{R}$, must satisfy (4.7). By symmetry the same holds if $\Delta \subseteq \overline{\mathbb{C} \setminus \overline{\mathbb{H}}}$.

Now we deal with Case (II). If Δ is any arbitrary triangle in $\overline{\mathbb{H}}$ such that $[z_1, z_2] \subseteq \mathbb{R}$, we can construct two right triangles Δ_1 and Δ_2 such that the three triangles taken together form a

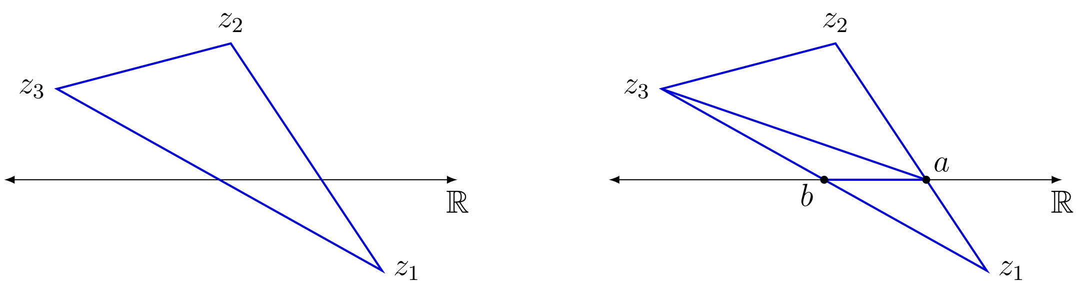

Figure 10.

rectangle R with bottom side in $\mathbb{R}$. Thus, applying (4.7) to the path integrals over Δ_1 and Δ_2, we obtain

$$\oint_\Delta f^* = \oint_\Delta f^* + \oint_{\Delta_1} f^* + \oint_{\Delta_2} f^* = \oint_R f^* = 0,$$

as desired. We now know that every arbitrary triangle $\Delta \subseteq \overline{\mathbb{H}}$ must satisfy (4.7). By symmetry the same is true if $\Delta \subseteq \overline{\mathbb{C} \setminus \overline{\mathbb{H}}}$.

Finally we examine the case when $\Delta \cap \mathbb{R}$ consists of precisely two points. This can only occur if $\overline{\mathbb{H}}$ contains two vertices of Δ and $\mathbb{C} \setminus \overline{\mathbb{H}}$ one vertex, or $\overline{\mathbb{C} \setminus \overline{\mathbb{H}}}$ contains two vertices and $\mathbb{H}$ one vertex. We will examine the case shown at left in Figure 10, where two vertices lie in $\mathbb{H}$ and one in $\mathbb{C} \setminus \overline{\mathbb{H}}$. Other scenarios (including those in which a vertex lies on $\mathbb{R}$) are handled similarly. Set $\Delta = [z_1, z_2, z_3, z_1]$, with $z_1 \in \mathbb{C} \setminus \overline{\mathbb{H}}$ and $z_2, z_3 \in \mathbb{H}$. Let $a \in [z_1, z_2] \cap \mathbb{R}$ and $b \in [z_2, z_3] \cap \mathbb{R}$ be the two points of intersection between Δ and $\mathbb{R}$. Define the triangles

$$\Delta_1 = [a, z_2, z_3, a], \quad \Delta_2 = [a, z_3, z_1, a], \quad \text{and} \quad \Delta_3 = [a, z_3, b, a],$$

as at right in Figure 10. For each k, Δ_k is a triangle that is a subset of either $\overline{\mathbb{H}}$ or $\overline{\mathbb{C} \setminus \overline{\mathbb{H}}}$, and so

$$\oint_{\Delta_k} f^* = 0.$$

Therefore

$$\oint_\Delta f^* = \oint_{\Delta_1} f^* + \oint_{\Delta_2} f^* + \oint_{\Delta_3} f^* = 0.$$

By symmetry the same outcome will be realized if $z_1 \in \mathbb{H}$ and $z_2, z_3 \in \mathbb{C} \setminus \overline{\mathbb{H}}$.

At this juncture we have shown that $\int_\Delta f^* = 0$ for any triangle $\Delta = [z_1, z_2, z_3, z_1] \subseteq \mathbb{C}$. Therefore f^* is analytic on $\mathbb{C}$ by Morera's Theorem. ∎

Exercise 4.27 (AN2.2.15). Use Morera's Theorem to prove the following. Suppose $f_n : \Omega \to \mathbb{C}$ is analytic on Ω for each $n \in \mathbb{N}$. If $f : \Omega \to \mathbb{C}$ is such that the sequence $(f_n|_K)$ converges uniformly to $f|_K$ on every compact set $K \subseteq \Omega$, then f is analytic on Ω.

Solution. Fix $z \in \Omega$. If $r > 0$ is such that $B_{2r}(z) \subseteq \Omega$, then $K = \overline{B}_r(z)$ is a compact subset of Ω so that $(f_n|_K)$ converges uniformly to $f|_K$. Since each $f_n|_K$ is continuous on K, by Theorem 2.54 it follows that $f|_K$ is also continuous on K. Hence f is continuous at z, and since $z \in \Omega$ is arbitrary we conclude that f is continuous on Ω.

Let $\Delta = [z_0, z_1, z_2, z_0] \subseteq \Omega$ such that $\blacktriangle = \text{Conv}(\Delta) \subseteq \Omega$. It is clear that $\int_\Delta f = 0$ if $\Delta = \{z_0\}$, so we discount the case when Δ is a point and assume that $\mathcal{L}(\Delta) > 0$. Fix $\epsilon > 0$. By Goursat's Theorem we have

$$\oint_\Delta f_n = 0$$

for all $n \in \mathbb{N}$. Also, since $f_n \to f$ uniformly on $\blacktriangle$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $z \in \blacktriangle$,

$$|f(z) - f_n(z)| < \frac{\epsilon}{\mathcal{L}(\Delta)}.$$

Now, letting $n \geq n_0$ and applying Theorem 3.23, we obtain

$$\begin{aligned} \left|\oint_\Delta f\right| &= \left|\oint_\Delta f - \oint_\Delta f_n\right| = \left|\oint_\Delta (f - f_n)\right| \leq \mathcal{L}(\Delta) \sup_{z\in\Delta} \big|(f - f_n)(z)\big| \\ &\leq \mathcal{L}(\Delta) \sup_{z\in\blacktriangle} \big|(f - f_n)(z)\big| \leq \mathcal{L}(\Delta) \cdot \frac{\epsilon}{\mathcal{L}(\Delta)} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\oint_\Delta f = 0.$$

It is now established that $f : \Omega \to \mathbb{C}$ is a continuous function such that $\int_\Delta f = 0$ for every triangle Δ such that $\text{Conv}(\Delta) \subseteq \Omega$. Therefore f is analytic on Ω by Morera's Theorem. ∎

Theorem 4.28. *Let $u, v : \Omega \to \mathbb{R}$. If $f = u + iv$ is analytic on Ω, then u and v are harmonic on Ω.*

Proof. Suppose $f = u+iv$ is analytic on Ω. Then, by Theorem 3.7, $u_x = v_y$ and $u_y = -v_x$ with $f' = u_x + iv_x = u_x - iu_y$ on Ω. Since f has derivatives of all orders by Corollary 4.20, both f' and f'' are differentiable—and hence continuous—on Ω. By Theorem 2.23 it follows that the real and imaginary parts of f' and f'' are continuous on Ω. Thus u_x and u_y are continuous on Ω in particular. Applying Theorem 3.7 to f', we have $f'' = u_{xx} + iv_{xx}$ with $u_{xx} = v_{xy}$ and $u_{xy} = -v_{xx}$, which shows that u_{xx} and u_{xy} are continuous. Making use of the Cauchy-Riemann equations, from $u_{yx} = -v_{xx}$ and $u_{yy} = -v_{xy} = -u_{xx}$ we see that u_{yx} and u_{yy} are also continuous. Hence all first- and second-order partial derivatives of u are continuous, and also $u_{xx} + u_{yy} = 0$, so that u is seen to be harmonic on Ω. Similar arguments will show v to be likewise harmonic. ∎

4.3 – Taylor Series Representation

A function f is **representable by power series** on Ω if, for each $z_0 \in \Omega$, there exists a sequence $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$ such that, for any $r > 0$ with $B_r(z_0) \subseteq \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in B_r(z_0)$.

Theorem 4.29. *If f is analytic on Ω, then f is representable by power series on Ω. In fact, for any $B_r(z_0) \subseteq \Omega$ we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{4.8}$$

*for all $z \in B_r(z_0)$. The series, called the **Taylor series** for f at z_0, converges absolutely on $B_r(z_0)$ and uniformly on compact subsets of $B_r(z_0)$.*

Proof. Suppose f is analytic on Ω, fix $z_0 \in \Omega$, and define the sequence $(a_n)_{n=0}^{\infty}$ by $a_n = f^{(n)}(z_0)/n!$. Let $r > 0$ be such that $B_(z_0) \subseteq \Omega$, and let $z \in B_r(z_0)$. Let $|z - z_0| < \rho < r$, so

$$z \in B_\rho(z_0) \subseteq \overline{B}_\rho(z_0) \subseteq B_r(z_0) \subseteq \Omega.$$

By Theorem 4.17,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(w)}{w - z}\,dw = \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(w)}{(w - z_0) - (z - z_0)}\,dw \\ &= \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}\,dw = \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty}\left(\frac{z - z_0}{w - z_0}\right)^n dw, \end{aligned}$$

since

$$\left|\frac{z - z_0}{w - z_0}\right| = \frac{|z - z_0|}{|w - z_0|} < \frac{\rho}{r} < 1.$$

Making use of Corollary 3.38 in the same manner as in the proof of Theorem 4.17, we next obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \sum_{n=0}^{\infty}\left[\frac{f(w)}{w - z_0}\left(\frac{z - z_0}{w - z_0}\right)^n\right] dw = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{C_\rho(z_0)} \frac{f(w)}{w - z_0}\left(\frac{z - z_0}{w - z_0}\right)^n dw \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(w)}{(w - z_0)^{n+1}}\,dw\right](z - z_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \end{aligned}$$

the last equality given by Corollary 4.20. This shows that f is representable by power series on Ω, and specifically it is representable by the power series in (4.8).

Now, since (4.8) holds for all $z \in B_r(z_0)$, it is immediate that the series is convergent on $B_r(z_0)$, and by Proposition 4.3 it follows that the series converges absolutely on $B_r(z_0)$. Thus by Theorem 4.2(1) the radius of convergence of the series is at least r, and therefore by Theorem 4.2(4) we conclude that the series converges uniformly on compact subsets of $B_r(z_0)$. ∎

Theorem 4.30. *Suppose (f_n) is a sequence of analytic functions on Ω. Suppose $f : \Omega \to \mathbb{C}$ is such that $f_n \xrightarrow{\text{u}} f$ on compact subsets of Ω. Then f is analytic on Ω, and $f_n^{(k)} \xrightarrow{\text{u}} f^{(k)}$ on compact subsets of Ω for each $k \in \mathbb{N}$.*

Proposition 4.31. *Let $r > 0$, and let Ω be an open set.*
1. *If f is a complex-valued function given by*
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
for all $z \in B_r(z_0)$, then f is analytic on $B_r(z_0)$.
2. *If f is representable on Ω by power series, then f is analytic on Ω.*

Proof.
Proof of Part (1). By Theorem 4.2 the given series converges uniformly on compact subsets of $B_r(z_0)$, which is to say if (σ_n) is the sequence of functions $B_r(z_0) \to \mathbb{C}$ given by
$$\sigma_n(z) = \sum_{k=0}^{n} a_k (z - z_0)^k,$$
then on each compact $K \subseteq B_r(z_0)$ the sequence (σ_n) converges uniformly to some function $\sigma_K : K \to \mathbb{C}$. Now, since for each $z \in B_r(z_0)$ we have
$$\lim_{n\to\infty} \sigma_n(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = f(z),$$
so that (σ_n) is seen to converge pointwise to f on $B_r(z_0)$, it follows that $\sigma_K = f|_K$, and hence (σ_n) converges uniformly to f on every compact subset of $B_r(z_0)$. Clearly each σ_n, being a polynomial function, is analytic on $B_r(z_0)$, and thus the function f is analytic on $B_r(z_0)$ by Theorem 4.30.

Proof of Part (2). Suppose f is representable on Ω by power series. Fix $z_0 \in \Omega$. There exists some $\epsilon > 0$ such that $B_\epsilon(z_0) \subseteq \Omega$, and so there exists a sequence $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$ such that
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
for all $z \in B_\epsilon(z_0)$. By Part (1) it follows that f is analytic on $B_\epsilon(z_0)$, and so in particular f is complex-differentiable at z_0. Since $z_0 \in \Omega$ is arbitrary we conclude that f is analytic on Ω. ∎

Suppose f is analytic on Ω. Then f has derivatives of all orders on Ω by Corollary 4.20, and so $f^{(k)}$ is analytic on Ω for any $k \in \mathbb{N}$. It follows by Theorem 4.29 that for any $B_r(z_0) \subseteq \Omega$ we have
$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(f^{(k)})^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(z_0)}{n!}(z - z_0)^n.$$
Reindexing and applying a little algebra then gives
$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{(n-k)!}(z - z_0)^{n-k} = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} n(n-1)\cdots(n-k+1)(z - z_0)^{n-k}.$$

In particular we have

$$f'(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} n(z - z_0)^{n-1}, \tag{4.9}$$

which shows that on $B_r(z_0)$ the series for $f'(z)$ may be obtained from the series for $f(z)$ by differentiating termwise:

$$f'(z) = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \frac{d}{dz}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} n(z - z_0)^{n-1},$$

where (4.9) follows by noting that the 0th term is zero.

The next theorem establishes the same series formula for $f^{(k)}$ derived above, but also makes clear that the coefficients of the series found for f in Theorem 4.29 are unique. That is, if f is given by a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on $B_r(z_0) \subseteq \Omega$, then the coefficients a_n can only be $f^{(n)}(z_0)/n!$ for $n \geq 0$.

Theorem 4.32. *Define $f : B_r(z_0) \to \mathbb{C}$ by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then f has derivatives of all orders on $B_r(z_0)$, with $f^{(k)} : B_r(z_0) \to \mathbb{C}$ given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z - z_0)^{n-k}.$$

Consequently, $a_n = f^{(n)}(z_0)/n!$ for each $n \geq 0$.

Proof. By Proposition 4.31(1) the function f is analytic on $B_r(z_0)$, and hence f has derivatives of all orders on $B_r(z_0)$ by Corollary 4.20. The rest of the argument begins where the proof of Proposition 4.31(1) left off. By Theorem 4.30,

$$\sigma_n^{(k)} \xrightarrow{\text{u}} f^{(k)}$$

on compact subsets of $B_r(z_0)$ for each $k \in \mathbb{N}$, which implies that $(\sigma_n^{(k)})$ converges pointwise to the function $f^{(k)}$ on $B_r(z_0)$. That is, for each $z \in B_r(z_0)$ and $k \in \mathbb{N}$,

$$\begin{aligned} f^{(k)}(z) &= \lim_{n\to\infty} \sigma_n^{(k)}(z) = \lim_{n\to\infty} \sum_{j=k}^{n} j(j-1)\cdots(j-k+1)a_j(z - z_0)^{j-k} \\ &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z - z_0)^{n-k}, \end{aligned} \tag{4.10}$$

as was to be shown.

Setting $z = z_0$ in the series for $f(z)$ readily yields $a_0 = f(z_0)$, and for each $k \geq 1$ we may set $z = z_0$ in (4.10) to obtain

$$f^{(k)}(z_0) = k(k-1)\cdots(1)a_k = k!a_k,$$

whence $a_k = f^{(k)}(z_0)/k!$ obtains. Therefore $a_n = f^{(n)}(z_0)/n!$ for each $n \geq 0$. ■

Corollary 4.33. *If*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \in \mathbb{C}$$

for all $z \in B_r(z_0)$, then $a_n = b_n$ for all $n \geq 0$.

Proof. Define $f : B_r(z_0) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for all $z \in B_r(z_0)$. Then

$$f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

for all $z \in B_r(z_0)$ also, and so

$$a_n = b_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$ by Theorem 4.32. ■

Exercise 4.34 (L2.1.2). Let $f : B_r(0) \to \mathbb{C}$ be given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

so that

$$f(-z) = \sum_{n=0}^{\infty} a_n(-z)^n = \sum_{n=0}^{\infty} (-1)^n a_n z^n$$

on $B_r(0)$. Define f to be **even** if $a_n = 0$ for n odd, and **odd** if $a_n = 0$ for n even. Show that f is even iff $f(-z) = f(z)$, and f is odd iff $f(-z) = -f(z)$.

Solution. Suppose f is even. If n is odd, then $a_n = 0$ and so $a_n = 0 = (-1)^n a_n$. If n is even, then $(-1)^n = 1$ and so $a_n = (-1)^n a_n$. Hence $(-1)^n a_n = a_n$ for all $n \geq 0$, and therefore

$$f(-z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n = \sum_{n=0}^{\infty} a_n z^n = f(z).$$

For the converse, suppose $f(-z) = f(z)$ on $B_r(0)$; that is,

$$\sum_{n=0}^{\infty} (-1)^n a_n z^n = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in B_r(0)$. Then $a_n = (-1)^n a_n$ for all $n \geq 0$ by Corollary 4.33. If n is odd, then $(-1)^n = -1$ implies that $a_n = -a_n$, whence $a_n = 0$ obtains and we conclude that $f(z)$ is even.

Next, suppose f is odd. If n is even, then $a_n = 0$ and so $-a_n = 0 = (-1)^n a_n$. If n is odd, then $(-1)^n = -1$ and so $-a_n = (-1)^n a_n$. Hence $(-1)^n a_n = -a_n$ for all $n \geq 0$, and therefore

$$f(-z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n = \sum_{n=0}^{\infty} -a_n z^n = -\sum_{n=0}^{\infty} a_n z^n = -f(z).$$

For the converse, suppose $f(-z) = -f(z)$ on $B_r(0)$; that is,

$$\sum_{n=0}^{\infty} (-1)^n a_n z^n = -\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} -a_n z^n$$

for all $z \in B_r(0)$. Then $-a_n = (-1)^n a_n$ for all $n \geq 0$ by Corollary 4.33. If n is even, then $(-1)^n = 1$ implies that $-a_n = a_n$, whence $a_n = 0$ obtains and we conclude that $f(z)$ is odd. ■

Exercise 4.35 (AN2.2.3). Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

We find that f has derivatives of all orders on $\mathbb{R}$, with $f^{(n)}(0) = 0$ for all $n \geq 0$. Hence the Taylor series for f about 0 does not converge to f.

Fix $r > 0$, and suppose there exists a function g that is analytic on $B_r(0)$ such that $g = f$ on $(-r, r)$. By a theorem it follows that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

for all $z \in B_r(0)$.

Observe that since $g = f$ on $(-r, r)$, we have

$$g^{(0)}(x) = g(x) = f(x) = f^{(0)}(x)$$

for all $x \in (-r, r)$. Let $n \geq 0$ be arbitrary and suppose that $g^{(n)}(x) = f^{(n)}(x)$ for all $x \in (-r, r)$. By Corollary 4.20 the function g has derivatives of all orders on $B_r(0)$, so for any $x \in (-r, r)$ we find that

$$g^{(n+1)}(x) = \lim_{h \to 0} \frac{g^{(n)}(h) - g^{(n)}(x)}{h}$$

exists in $\mathbb{C}$. That is, there is some complex number $g^{(n+1)}(x)$ such that, for all $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\left| \frac{g^{(n)}(h) - g^{(n)}(x)}{h} - g^{(n+1)}(x) \right| < \epsilon$$

whenever $0 < |h| < \delta$. In particular this implies that

$$\left| \frac{f^{(n)}(t) - f^{(n)}(x)}{t} - g^{(n+1)}(x) \right| < \epsilon$$

whenever $t \in (-r, r)$ with $0 < |t| < \delta$, using the inductive hypothesis that $g^{(n)} = f^{(n)}$ on $(-r, r)$. Hence

$$f^{(n+1)}(x) = \lim_{t \to 0} \frac{f^{(n)}(t) - f^{(n)}(x)}{t} = g^{(n+1)}(x),$$

and we conclude that $g^{(n+1)} = f^{(n+1)}$ on $(-r, r)$.

Therefore $g^{(n)} = f^{(n)}$ on $(-r, r)$ for all $n \geq 0$ by induction, and so for $z \in B_r(0)$ we find that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = 0.$$

That is, $g = 0$ on $B_r(0)$, which implies that $f = 0$ on $(-r, r)$, which is a contradiction. Thus, for any $r > 0$ there can be no analytic function $B_r(0) \to \mathbb{C}$ which equals f on $(-r, r)$. ■

Exercise 4.36 (AN2.2.5). Prove that if f is analytic at z_0, then it is not possible that

$$|f^{(n)}(z_0)| > n! b_n$$

for all $n \in \mathbb{N}$, where $(b_n)^{1/n} \to \infty$ as $n \to \infty$.

Solution. We will prove the contrapositive of the statement. Suppose $|f^{(n)}(z_0)| > n! b_n$ for all $n \in \mathbb{N}$, where $(b_n)^{1/n} \to \infty$ as $n \to \infty$. For any $z \neq z_0$,

$$\begin{aligned}\rho = \limsup_{n \to \infty} \left| \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \right|^{1/n} &= \limsup_{n \to \infty} \left(\frac{|f^{(n)}(z_0)|}{n!} \right)^{1/n} |z - z_0| \\ &\geq \limsup_{n \to \infty} \left(\frac{n! b_n}{n!} \right)^{1/n} |z - z_0| = \limsup_{n \to \infty} (b_n)^{1/n} |z - z_0| = \infty,\end{aligned}$$

and so by the Root Test the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

is not convergent on $B_r(z_0)$ for any $r > 0$. Thus, if Ω is any open set containing z_0, there exists some $B_r(z_0) \subseteq \Omega$ for which

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

fails to hold, and so f is not analytic on Ω by Theorem 4.29. Since Ω is an arbitrary open set containing z_0, we conclude that f is not analytic at z_0. ■

Exercise 4.37 (AN2.2.6). Let $f : \Omega \to \mathbb{C}$ be analytic, and for $z_0 \in \Omega$ let $r > 0$ be such that $B_r(z_0) \subseteq \Omega$. For each $n \geq 0$ define $R_n : B_r(z_0) \to \mathbb{C}$ by

$$R_n(z) = f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Show the following.

(a) If $z \in \mathbb{C}$ is such that $|z - z_0| < \rho < r$, then

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_{C_\rho(z_0)} \frac{f(w)}{(w-z)(w-z_0)^{n+1}}\,dw.$$

(b) If $z \in \mathbb{C}$ is such that $|z - z_0| \le s < \rho < r$, then

$$|R_n(z)| \le \frac{\rho}{\rho - s}\left(\frac{s}{\rho}\right)^{n+1} \max_{w \in C_\rho(z_0)} |f(w)|.$$

Solution. (a) Let $z \in \mathbb{C}$ be such that $|z - z_0| < \rho < r$. Let $C = C_\rho(z_0)$. Since $\overline{B}_\rho(z_0) \subseteq \Omega$, by Cauchy's Integral Formula for a Circle and Corollary 4.20,

$$\begin{aligned} R_n(z) &= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\sum_{k=0}^{n}\oint_C \frac{(z-z_0)^k f(w)}{(w-z_0)^{k+1}}\,dw \\ &= \frac{1}{2\pi i}\oint_C f(w)\left(\frac{1}{w-z} - \sum_{k=0}^{n}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}\right)dw. \end{aligned}$$

Now,

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\dfrac{z-z_0}{w-z_0}} = \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}},$$

and so

$$\begin{aligned} R_n(z) &= \frac{1}{2\pi i}\oint_C f(w)\left(\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}} - \sum_{k=0}^{n}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}\right)dw \\ &= \frac{1}{2\pi i}\oint_C \left(f(w)\sum_{k=n+1}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}\right)dw \\ &= \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \left(\frac{f(w)}{(w-z_0)^{n+1}}\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}\right)dw \\ &= \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(w)}{(w-z)(w-z_0)^{n+1}}\,dw. \end{aligned}$$

(b) Let $z \in \mathbb{C}$ be such that $z \in \mathbb{C}$ is such that $|z - z_0| \le s < \rho < r$. For any $w \in C_\rho(z_0)$ we have

$$|w-z| = |(w-z_0)+(z_0-z)| \ge \big||w-z_0| - |z_0-z|\big| = \big|\rho - |z_0-z|\big| = \rho - |z-z_0| \ge \rho - s,$$

so that

$$\frac{1}{|w-z|} \le \frac{1}{\rho-s},$$

and then part (a) and Theorem 3.23 gives

$$\begin{aligned} |R_n(z)| &\le \frac{|z-z_0|^{n+1}}{2\pi}\left|\oint_{C_\rho(z_0)}\frac{f(w)}{(w-z)(w-z_0)^{n+1}}\,dw\right| \le \rho s^{n+1}\sup_{w\in C_\rho(z_0)}\frac{|f(w)|}{|w-z||w-z_0|^{n+1}} \\ &= \rho s^{n+1}\sup_{w\in C_\rho(z_0)}\frac{|f(w)|}{(\rho-s)\rho^{n+1}} = \frac{\rho}{\rho-s}\left(\frac{s}{\rho}\right)^{n+1}\max_{w\in C_\rho(z_0)}|f(w)|. \end{aligned}$$

■

Exercise 4.38 (AN2.2.12). Define $F : \mathbb{C} \setminus C_r(z_0) \to \mathbb{C}$ by

$$F(z) = \oint_{C_r(z_0)} \frac{1}{w-z}\, dw.$$

Show that $F \equiv 2\pi i$ on $B_r(z_0)$.

Solution. By Theorem 4.19 F is analytic on $\mathbb{C} \setminus C_r(z_0)$, and $F' : \mathbb{C} \setminus C_r(z_0) \to \mathbb{C}$ is given by

$$F'(z) = \oint_{C_r(z_0)} \frac{1}{(w-z)^2}\, dw.$$

Fix $z \in B_r(z_0)$, define $f : \mathbb{C} \setminus \{z\} \to \mathbb{C}$ by

$$f(w) = \frac{1}{(w-z)^2},$$

and define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = z_0 + re^{it}$ so that $\gamma^* = C_r(z_0)$. Clearly f is continuous and has primitive

$$\varphi(w) = -\frac{1}{w-z}$$

on $\mathbb{C} \setminus \{z\}$, and since γ is a closed path in $\mathbb{C} \setminus \{z\}$ it follows by the Fundamental Theorem for Path Integrals that

$$F'(z) = \oint_{C_r(z_0)} \frac{1}{(w-z)^2}\, dw = \oint_{C_r(z_0)} f = \oint_\gamma f = 0.$$

Thus $F' \equiv 0$ on $B_r(z_0)$, and it follows that F is constant on $B_r(z_0)$. Now,

$$F(z_0) = \oint_\gamma \frac{1}{w-z_0}\, dw = \int_0^{2\pi} \frac{ire^{it}}{(z_0 + re^{it}) - z_0} dt = \int_0^{2\pi} i\, dt = 2\pi i,$$

and therefore $F \equiv 2\pi i$. ■

Exercise 4.39 (AN2.2.13a). Suppose f is analytic on $B_R(z_0)$. Prove that for $0 < r < R$

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi r^n} \int_0^{2\pi} |f(z_0 + re^{it})| dt.$$

Solution. Fix $r \in (0, R)$. Since f is analytic on $B_R(z_0)$ and $\overline{B}_r(z_0) \subseteq B_R(z_0)$, by Corollary 4.20

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}$. Defining $\gamma : [0, 2\pi] \to \mathbb{C}$ to be the usual parameterization of $C_r(z_0)$ given by $\gamma(t) = z_0 + re^{it}$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{(\gamma(t)-z)^{n+1}} \gamma'(t)\, dt,$$

whence

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))\gamma'(t)}{(\gamma(t)-z_0)^{n+1}} dt = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{ire^{it} f(z_0 + re^{it})}{(re^{it})^{n+1}} dt,$$

and finally

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi}\int_0^{2\pi}\frac{f(z_0+re^{it})}{r^n(e^{it})^n}dt\right| \leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{|f(z_0+re^{it})|}{r^n|e^{it}|^n}dt = \frac{n!}{2\pi r^n}\int_0^{2\pi}|f(z_0+re^{it})|dt$$

as desired. ∎

Exercise 4.40 (AN2.2.13b). Let f be an entire function. Prove that if there exists some $M > 0$, $k \in \mathbb{N}$, and $\rho > 0$ such that $|f(z)| \leq M|z|^k$ for all $z \in \mathbb{C}$ for which $|z| > \rho$, then f is a polynomial function with $\deg(f) \leq k$.

Solution. Since f is analytic on $\mathbb{C}$, by Theorem 4.29 we have

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n \tag{4.11}$$

for all $z \in \mathbb{C}$. Let $r > \rho$ be arbitrary. Certainly f is analytic on $B_R(0)$ for any $R > r$, and so by the previous exercise

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi r^n}\int_0^{2\pi}|f(re^{it})|dt.$$

Observing that $|re^{it}| = r > \rho$, by hypothesis

$$|f(re^{it})| \leq M|re^{it}|^k = Mr^k,$$

so that

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi r^n}\int_0^{2\pi}Mr^k dt = \frac{n!M}{r^{n-k}},$$

and thus

$$0 \leq \frac{|f^{(n)}(0)|}{n!} \leq \frac{M}{r^{n-k}}$$

for all $r > \rho > 0$. If $n > k$, then $M/r^{n-k} \to 0$ as $r \to \infty$, and so by the Squeeze Theorem

$$\frac{|f^{(n)}(0)|}{n!} = \lim_{r\to\infty}\frac{|f^{(n)}(0)|}{n!} = 0.$$

Therefore $f^{(n)}(0)/n! = 0$ for $n > k$, and (4.11) yields

$$f(z) = \sum_{n=0}^{k}\frac{f^{(n)}(0)}{n!}z^n.$$

That is, f is a polynomial function of degree at most k. ∎

Exercise 4.41 (AN2.2.13c). Let f be an entire function such that $|f(z)| \leq 1 + |z|^{3/2}$ for all $z \in \mathbb{C}$. Prove that there exist $a_0, a_1 \in \mathbb{C}$ such that $f(z) = a_0 + a_1 z$.

Solution. Let $M = 2$ and $\rho = 1$. Let z be such that $|z| > \rho$. Then $|z|^2, |z|^{1/2} > 1$, so that

$$\frac{1}{|z|^2} + \frac{1}{|z|^{1/2}} < 1 + 1 = 2,$$

and thus (since $z \neq 0$) we have

$$|f(z)| \leq 1 + |z|^{3/2} = \left(\frac{1}{|z|^2} + \frac{1}{|z|^{1/2}}\right)|z|^2 \leq 2|z|^2.$$

Thus there exists some $M, \rho > 0$ such that $|f(z)| \leq M|z|^2$ for all z with $|z| > \rho$, and so by Exercise 4.40 we conclude that f is a polynomial function of degree at most 2; that is, there exists $a_0, a_1, a_2 \in \mathbb{C}$ such that

$$f(z) = a_0 + a_1 z + a_2 z^2.$$

Now we have

$$|a_0 + a_1 z + a_2 z^2| \leq 1 + |z|^{3/2}$$

for all z, whence

$$\begin{aligned} |a_2 z^2| - |a_0 + a_1 z| \leq 1 + |z|^{3/2} \quad &\Rightarrow \quad |a_2 z^2| \leq 1 + |a_0 + a_1 z| + |z|^{3/2} \\ &\Rightarrow \quad |a_2||z|^2 \leq 1 + |a_0| + |a_1||z| + |z|^{3/2}, \end{aligned}$$

and so for all $z \neq 0$

$$|a_2||z|^{1/2} \leq 1 + \frac{|a_1|}{|z|^{1/2}} + \frac{1 + |a_0|}{|z|^{3/2}}. \tag{4.12}$$

If we suppose $a_2 \neq 0$, then we may choose $z \neq 0$ such that $|z|^{1/2} > 3/|a_2|$, $|z|^{1/2} > 2|a_1|$, and $|z|^{3/2} > 2(1 + |a_0|)$, in which case (4.12) yields the contradiction $3 < 2$. Therefore $a_2 = 0$ and we obtain $f(z) = a_0 + a_1 z$. ■

4.4 – The Exponential Function

Theorem 4.42. *The exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ *given by*

$$\exp(x + iy) = e^x(\cos y + i \sin y)$$

has the following properties.

1. *The function* $\exp$ *is entire, with* $\exp^{(n)} = \exp$ *on* $\mathbb{C}$ *for all* $n \in \mathbb{N}$.
2. *For all* $z \in \mathbb{C}$,
$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
3. *For all* $z, w \in \mathbb{C}$,
$$\exp(z + w) = \exp(z)\exp(w).$$
4. *There are no zeros for* $\exp$ *in* $\mathbb{C}$, *and*
$$\exp(-z) = \frac{1}{\exp(z)}$$
for all $z \in \mathbb{C}$.
5. $\exp(z) = 1$ *if and only if* $z = 2\pi ni$ *for some* $n \in \mathbb{Z}$.
6. *For all* $z \in \mathbb{C}$,
$$|\exp(z)| = e^{\operatorname{Re} z}.$$
7. *For all* $z \in \mathbb{C}$, $\exp(z) = \exp(z + w)$ *if and only if* $w = 2\pi ni$ *for some* $n \in \mathbb{Z}$. *In particular if* $e^{is} = e^{it}$ *for* $s, t \in \mathbb{R}$, *then* $s - t = 2\pi n$ *for some* $n \in \mathbb{Z}$.
8. $\exp$ *maps* $\{z : \operatorname{Re} z = r\}$ *onto* $C_{e^r}(0)$, *and maps* $\{z : \operatorname{Im} z = \theta\}$ *onto the open ray from* 0 *through* $\exp(i\theta)$.
9. *For each* $\theta \in \mathbb{R}$, *the map*
$$\exp : \{z : \theta \le \operatorname{Im} z < \theta + 2\pi\} \to \mathbb{C}_*$$
is a bijection.

Proof.
Proof of Part (1). By Proposition 3.12 we find that $\exp'(z) = \exp(z)$ for each $z \in \mathbb{C}$, and so the exponential function is entire. That $\exp^{(n)}(z) = \exp(z)$ for every $z \in \mathbb{C}$ and $n \in \mathbb{N}$ follows from Proposition 3.12 via induction.

Proof of Part (2). Fix $z \in \mathbb{C}$. Let $r > 0$ be such that $z \in B_r(0)$. Since $B_r(0) \subseteq \mathbb{C}$ and $\exp$ is analytic on $\mathbb{C}$ by part (1), we have

$$\exp(z) = \sum_{n=0}^{\infty} \frac{\exp^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{\exp(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

by Theorem 4.29 and another application of part (1).

Proof of Part (3). Fix $z, w \in \mathbb{C}$. By Theorem 4.29 and part (1) we have

$$\exp(\zeta) = \sum_{n=0}^{\infty} \frac{\exp^{(n)}(w)}{n!} (\zeta - w)^n = \sum_{n=0}^{\infty} \frac{\exp(w)}{n!} (\zeta - w)^n$$

for all $\zeta \in \mathbb{C}$, and thus if we set $\zeta = z + w$ we obtain

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{\exp(w)}{n!} z^n = \exp(w) \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(w)\exp(z)$$

by part (2), observing that the series $\sum z^n/n!$ is absolutely convergent on $\mathbb{C}$.

Proof of Part (4). If $\exp(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then part (3) implies that

$$1 = \exp(0) = \exp(-z_0 + z_0) = \exp(-z_0)\exp(z_0) = 0,$$

which is a contradiction. Therefore $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.

Next, for any $z \in \mathbb{C}$,

$$1 = \exp(0) = \exp(-z + z) = \exp(-z)\exp(z) \quad \Rightarrow \quad \exp(-z) = \frac{1}{\exp(z)},$$

since $\exp(z) \neq 0$.

Proof of Part (5). Suppose $z = x + iy$ is such that $\exp(z) = 1$, so

$$e^x \cos y + ie^x \sin y = 1,$$

and hence $\sin y = 0$ and $e^x \cos y = 1$. From $\sin y = 0$ we obtain $y = m\pi$ for some $m \in \mathbb{Z}$, where m must be an even integer since otherwise $\cos y = -1$ and we conclude that $e^x = -1$ for $x \in \mathbb{R}$, a contradiction. Therefore $z = 2\pi ni$ for some $n \in \mathbb{Z}$.

Proof of the converse is straightforward: If $z = 2\pi ni$ for some $n \in \mathbb{Z}$, then $\exp(z) = \cos(2\pi n) + i\sin(2\pi n) = 1$.

Proof of Part (6). If $z = x + iy$, then

$$|\exp(z)| = |e^x \cos y + ie^x \sin y| = e^x |\cos y + i\sin y| = e^x = e^{\operatorname{Re} z}.$$

Proof of Part (7). If $\exp(z + w) = \exp(z)$ for all $z \in \mathbb{C}$, then we have $\exp(w) = \exp(0) = 1$ in particular, and so $w = 2\pi ni$ for some $n \in \mathbb{Z}$ by part (5).

If $w = 2\pi ni$ for some $n \in \mathbb{Z}$, then for any $z \in \mathbb{C}$ we find that

$$\exp(z + p) = \exp(z)\exp(p) = \exp(z)$$

by parts (3) and (5).

Finally, suppose $e^{is} = e^{it}$ for $s, t \in \mathbb{R}$. Then $e^{it} = e^{it+(is-it)}$, and by the result just proven it follows that $is - it = 2\pi ni$ for some $n \in \mathbb{Z}$, and therefore $s - t = 2\pi n$.

Proof of Part (8). Let $L_r = \{z : \operatorname{Re} z = r\}$. If $z \in L_r$, so that $z = r + iy$ for some $y \in \mathbb{R}$, then

$$|\exp(z)| = |e^r(\cos y + i\sin y)| = e^r$$

shows that $\exp(z) \in C_{e^r}(0)$, and so $\exp(L_r) \subseteq C_{e^r}(0)$. On the other hand any $w \in C_{e^r}(0)$ is expressible as

$$w = e^r \cos\theta + ie^r \sin\theta$$

for some $\theta \in [0, 2\pi)$, so that $z \in L_r$ given by $z = r + i\theta$ is such that $\exp(z) = w$, and thus $C_{e^r}(0) \subseteq \exp(L_r)$. ■

Definition 4.43. *Define the **cosine** function* $\cos : \mathbb{C} \to \mathbb{C}$ *and the **sine** function* $\sin : \mathbb{C} \to \mathbb{C}$ *by*

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for all $z \in \mathbb{C}$.

Clearly the cosine and sine functions are entire functions, and thus by Theorem 4.29 there exist sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ such that $\cos(z) = \sum a_n z^n$ and $\sin(z) = \sum b_n z^n$ for all $z \in \mathbb{C}$. Since $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$ for all $z \in \mathbb{C}$, by Exercise 4.34 we conclude that $\cos(z)$ is an even function with $a_n = 0$ for n odd, and $\sin(z)$ is an odd function with $b_n = 0$ for n even. More explicitly, by Theorem 4.42(2),

$$\cos(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \sum_{n=0}^{\infty} \frac{(iz)^n + (-iz)^n}{2n!} = \sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{2n!} z^n.$$

Corollary 4.33 requires

$$a_n = \frac{i^n + (-i)^n}{2n!}$$

for all $n \geq 0$. We have $a_{2k+1} = 0$ and

$$a_{2k} = \frac{i^{2k} + (-i)^{2k}}{2(2k)!} = \frac{(-1)^k + (-1)^k}{2(2k)!} = \frac{(-1)^k}{(2k)!}$$

for all $k \geq 0$, and in this fashion obtain

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

on $\mathbb{C}$. A similar argument leads to a power series representation for $\sin(z)$ on $\mathbb{C}$, which is given in the following proposition.

Proposition 4.44. *For all* $z \in \mathbb{C}$,

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \textit{and} \quad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

We define the **tangent**, **cotangent**, **secant**, and **cosecant** functions by

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \quad \cot(z) = \frac{\cos(z)}{\sin(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad \csc(z) = \frac{1}{\sin(z)},$$

respectively. None of these four functions is entire, but note that $\sec(z)$ is even while the other three functions are odd.

Exercise 4.45 (AN2.3.3). Evaluate

$$\oint_{\partial \mathbb{B}} \frac{\sin z}{z^4} dz.$$

Solution. The series
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
is convergent for any $z \in \mathbb{C}$, and thus by the sequential limit equivalent of Theorem 2.15 we obtain
$$\frac{\sin z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{z^4(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!}$$
for any $z \neq 0$. This new series converges for all $z \in \partial\mathbb{B}$, and so
$$\begin{aligned}\oint_{\partial\mathbb{B}} \frac{\sin z}{z^4} dz &= \oint_{\partial\mathbb{B}} \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!} \right) dz \\ &= \oint_{\partial\mathbb{B}} \left(\frac{1}{z^3} - \frac{1}{6z} + \sum_{n=2}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!} \right) dz.\end{aligned}$$
Letting
$$f(z) = \sum_{n=2}^{\infty} (-1)^n \frac{z^{2n-3}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+5)!},$$
we write
$$\oint_{\partial\mathbb{B}} \frac{\sin z}{z^4} dz = \oint_{\partial\mathbb{B}} \left(\frac{1}{z^3} - \frac{1}{6z} + f(z) \right) dz.$$
Now, since f is analytic on $\mathbb{C}$, by Cauchy's Theorem for Starlike Regions it has a primitive on $\mathbb{C}$, and hence
$$\oint_{\partial\mathbb{B}} f(z)\, dz = 0$$
by the Fundamental Theorem of Path Integrals. As for z^{-3}, it is analytic on $\Omega = \mathbb{C}_*$ and has primitive $-z^{-2}/2$ there, and since $\partial\mathbb{B} \subseteq \Omega$ we conclude once more by the Fundamental Theorem of Path Integrals that
$$\oint_{\partial\mathbb{B}} \frac{1}{z^3} dz = 0.$$
Finally we have
$$\oint_{\partial\mathbb{B}} \frac{1}{6z} dz = \frac{1}{6} \oint_{\partial\mathbb{B}} \frac{1}{z} dz = \frac{2\pi i}{6} = \frac{\pi i}{3}$$
by direct computation, and so
$$\oint_{\partial\mathbb{B}} \frac{\sin z}{z^4} dz = -\frac{\pi i}{3}$$
obtains. ∎

Exercise 4.46 (AN2.3.4). Show that for every $r > 0$ there exists some $n_0 \geq 0$ such that, for all $n \geq n_0$, the zeros of
$$\sigma_n(z) = \sum_{k=0}^{n} \frac{1}{k!} z^k$$
lie in $\mathbb{C} \setminus B_r(0)$.

Solution. Fix $r > 0$. By Example 4.4 together with Theorem 4.42(2) the sequence of functions (σ_n) converges uniformly to exp on the compact set $\overline{B}_r(0)$, and so there exists some n_0 such that

$$|\sigma_n(z) - \exp(z)| < \frac{1}{2e^r}$$

for all $n \geq n_0$ and $z \in \overline{B}_r(0)$. Now, for any $z = x + iy \in \overline{B}_r(0)$, since

$$|z| \leq r \quad \Rightarrow \quad |x| \leq r \quad \Rightarrow \quad -r \leq x \leq r,$$

it follows that

$$|\exp(z)| = |e^x(\cos y + i \sin y)| = e^x \in [e^{-r}, e^r],$$

and so in particular $|\exp(z)| \geq e^{-r}$. Thus, for all $n \geq n_0$ and $z \in \overline{B}_r(0)$,

$$\begin{aligned} |\sigma_n(z) - \exp(z)| < \frac{1}{2e^r} \quad &\Rightarrow \quad \big||\sigma_n(z)| - |\exp(z)|\big| < \frac{1}{2e^r} \\ &\Rightarrow \quad |\exp(z)| - \frac{1}{2}e^{-r} < |\sigma_n(z)| < |\exp(z)| + \frac{1}{2}e^{-r}, \end{aligned}$$

and hence

$$|\sigma_n(z)| > e^{-r} - \frac{1}{2}e^{-r} = \frac{1}{2}e^{-r} > 0.$$

That is, $\sigma_n(z) \neq 0$ for all $n \geq n_0$ and $z \in \overline{B}_r(0)$, and therefore any zeros of σ_n must lie in $\mathbb{C} \setminus B_r(0)$ for $n \geq n_0$. ■

Exercise 4.47 (AN2.3.5). Let f be an entire function that satisfies the initial value problem

$$f'' + f = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

Prove that $f(z) = \sin(z)$ for all $z \in \mathbb{C}$.

Solution. It is easy to verify that $\sin(z)$ is a solution to the IVP; however, the trick is to show that there can be no other entire function that works. That is, we must show that $\sin(z)$ is a unique solution.

Since f is an entire function, by Theorem 4.29 it is representable in $\mathbb{C}$ by power series, and in particular there exists a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{C}$. Now, f has derivatives of all orders by Corollary 4.20, so from $f'' + f = 0$ we obtain $f^{(n+2)} + f^{(n)} = 0$ for all $n \geq 0$, and thus

$$f^{(n+2)}(0) + f^{(n)}(0) = 0.$$

By Theorem 4.32 we have $a_n = f^{(n)}(0)/n!$, and so the sequence $(a_n)_{n=0}^{\infty}$ is defined recursively by

$$(n+2)!a_{n+2} + n!a_n = 0, \quad a_0 = 0, \quad a_1 = 1,$$

or equivalently

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \quad a_0 = 0, \quad a_1 = 1.$$

This recursion relation uniquely determines the sequence $(a_n)_{n=0}^{\infty}$, and therefore $\sin(z)$ must be the unique solution to the IVP. ■

Exercise 4.48 (AN2.3.6). Let f be an entire function such that $f' = f$ and $f(0) = 1$. What follows and why?

Solution. Since f is an entire function we have, by Theorem 4.29, that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. From $f' = f$ we obtain $f^{(n+1)} = f^{(n)}$ for all $n \geq 0$, and thus $f^{(n)}(0) = f(0) = 1$ for all $n \geq 0$. Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

and we see that $f(z) = \exp(z)$. ■

4.5 – Bernoulli Numbers

Consider the function $\varphi(z) = z(e^z - 1)^{-1}$. Clearly φ is analytic on $\Omega = \{z \in \mathbb{C} : e^z \neq 1\}$, and since, by Theorem 4.42(5), $e^z = 1$ if and only if $z = 2\pi ni$ for some $n \in \mathbb{Z}$, we see that φ is analytic on $B'_{2\pi}(0) \subseteq \Omega$. Now define $f : B_{2\pi}(0) \to \mathbb{C}$ by

$$f(z) = \begin{cases} \dfrac{z}{e^z - 1}, & 0 < |z| < 2\pi \\ 1, & z = 0. \end{cases}$$

It can be shown that $f(z) \to 1$ as $z \to 0$, so f is analytic on $B'_{2\pi}(0)$ and continuous on $B_{2\pi}(0)$, and hence f is analytic on $B_{2\pi}(0)$ by Corollary 4.22. By Theorem 4.29 we conclude that f has a power series representation $\sum a_n z^n$ on $B_{2\pi}(0)$. This justifies the following definition.

Definition 4.49. *The* ***Bernoulli numbers*** *are the numbers B_n for which*

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} \tag{4.13}$$

for all $z \in B_{2\pi}(0)$.

By Theorem 4.2(1) the series in (4.13) converges absolutely on $B_{2\pi}(0)$, and so in particular we have the useful fact that $\sum_{n=0}^{\infty} |B_n|/n!$ converges.

By Theorem 4.29 we see that

$$B_n = f^{(n)}(0)$$

for all $n \geq 0$. This formula could be used to find Bernoulli numbers, though in the following exercise we will instead make use of the Cauchy product of series formula.

Exercise 4.50 (L2.1.3). Show that

$$\sum_{k=1}^{n} \frac{B_{n-k}}{k!(n-k)!} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

Also show that $B_n = 0$ for all odd $n \geq 3$.

Solution. Multiply both sides of (4.13) by $e^z - 1$ to obtain

$$(e^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = z.$$

That is,

$$z = \left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right), \tag{4.14}$$

where

$$a_n = \begin{cases} 0, & \text{if } n = 0 \\ 1/n!, & \text{if } n > 0 \end{cases}$$

by Theorem 4.42(2), and $b_n = B_n/n!$ for $n \geq 0$. The first series in (4.14) is absolutely convergent for all $z \in \mathbb{C}$, whereas the second series is convergent for all $z \in B_{2\pi}(0)$. Thus by Mertens' Theorem,

$$z = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k z^k \cdot b_{n-k} z^{n-k}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \frac{1}{k!} \cdot \frac{B_{n-k}}{(n-k)!}\right) z^n$$

for $z \in B_{2\pi}(0)$, and so

$$\sum_{k=1}^{n} \frac{B_{n-k}}{k!(n-k)!} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

by Corollary 4.33. Using this recursion relation, we readily obtain

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}.$$

To show that $B_n = 0$ for all odd $n \geq 3$, we use the result of Exercise 4.34. We have

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = -\frac{z}{2} + \sum_{n \neq 1} \frac{B_n}{n!} z^n,$$

and so

$$\begin{aligned}\sum_{n \neq 1} \frac{B_n}{n!} z^n &= \frac{z}{2} + \frac{z}{e^z - 1} = \frac{z}{2}\left(\frac{e^z + 1}{e^z - 1}\right) \\ &= \frac{z}{2}\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} \cdot \frac{e^{z/2}}{e^{z/2}}\right) = \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}},\end{aligned}$$

where we make use of Theorem 4.42(3) to secure the third equality. Now, setting

$$g(z) = \sum_{n \neq 1} \frac{B_n}{n!} z^n,$$

we have

$$\begin{aligned} g(-z) &= \sum_{n \neq 1} \frac{B_n}{n!}(-z)^n = \frac{-z}{2} \cdot \frac{e^{-z/2} + e^{-(-z)/2}}{e^{-z/2} - e^{-(-z)/2}} = \frac{-z}{2} \cdot \frac{e^{-z/2} + e^{z/2}}{e^{-z/2} - e^{z/2}} \\ &= \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = g(z). \end{aligned}$$

Thus $g(z)$ is even, and by Exercise 4.34 we conclude that $B_n/n! = 0$ for $n = 3, 5, 7, \ldots$, and therefore $B_n = 0$ for odd $n \geq 3$. ■

Exercise 4.51 (L2.1.4). Show that

$$\frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n},$$

and then obtain

$$\pi z \cot(\pi z) = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} B_{2n} z^{2n}.$$

Solution. From Exercise 4.50 we have

$$\begin{aligned}\frac{z}{e^z-1} &= B_0 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \frac{B_4}{4!}z^4 + \frac{B_5}{5!}z^5 + \frac{B_6}{6!}z^6 + \cdots \\ &= B_0 - \frac{1}{2}z + \frac{B_2}{2!}z^2 + \frac{B_4}{4!}z^4 + \frac{B_6}{6!}z^6 + \cdots = -\frac{1}{2}z + \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n},\end{aligned}$$

and thus

$$\frac{z}{2} + \frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}.$$

From this we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n} &= \frac{z}{2}\left(1+\frac{2}{e^z-1}\right) = \frac{z}{2}\left(\frac{e^z-1}{e^z-1}+\frac{2}{e^z-1}\right) = \frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right) \\ &= \frac{z}{2}\left(\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\cdot\frac{e^{z/2}}{e^{z/2}}\right) = \frac{z}{2}\cdot\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}},\end{aligned}$$

as was to be shown.

Recalling Definition 4.43 and the remarks that follow it,

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{\dfrac{e^{iz}+e^{-iz}}{2}}{\dfrac{e^{iz}-e^{-iz}}{2i}} = i\left(\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right),$$

and so

$$\begin{aligned}\pi z\cot(\pi z) &= \pi z\cdot i\left(\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}}\right) = \frac{2\pi iz}{2}\cdot\frac{e^{2\pi iz/2}+e^{-2\pi iz/2}}{e^{2\pi iz/2}-e^{-2\pi iz/2}} \\ &= \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}(2\pi iz)^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{(2\pi)^{2n}}{(2n)!}B_{2n}z^{2n}\end{aligned}$$

as desired. ■

Exercise 4.52 (L2.1.5). Express the power series centered at 0 for $z/\sin(z)$ and $z\cot(z)$ in terms of Bernoulli numbers.

Solution. For $z\cot(z)$ we need only replace z with z/π in the result of Exercise 4.51 to obtain

$$z\cot(z) = \pi\left(\frac{z}{\pi}\right)\cot\left[\pi\left(\frac{z}{\pi}\right)\right] = \sum_{n=0}^{\infty}(-1)^n\frac{(2\pi)^{2n}}{(2n)!}B_{2n}\left(\frac{z}{\pi}\right)^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}B_{2n}z^{2n}. \tag{4.15}$$

To find a recursion relation that will generate the coefficients of $z/\sin(z)$, we first observe that

$$z\cot(z) = z\cdot\frac{\cos(z)}{\sin(z)} = \frac{z}{\sin(z)}\cdot\cos(z). \tag{4.16}$$

Letting

$$\frac{z}{\sin(z)} = \sum_{n=0}^{\infty}a_n z^n \quad\text{and}\quad \cos(z) = \sum_{n=0}^{\infty}b_n z^n,$$

so that $b_n = 0$ for n odd and $b_n \neq 0$ for n even, we obtain

$$z\cot(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

from (4.16), and so by (4.15) we find that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!} B_{2n} z^{2n}. \tag{4.17}$$

For $n = 1$ we have

$$\sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{1} a_k b_{1-k} = a_0 b_1 + a_1 b_0 = 0,$$

whence $a_1 = 0$ since $b_1 = 0$ and $b_0 \neq 0$. Let n odd be arbitrary and suppose $a_k = 0$ for all odd $k \leq n$. We have

$$\sum_{k=0}^{n+2} a_k b_{n+2-k} = 0,$$

but since $a_k = b_k = 0$ for all odd $k \leq n$, and $n + 2 - k$ is even if and only if k is odd, the sum collapses to yield $a_{n+2} b_0 = 0$. Hence $a_{n+2} = 0$ and we conclude by the principle of induction that $a_n = 0$ for all odd n.

In contrast, from (4.17) we see that

$$\sum_{k=0}^{2n} a_k b_{2n-k} = (-1)^n \frac{2^{2n}}{(2n)!} B_{2n}$$

for all $n \in \mathbb{N}$. This recursion relation will deliver all the nonzero coefficients for the series $z/\sin(z)$. In particular we have

$$a_0 = B_0, \quad a_2 = \frac{1}{2}B_0 - 2B_2, \quad a_4 = \frac{5}{24}B_0 - B_2 + \frac{2}{3}B_4,$$

so that

$$\frac{z}{\sin(z)} = 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots .$$

To find an explicit formula for the coefficients a_n, we proceed as follows:

$$\begin{aligned}\frac{1}{\sin(2z)} = \frac{1}{2\sin(z)\cos(z)} &= \frac{\sec^2(z)}{2\tan(z)} = \frac{1 + \tan^2(z)}{2\tan(z)} \\ &= \frac{2 - (1 - \tan^2(z))}{2\tan(z)} = \frac{1}{\tan(z)} - \frac{1 - \tan^2(z)}{2\tan(z)} \\ &= \cot(z) - \cot(2z),\end{aligned}$$

whence

$$\frac{2z}{\sin(2z)} = 2z\cot(z) - 2z\cot(2z)$$

obtains, and so

$$\frac{z}{\sin(z)} = 2 \cdot \frac{z}{2}\cot\left(\frac{z}{2}\right) - z\cot(z) = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n)!} B_{2n}\left(\frac{z}{2}\right)^{2n} - \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!} B_{2n} z^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2 - 2^{2n}}{(2n)!} B_{2n} z^{2n}.$$

■

Exercise 4.53. Show that

$$\int_0^1 \frac{x^{s-1}}{e^x - 1}\,dx = \sum_{n=0}^{\infty} \frac{B_n}{n!(s+n-1)} \tag{4.18}$$

for all $s \in (1, \infty)$.

Solution. For $s > 1$, note that since

$$\frac{|B_n|}{n!(s+n-1)} \leq \frac{|B_n|}{n!}$$

for all $n \geq 1$, and $\sum |B_n|/n!$ converges, the series in (4.18) is absolutely convergent by the Direct Comparison Test.

Fix $s > 1$. By Definition 4.49, since $[0, 1] \subseteq B_{2\pi}(0)$,

$$\int_0^1 \frac{x^{s-1}}{e^x - 1}\,dx = \int_0^1 \left(x^{s-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \right) dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^{s+n-2} \right) dx.$$

Fix $\xi \in (0, 1)$, and define the sequence of functions $(\varphi_n : [\xi, 1] \to \mathbb{R})_{n \geq 0}$ by

$$\varphi_n(x) = \frac{B_n}{n!} x^{s+n-2}$$

for each $n \geq 0$ and $x \in [\xi, 1]$. Then

$$\|\varphi_n\|_{[\xi,1]} = \sup_{x \in [\xi,1]} \left(\frac{|B_n|}{n!} |x|^{s+n-2} \right) = \frac{|B_n|}{n!} \sup_{x \in [\xi,1]} \frac{|x|^{s+n}}{x^2} \leq \frac{|B_n|}{n!\xi^2} := M_n,$$

and since $\sum M_n$ is a convergent series, the Weierstrass M-Test implies that $\sum \varphi_n$ converges uniformly on $[\xi, 1]$. It is clear that $\varphi_n \in \mathcal{R}[\xi, 1]$, so

$$\int_\xi^1 \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^{s+n-2} \right) dx = \int_\xi^1 \sum_{n=0}^{\infty} \varphi_n = \sum_{n=0}^{\infty} \int_\xi^1 \varphi_n = \sum_{n=0}^{\infty} \int_\xi^1 \frac{B_n}{n!} x^{s+n-2}\,dx$$

by Corollary 3.36, and then

$$\int_0^1 \frac{x^{s-1}}{e^x - 1}\,dx = \lim_{\xi \to 0^+} \int_\xi^1 \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^{s+n-2} \right) dx = \lim_{\xi \to 0^+} \lim_{k \to \infty} \sum_{n=0}^{k} \int_\xi^1 \frac{B_n}{n!} x^{s+n-2}\,dx$$

$$= \lim_{\xi \to 0^+} \lim_{k \to \infty} \sum_{n=0}^{k} \frac{(1-\xi) B_n}{n!(s+n-1)}.$$

Define $(g_n : (0, 1] \to \mathbb{R})_{n \geq 0}$ by

$$g_n(\xi) = \frac{(1-\xi) B_n}{n!(s+n-1)}$$

for each $n \geq 0$ and $\xi \in (0,1]$, and let $s_k = \sum_{n=0}^{k}$. For $n \in \mathbb{N}$,

$$\|g_n\|_{(0,1]} = \sup_{\xi \in (0,1]} \frac{(1-\xi)|B_n|}{n!(s+n-1)} \leq \frac{|B_n|}{n!(s+n-1)} \leq \frac{|B_n|}{n!},$$

and since $\sum |B_n|/n!$ converges, the Weierstrass M-Test implies the series $\sum g_n$, and hence the sequence $(s_k)_{k\geq 0}$, is uniformly convergent on $(0,1]$. It then follows by Theorem 2.55 that

$$\int_0^1 \frac{x^{s-1}}{e^x-1}\,dx = \lim_{\xi \to 0^+} \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(1-\xi)B_n}{n!(s+n-1)} = \lim_{k\to\infty} \lim_{\xi \to 0^+} \sum_{n=0}^{k} \frac{(1-\xi)B_n}{n!(s+n-1)}$$

$$= \lim_{k\to\infty} \sum_{n=0}^{k} \frac{B_n}{n!(s+n-1)} = \sum_{n=0}^{\infty} \frac{B_n}{n!(s+n-1)},$$

which proves (4.18) for all $s \in (1,\infty)$. ■

5

Maximum Modulus Principle

5.1 – Liouville's Theorem

Theorem 5.1 (Cauchy's Estimate). *Let $\Omega \subseteq \mathbb{C}$ be an open set. If f is analytic on Ω and $\overline{B}_r(z_0) \subseteq \Omega$, then*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{z \in C_r(z_0)} |f(z)|$$

Proof. Suppose that f is analytic on Ω and $\overline{B}_r(z_0) \subseteq \Omega$ for some $r > 0$. Fix $n \in \mathbb{N}$. By Corollary 4.20 we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}} dw$$

for all $z \in B_r(z_0)$

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{C_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{n!}{2\pi} \left| \oint_{C_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|.$$

Thus, by Theorem 3.23,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \mathcal{L}(C_r(z_0)) \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| = \frac{n!}{2\pi}(2\pi r) \sup_{z \in C_r(z_0)} \frac{|f(z)|}{|z-z_0|^{n+1}},$$

and hence

$$|f^{(n)}(z_0)| \leq n! r \sup_{z \in C_r(z_0)} \frac{|f(z)|}{r^{n+1}} = \frac{n!}{r^n} \sup_{z \in C_r(z_0)} |f(z)|,$$

as was to be shown. ∎

Theorem 5.2 (Liouville's Theorem). *If f is a bounded entire function, then f is a constant function.*

The following proof of Liouville's Theorem will make use of Exercise 4.40 instead of Cauchy's Estimate.

Proof. Let f be a bounded entire function, so there exists some $M > 0$ such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. Then for all z such that $|z| > 1$ we have $|f(z)| \le M|z|$, so by Exercise 4.40 f is a polynomial function of degree at most 1. Thus there exist $a, b \in \mathbb{C}$ such that $f(z) = a + bz$, and for all z we find that

$$|f(z)| \le M \quad \Rightarrow \quad |a + bz| \le M \quad \Rightarrow \quad |bz| - |a| \le M \quad \Rightarrow \quad |b||z| \le M + |a|,$$

which is only possible if $b = 0$. Therefore $f \equiv a$. ∎

Theorem 5.3 (Fundamental Theorem of Algebra). *If p is a polynomial function of degree $n \ge 1$, then p has a zero in $\mathbb{C}$.*

Proof. Suppose p is a polynomial function of degree $n \ge 1$, so

$$p(z) = a_n z^n + \cdots + a_1 z + a_0.$$

For $z \neq 0$,

$$|p(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \ge |z|^n \left(|a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right). \tag{5.1}$$

Let

$$g(z) = \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|,$$

so that

$$g(z) \le \frac{1}{|z|} \left(|a_{n-1}| + \frac{|a_{n-2}|}{|z|} + \cdots + \frac{|a_0|}{|z^{n-1}|} \right),$$

and it's clear there exists some $r > 0$ sufficiently large that $g(z) < |a_n|/2$ for all $z \in \mathbb{C} \setminus B_r(0)$. From (5.1) comes

$$|p(z)| \ge |z|^n (|a_n| - g(z)) \ge \frac{|a_n|}{2}|z|^n \ge \frac{|a_n| r^n}{2} := M > 0$$

for all $z \in \mathbb{C} \setminus B_r(0)$.

Now, suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Since p is an entire function, it then follows that $1/p$ is also entire. Moreover,

$$\big|(1/p)(z)\big| = \frac{1}{|p(z)|} \le \frac{1}{M}$$

for all $z \in \mathbb{C} \setminus B_r(0)$, and since the continuity of $1/p$ implies it is bounded on the compact set $\overline{B}_r(0)$, we conclude that $1/p$ is bounded on $\mathbb{C}$. Thus by Liouville's Theorem $1/p$ is a constant function, which in turns leads us to conclude that p is likewise constant: $p(z) = a_0$. But this contradicts the hypothesis that $\deg(p) \ge 1$. Therefore there must exist some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. ∎

Exercise 5.4 (AN2.4.13). If f is an entire function such that $|f(z)| \ge 1$ for all $z \in \mathbb{C}$, then f is a constant function.

Solution. Suppose f is an entire function such that $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Then $f(z) \neq 0$, and so

$$(1/f)'(z) = -\frac{f'(z)}{f^2(z)}$$

by the Quotient Rule. Since $z \in \mathbb{C}$ is arbitrary it follows that $1/f$ is an entire function; and since

$$\big|(1/f)(z)\big| = \left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|} \in (0,1],$$

we see that $1/f$ is also bounded. Hence $1/f$ is a constant function by Liouville's Theorem: there exists some $c \neq 0$ such that $1/f \equiv c$ on $\mathbb{C}$, and hence $f \equiv 1/c$ on $\mathbb{C}$. ■

Exercise 5.5 (AN2.4.16). Suppose that f is an entire function such that $\operatorname{Im} f \geq 0$ on $\mathbb{C}$. Prove that f is constant.

Solution. We have $f = u + iv$ with $v(z) \geq 0$ for all $z \in \mathbb{C}$. Define $g = e^{if}$, which is to say $g(z) = \exp(if(z))$. Since if and the exponential function are analytic on $\mathbb{C}$, by the Chain Rule it follows that the composition g is an entire function. Also g is bounded,

$$0 < |g(z)| = |e^{if(z)}| = |e^{-v(z)+iu(z)}| = e^{-v(z)} = \frac{1}{e^{v(z)}} \leq 1,$$

so g is constant by Liouville's Theorem and we have $g' \equiv 0$. Now, for any $z \in \mathbb{C}$,

$$0 = g'(z) = \exp'(if(z)) \cdot if'(z) = if'(z)\exp(if(z)),$$

and we obtain $f'(z) = 0$ since $\exp(if(z)) \neq 0$. Therefore $f' \equiv 0$ and we conclude by Theorem 3.26 that f is constant. ■

5.2 – The Identity Theorem

Definition 5.6. *Let f be analytic on Ω, $z_0 \in \Omega$, and $f(z_0) = 0$. If there exists some $m \in \mathbb{N}$ and analytic function $g : \Omega \to \mathbb{C}$ such that $g(z_0) \neq 0$ and*

$$f(z) = (z - z_0)^m g(z)$$

for all $z \in \Omega$, then f is said to have a ***zero of order m*** *at z_0, written $\operatorname{ord}(f, z_0) = m$. A zero of order 1 is called a* ***simple zero****.*

If a function f has a zero of order $m \in \mathbb{N}$ at a point z_0 where it is analytic, we also say that z_0 is a zero for f of **multiplicity** m. For the more elegant statement of certain theorems we define $\operatorname{ord}(f, z) = 0$ whenever $f(z) \neq 0$. Also it is convenient to define, for $f : \Omega \to \mathbb{C}$, the set

$$Z(f) = \{z \in \Omega : f(z) = 0\}$$

of all zeros of f in its domain. Finally,

$$Z(f, S) = Z(f) \cap S = \{z \in S : f(z) = 0\}$$

for any arbitrary $S \subseteq \Omega$.

Proposition 5.7. *Suppose f is analytic and not identically zero on any component of Ω. If $S = \{\zeta_1, \ldots, \zeta_n\} \subseteq Z(f)$ with $\operatorname{ord}(f, \zeta_j) = k_j$ for each $1 \leq j \leq n$, then there is an analytic function $g : \Omega \to \mathbb{C}$ that is nonvanishing on S such that*

$$f(z) = g(z) \prod_{j=1}^{n} (z - \zeta_j)^{k_j}$$

for all $z \in \Omega$.

Proof. Define the analytic function $\psi : \Omega \to \mathbb{C}$ by

$$\psi(z) = \prod_{j=1}^{n} (z - \zeta_j)^{k_j}.$$

The function f/ψ is analytic on $\Omega \setminus S$, with

$$(f/\psi)(z) = f(z) \prod_{j=1}^{n} \frac{1}{(z - \zeta_j)^{k_j}}$$

for all $z \in \Omega \setminus S$. Now, for each j there exists analytic $\psi_j : \Omega \to \mathbb{C}$ with $\psi_j(\zeta_j) \neq 0$ such that

$$f(z) = (z - \zeta_j)^{k_j} \psi_j(z)$$

for all $z \in \Omega$, and since there exists some $\epsilon_j > 0$ such that $B_{\epsilon_j}(\zeta_j) \subseteq \Omega$ and $B'_{\epsilon_j}(\zeta_j) \cap S = \varnothing$,

$$\lim_{z \to \zeta_j} (f/\psi)(z) = \lim_{z \to \zeta_j} \left[(z - \zeta_j)^{k_j} \psi_j(z) \prod_{\ell=1}^{n} \frac{1}{(z - \zeta_\ell)^{k_\ell}} \right]$$

$$= \lim_{z\to\zeta_j}\left[\psi_j(z)\prod_{\ell\neq j}\frac{1}{(z-\zeta_\ell)^{k_\ell}}\right] = \psi_j(\zeta_j)\prod_{\ell\neq j}\frac{1}{(\zeta_j-\zeta_\ell)^{k_\ell}} \in \mathbb{C}_*.$$

Define

$$(f/\psi)(\zeta_j) = \psi_j(\zeta_j)\prod_{\ell\neq j}\frac{1}{(\zeta_j-\zeta_\ell)^{k_\ell}}$$

for each j so as to make f/ψ continuous and nonzero at each $\zeta_j \in S$. By Corollary 4.22, the continuity of f/ψ on $B_{\epsilon_j}(\zeta_j)$ and analyticity on $B'_{\epsilon_j}(\zeta_j)$ implies analyticity on $B_{\epsilon_j}(\zeta_j)$, and in this way f/ψ is extended to a function $g : \Omega \to \mathbb{C}$ that is analytic on Ω and nonzero on S. Specifically,

$$g(z) = \begin{cases} f(z)/\psi(z), & z \in \Omega \setminus S \\ \psi_j(\zeta_j)\prod_{\ell\neq j}(\zeta_j-\zeta_\ell)^{-k_\ell}, & z = \zeta_j. \end{cases}$$

For each $z \in \Omega \setminus S$,

$$g(z)\psi(z) = \frac{f(z)}{\psi(z)}\cdot\psi(z) = f(z),$$

and for each $\zeta_j \in S$,

$$g(\zeta_j)\psi(\zeta_j) = \psi_j(\zeta_j)\prod_{\ell\neq j}\frac{1}{(\zeta_j-\zeta_\ell)^{k_\ell}}\cdot 0 = 0 = f(\zeta_j).$$

Therefore $f = g\psi$ on Ω, as desired. ∎

Proposition 5.8. *Let f be analytic and not identically zero on any component of Ω, with $Z(f) = \{\zeta_1, \dots, \zeta_n\}$ and $\operatorname{ord}(f,\zeta_j) = k_j$ for $1 \le j \le n$. If $1 \le m \le n$ and g is the analytic function for which $g(\zeta_m) \neq 0$ and*

$$f(z) = (z-\zeta_m)^{k_m} g(z)$$

for all $z \in \Omega$, then $Z(g) = Z(f) \setminus \{\zeta_m\}$ with $\operatorname{ord}(g,\zeta_j) = k_j$ for each $j \neq m$.

Proof. By Proposition 5.7 there exists some analytic function $h : \Omega \to \mathbb{C}$ that is nonvanishing on $Z(f)$ and given by

$$f(z) = h(z)\prod_{j=1}^{n}(z-\zeta_j)^{k_j} \tag{5.2}$$

for all $z \in \Omega$. Suppose $1 \le m \le n$ and g is the analytic function for which $g(\zeta_m) \neq 0$ and

$$f(z) = (z-\zeta_m)^{k_m} g(z) \tag{5.3}$$

for all $z \in \Omega$. Equating the right-hand sides of (5.2) and (5.3) and dividing by $(z-\zeta_m)^{k_m}$ yields

$$g(z) = h(z)\prod_{j\neq m}(z-\zeta_j)^{k_j}$$

for all $z \neq \zeta_m$. This equation must hold for all $z \in \Omega$, including at ζ_m, since g and h are known to be continuous on Ω. In particular we have, for any $\ell \neq m$,

$$g(z) = (z-\zeta_\ell)^{k_\ell}\cdot h(z)\prod_{j\neq \ell,m}(z-\zeta_j)^{k_j},$$

where of course

$$\varphi(z) = h(z) \prod_{j \neq \ell, m} (z - \zeta_j)^{k_j}$$

is analytic on Ω with $\varphi(\zeta_\ell) \neq 0$. Therefore $\operatorname{ord}(g, \zeta_\ell) = k_\ell$ for any $\ell \neq m$. ■

Exercise 5.9 (**AN2.4.1**)**.** Give an example of a non-constant analytic function f on an open connected set Ω such that $Z(f)$ has a limit point on $\mathbb{C} \setminus \Omega$

Solution. Since $1/z$ is analytic on $\Omega = \mathbb{C}_*$ and $\sin(z)$ is analytic on $\mathbb{C}$, by the Chain Rule it follows that $f(z) = \sin(1/z)$ is analytic on Ω. Define the sequence $(z_n)_{n=1}^{\infty} \subseteq \Omega$ by $z_n = 1/(n\pi)$. Then $z_n \to 0$ as $n \to \infty$, yet

$$f(z_n) = \sin(1/z_n) = \sin(n\pi) = 0$$

for all n. Hence 0 is a limit point for $Z(f)$ on $\mathbb{C} \setminus \Omega = \{0\}$. ■

Exercise 5.10 (**AN2.4.2**)**.** Suppose f is analytic on Ω and $z_0 \in \Omega$, so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on $B_r(z_0) \subseteq \Omega$. Then $\operatorname{ord}(f, z_0) = m$ if and only if $a_n = 0$ for $0 \leq n \leq m - 1$ and $a_m \neq 0$.

Solution. Suppose that f has zero of order m at z_0, so there exists some analytic function $g : \Omega \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in \Omega$. By Theorem 4.29 we have

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

where $b_0 = g(z_0)$. Now,

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^{n+m} = \sum_{n=m}^{\infty} b_{n-m} (z - z_0)^n.$$

Defining $c_n = 0$ for $0 \leq n \leq m - 1$ and $c_n = b_{n-m}$ for $n \geq m$, we obtain

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for all $z \in B_r(z_0)$. Thus $a_n = c_n$ for all $n \geq 0$ by Corollary 4.33, and therefore $a_n = c_n = 0$ for $0 \leq n \leq m - 1$ and $a_m = c_m = b_0 = g(z_0) \neq 0$. In fact, by Theorem 4.32 we have

$$f^{(n)}(z_0) = n! a_n = 0$$

for $0 \leq n \leq m - 1$ and $f^{(m)}(z_0) \neq 0$.

Conversely, suppose $a_n = 0$ for $0 \leq n \leq m - 1$ and $a_m \neq 0$, so that

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n$$

on $B_r(z_0)$. Define $g : B_r(z_0) \to \mathbb{C}$ by

$$g(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m} = \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n.$$

The series is absolutely convergent on $B_r(z_0)$, and so g is analytic on $B_r(z_0)$ by Proposition 4.31(1). Then $g(z_0) = a_m \neq 0$, and

$$f(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m} = (z - z_0)^m g(z)$$

for $z \in B_r(z_0)$. In fact if we define

$$\hat{g}(z) = \begin{cases} g(z), & \text{if } z \in B_r(z_0) \\ f(z)/(z - z_0)^m, & \text{if } z \in \Omega \setminus B_r(z_0) \end{cases}$$

then $\hat{g} : \Omega \to \mathbb{C}$ is an analytic function such that $\hat{g}(z_0) \neq 0$ and $f(z) = (z - z_0)\hat{g}(z)$ on Ω. Therefore f is of order m at z_0. ■

Proposition 5.11. *Suppose that f is analytic at z_0 and $m \in \mathbb{N}$. Then* $\operatorname{ord}(f, z_0) = m$ *if and only if*

$$\min\{n \in \mathbb{Z} : f^{(n)}(z_0) \neq 0\} = m. \tag{5.4}$$

Proof. Suppose $\operatorname{ord}(f, z_0) = m$. Since f is analytic on $B_r(z_0)$ for some $r > 0$, by Theorem 4.29 we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for all $z \in B_r(z_0)$. By Exercise 5.10, $f^{(n)}(z_0)/n! = 0$ for $0 \leq n \leq m - 1$ and $f^{(m)}(z_0)/m! \neq 0$, and (5.4) readily follows. The proof of the converse is clear. ■

Lemma 5.12. *Let f be analytic on a region Ω. If L is the set of limit points of $Z(f)$ that lie in Ω, then L is both an open and closed set in Ω.*

Proof. Suppose $z_0 \in \Omega$ is such that $z_0 \notin Z(f)$. Then $|f(z_0)| > 0$, and since f is continuous at z_0, there exists some $\delta > 0$ such that

$$|f(z_0) - f(z)| < \tfrac{1}{2}|f(z_0)|,$$

and hence $|f(z)| > \frac{1}{2}|f(z_0)|$, for all $z \in B_\delta(z_0)$. Thus $B_\delta(z_0) \cap Z(f) = \varnothing$, which implies that $z_0 \notin L$. Therefore $L \subseteq Z(f)$.

Next, let $z_0 \in \Omega \setminus L$, so $z_0 \in \Omega$ is not a limit point of $Z(f)$ in Ω, and hence z_0 is not a limit point of $Z(f)$ in $\mathbb{C}$. Then there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \subseteq \Omega$ and $B_\epsilon(z_0) \cap Z(f) = \varnothing$. Now, for any $z \in B_\epsilon(z_0)$ there exists $\epsilon_z > 0$ such that $B_{\epsilon_z}(z) \subseteq B_\epsilon(z_0)$, and so $B_{\epsilon_z}(z) \cap Z(f) = \varnothing$. Then $z \in \Omega \setminus L$, which implies that $B_\epsilon(z_0) \subseteq \Omega \setminus L$. This shows that $\Omega \setminus L$ is open in $\mathbb{C}$, so $(\Omega \setminus L)^c$ is closed in $\mathbb{C}$, and then since $L = \Omega \cap (\Omega \setminus L)^c$ it follows that L is closed in Ω.

Finally, fix $z_0 \in L$. Since $L \subseteq \Omega$, by Theorem 4.29 there exists $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in B_r(z_0) \subseteq \Omega$. Also $f(z_0) = 0$ since $L \subseteq Z(f)$. By Exercise 5.10 either $a_n = 0$ for all n, or $\operatorname{ord}(f, z_0) = m \in \mathbb{N}$. Suppose $\operatorname{ord}(f, z_0) = m \in \mathbb{N}$, so there exists analytic $g : \Omega \to \mathbb{C}$ with $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in \Omega$. Since g is continuous at z_0, there exists $0 < \delta < r$ such that $g(z) \neq 0$ for all $z \in B_\delta(z_0)$, and hence $f(z) \neq 0$ for all $z \in B'_\delta(z_0)$. Thus $B'_\delta(z_0) \cap Z(f) = \varnothing$, implying z_0 is not a limit point of $Z(f)$; that is, $z_0 \notin L$, which is a contradiction. We conclude that $a_n = 0$ for all n, so $f \equiv 0$ on $B_r(z_0) \subseteq \Omega$, and hence $B_r(z_0) \subseteq L$. This shows that L is open in $\mathbb{C}$, and since $L \subseteq \Omega$, it follows that L is open in Ω. ■

Theorem 5.13 (**Identity Theorem**). *Suppose f is analytic on a region Ω. If $Z(f)$ has a limit point in Ω, then $f \equiv 0$ on Ω.*

Proof. Suppose $Z(f)$ has a limit point in Ω, which is to say $L \neq \varnothing$, where L is the set of limit points of $Z(f)$ that lie in Ω. By Lemma 5.12, L is both open and closed in Ω. In particular there exists a closed set $F \subseteq \mathbb{C}$ such that $L = \Omega \cap F$, and then $\Omega \setminus L = \Omega \cap F^c$ shows that $\Omega \setminus L$ is open in Ω. Since $(\Omega \setminus L) \cap L = \varnothing$ and $(\Omega \setminus L) \cup L = \Omega$, the connectedness of Ω implies that either $\Omega \setminus L = \varnothing$ or $L = \varnothing$. However, $L \neq \varnothing$ by hypothesis, and so $\Omega \setminus L = \varnothing$ obtains. Hence $\Omega \subseteq L$, and therefore $L = \Omega$. In the proof of Lemma 5.12 we found that $L \subseteq Z(f) \subseteq \Omega$, and so it follows that $Z(f) = \Omega$. That is, $f \equiv 0$ on Ω. ■

Corollary 5.14. *Suppose f and g are analytic on a region Ω. If $Z(f - g)$ has a limit point in Ω, then $f \equiv g$ on Ω.*

Proposition 5.15. *Suppose f is analytic on a region Ω, $z_0 \in \Omega$, and $f(z_0) = 0$. If f is not identically zero on Ω, then* $\operatorname{ord}(f, z_0) = m$ *for some $m \in \mathbb{N}$.*

Proof. Suppose f is not identically zero on Ω. Let $r > 0$ be such that $B = B_r(z_0) \subseteq \Omega$. If $f|_B \equiv 0$, then $Z(f)$ has a limit point in Ω (for instance z_0 itself), and so by the Identity Theorem we arrive at the contradiction that f must be identically zero on Ω. Hence $f|_B$ is not identically zero, and since

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for all $z \in B$ by Theorem 4.29, it follows that $f^{(n)}(z_0)/n! \neq 0$ for some $n \in \mathbb{N}$ (note $n > 0$ since $f(z_0) = 0$ by hypothesis). The Well-Ordering Principle now implies that

$$m = \min\left\{ n \in \mathbb{N} : \frac{f^{(n)}(z_0)}{n!} \neq 0 \right\}$$

exists in $\mathbb{N}$, so that $f^{(n)}(z_0) = 0$ for $0 \leq n \leq m-1$ and $f^{(m)}(z_0) \neq 0$, and therefore $\operatorname{ord}(f, z_0) = m$ by Proposition 5.11. ■

Exercise 5.16 (**AN2.4.11**). An open set Ω is connected if and only if, for any functions f and g analytic on Ω, $fg \equiv 0$ implies that $f \equiv 0$ or $g \equiv 0$.

Solution. Suppose Ω is connected. Let $f, g : \Omega \to \mathbb{C}$ be analytic, and suppose that $fg \equiv 0$. Fix $z_0 \in \Omega$, and let $r > 0$ be such that $K = \overline{B}_r(z_0) \subseteq \Omega$. Let

$$Z(f, K) = \{z \in K : f(z) = 0\} \quad \text{and} \quad Z(g, K) = \{z \in K : g(z) = 0\}.$$

Since $f(z)g(z) = 0$ for all $z \in K$, every point in K is in either $Z(f, K)$ or $Z(g, K)$, which implies that $Z(f, K) \cup Z(g, K) = K$ and either $Z(f, K)$ or $Z(g, K)$ is an infinite set. If $Z(f, K)$ is infinite, then since K is compact and $Z(f, K) \subseteq K$, we conclude that $Z(f, K)$ has a limit point in K, and thus $Z(f, K)$ has a limit point in Ω and it follows by the Identity Theorem that $f \equiv 0$. If $Z(g, K)$ is infinite, then $g \equiv 0$. Therefore either $f \equiv 0$ or $g \equiv 0$.

For the converse, suppose that Ω is not connected, so that Ω consists of at least two components. Let Ω_1 be one component, and let $\Omega_2 = \Omega \setminus \Omega_1$ be the union of all other components, so that Ω_1 and Ω_2 are both open sets. Define $f : \Omega \to \mathbb{C}$ by

$$f(z) = \begin{cases} 0, & \text{if } z \in \Omega_1 \\ 1, & \text{if } z \in \Omega_2 \end{cases}$$

and define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \begin{cases} 1, & \text{if } z \in \Omega_1 \\ 0, & \text{if } z \in \Omega_2 \end{cases}$$

Then f and g are analytic on Ω such that $fg \equiv 0$, yet neither $f \equiv 0$ nor $g \equiv 0$ is the case. ■

Exercise 5.17 (AN2.4.14). Does there exist an entire function f, not identically zero, for which the set $Z(f)$ is uncountable?

Solution. Suppose a set $S \subseteq \mathbb{C}$ is uncountable. For each $n \in \mathbb{N}$ define $B_n = B_n(0)$. Claim: there exists some $n \in \mathbb{N}$ such that $S \cap B_n$ is uncountable. To verify the claim, suppose that $S \cap B_n$ is at most countable (i.e. empty, finite, or countable) for all n. Since a countable union of at most countable sets is at most countable, it follows that

$$\bigcup_{n=1}^{\infty} (S \cap B_n)$$

is at most countable. But this is impossible, since

$$\bigcup_{n=1}^{\infty} (S \cap B_n) = S \cap \left(\bigcup_{n=1}^{\infty} B_n \right) = S \cap \mathbb{C} = S,$$

and S is uncountable by hypothesis. The claim must be true.

Now, suppose f is an entire function such that $Z(f)$ is uncountable. Then there exists some $n \in \mathbb{N}$ such that $Z_n(f) := Z(f) \cap \overline{B}_n$ is uncountable. In particular $Z_n(f)$ is an infinite subset of the compact set $\overline{B}_n$, and hence $Z_n(f)$ has a limit point $w \in \overline{B}_n$ by Theorem 2.39. Since $Z_n(f) \subseteq Z(f)$, it follows that w is a limit point of $Z(f)$ as well. We now see that $Z(f)$ has a limit point in the open connected set $\mathbb{C}$ on which f is analytic, and so by the Identity Theorem we must have $f \equiv 0$. Therefore there does not exist an entire function with uncountably many zeros. ■

5.3 – The Maximum Principle for Analytic Functions

Lemma 5.18. *Suppose $\varphi : [a,b] \to \mathbb{R}$ is continuous, there exists $\alpha \in \mathbb{R}$ such that $\varphi \leq \alpha$ on $[a,b]$, and*

$$\frac{1}{b-a}\int_a^b \varphi \geq \alpha. \tag{5.5}$$

Then $\varphi \equiv \alpha$ on $[a,b]$.

Proof. Suppose $\varphi(c) < \alpha$ for some $c \in [a,b]$, so there exists $\delta > 0$ such that $\varphi(c) = \alpha - 2\delta$. By continuity there exists some $\epsilon > 0$ such that $\varphi(c) \leq \alpha - \delta$ for all $x \in (c-\epsilon, c+\epsilon) \cap [a,b]$. Now,

$$\int_a^b \varphi = \int_a^{c-\epsilon} \varphi + \int_{c-\epsilon}^{c+\epsilon} \varphi + \int_{c+\epsilon}^b \varphi \leq \int_a^{c-\epsilon} \alpha + \int_{c-\epsilon}^{c+\epsilon} (\alpha - \delta) + \int_{c+\epsilon}^b \alpha = \alpha(b-a) - 2\delta\epsilon,$$

whence $\int_a^b \varphi < \alpha(b-a)$, and finally

$$\frac{1}{b-a}\int_a^b \varphi < \alpha.$$

Thus (5.5) implies that $\varphi(x) \geq \alpha$ holds for all $x \in [a,b]$, and since $\varphi \leq \alpha$ on $[a,b]$ by hypothesis, we conclude that $\varphi \equiv \alpha$ on $[a,b]$. ■

We now present the maximum principle for analytic functions, henceforth to be referred to simply as the Maximum Principle. The proof of the first part makes use of two easily checked facts: $\operatorname{Re} z \leq |z|$, and if $\operatorname{Re} z = |z|$ then $z = \operatorname{Re} z$.

Theorem 5.19 (**Maximum Principle**)**.** *Let f be analytic on a region Ω.*

1. *If $|f|$ has a local maximum at some $z_0 \in \Omega$, then f is constant on Ω.*
2. *Let*
$$\alpha = \sup_{z \in \Omega} |f(z)|.$$
Then either $|f| < \alpha$ on Ω or f is constant on Ω.
3. *Suppose Ω is bounded. If*
$$\limsup |f(z_n)| \leq \alpha$$
for every sequence $(z_n) \subseteq \Omega$ that converges to some $z \in \partial\Omega$, then either $|f| < \alpha$ on Ω or f is constant on Ω.
4. *Suppose Ω is bounded and f is continuous on $\overline{\Omega}$. If $M = \max\{|f(z)| : z \in \partial\Omega\}$, then either $|f| < M$ on Ω or f is constant on $\overline{\Omega}$. Moreover,*
$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|. \tag{5.6}$$

Proof.
Proof of Part (1). Suppose $|f|$ has a local maximum at some $z_0 \in \Omega$. Thus there exists some $r > 0$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \overline{B}_r(z_0) \subseteq \Omega$. If $f(z_0) = 0$, then $f \equiv 0$ on $\overline{B}_r(z_0)$, and hence $f \equiv 0$ (i.e. f is constant) on Ω by the Identity Theorem.

Assume $f(z_0) = w_0 \neq 0$. Define $\gamma : [0, 2\pi] \to \partial B_r(z_0)$ by $\gamma(t) = z_0 + re^{it}$. Set $\varphi = |f \circ \gamma|/|w_0|$. Then $\varphi : [0, 2\pi] \to \mathbb{R}$ is a continuous function such that $\varphi \leq 1$ on $[0, 2\pi]$. Now by Corollary 4.18,

$$w_0 = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} (f \circ \gamma)(t)\, dt,$$

so

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{(f \circ \gamma)(t)}{w_0}\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|(f \circ \gamma)(t)|}{|w_0|}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)\, dt$$

and Lemma 5.18 implies that $\varphi \equiv 1$ on $[0, 2\pi]$. That is, $|f(z)/w_0| = 1$ for all $z \in \partial B_r(z_0)$.

Next, define $\psi : [0, 2\pi] \to \mathbb{R}$ by

$$\psi(t) = \mathrm{Re}\left(\frac{(f \circ \gamma)(t)}{w_0}\right),$$

which is continuous. For any $t \in [0, 2\pi]$ we have $\psi(t) \leq \varphi(t) \leq 1$ by the general property $\mathrm{Re}\, z \leq |z|$, while

$$\frac{1}{2\pi} \int_0^{2\pi} \psi(t)\, dt = \mathrm{Re}\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{(f \circ \gamma)(t)}{w_0}\, dt\right) = \mathrm{Re}(1) = 1 \geq 1.$$

Thus $\psi \equiv 1$ on $[0, 2\pi]$ by Lemma 5.18. That is, $\mathrm{Re}[f(z)/w_0] = 1$ for all $z \in \partial B_r(z_0)$.

We now have $|f(z)/w_0| = 1 = \mathrm{Re}[f(z)/w_0]$ for all $z \in \partial B_r(z_0)$, giving $f(z)/w_0 = 1$ and hence $f(z) = w_0$ for all $z \in C_r(z_0)$. This shows that $Z(f - w_0) \supseteq C_r(z_0)$, and since every point on $C_r(z_0)$ is a limit point of $C_r(z_0)$, we conclude by Corollary 5.14 that $f \equiv w_0$ on Ω.

Proof of Part (2). Suppose $|f| < \alpha$ on Ω is not the case. Then there exists some $z_0 \in \Omega$ such that $|f(z_0)| = \alpha$, and thus $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. This implies $|f|$ has a local maximum at $z_0 \in \Omega$, and by part (1) it follows that f is constant on Ω. Therefore either $|f| < \alpha$ on Ω or f is constant on Ω.

Proof of Part (3). Suppose $\limsup |f(z_n)| \leq \alpha$ for every sequence $(z_n)_{n \in \mathbb{N}}$ in Ω such that $z_n \to z_b$ for some $z_b \in \partial\Omega$. Let $\beta = \sup_{z \in \Omega} |f(z)|$. Thus for each $n \in \mathbb{N}$ there exists some $z_n \in \Omega$ such that $\beta - n^{-1} < |f(z_n)| \leq \beta$, and then $\lim_{n \to \infty} |f(z_n)| = \beta$. Now, $\overline{\Omega}$ is compact since Ω is bounded, and so by Theorem 2.39 the sequence $(z_n)_{n \in \mathbb{N}}$ has a subsequence $(z_{n_k})_{k \in \mathbb{N}}$ that converges to some $z_0 \in \overline{\Omega}$.

If $z_0 \in \Omega$, then by the continuity of $|f|$ we have

$$\beta = \lim_{k \to \infty} |f(z_{n_k})| = |f(z_0)|.$$

Hence $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, showing that $|f|$ has a local maximum at z_0, and therefore f is constant on Ω by part (1).

Suppose $z_0 \notin \Omega$. Then $z_0 \in \partial\Omega$, and so

$$\beta = \lim_{k \to \infty} |f(z_{n_k})| = \limsup |f(z_{n_k})| \leq \alpha.$$

By part (2) either f is constant on Ω or $|f| < \beta$, and therefore either f is constant on Ω or $|f| < \alpha$.

Proof of Part (4). We note first that, since $\partial\Omega$ is compact and $|f||_{\partial\Omega}$ is continuous, there exists some $z_b \in \partial\Omega$ such that

$$|f(z_b)| = \max_{z\in\partial\Omega} |f(z)| := M.$$

Also $\overline{\Omega}$ is compact and $|f| : \overline{\Omega} \to \mathbb{R}$ is continuous, and so there exists $z_0 \in \overline{\Omega}$ for which

$$|f(z_0)| = \max_{z\in\overline{\Omega}} |f(z)| := N.$$

Suppose $z_0 \in \Omega$. Then $|f|$ has a local maximum at a point in Ω, and part (1) implies that f is constant on Ω. In particular $|f| \equiv N$ on Ω, so by the continuity of $|f| : \overline{\Omega} \to \mathbb{R}$ we have $|f| \equiv N$ on $\overline{\Omega}$, whence $M = N$ obtains and (5.6) is affirmed.

Next suppose $z_0 \in \partial\Omega$. Then $N \leq M$, and since it is clear that $M \leq N$, we once again obtain $M = N$. Now, by part (2) either $|f| < \sup_{z\in\Omega} |f(z)|$ or f is constant on Ω, and since

$$\sup_{z\in\Omega} |f(z)| \leq \max_{z\in\overline{\Omega}} |f(z)| = M,$$

we conclude that either $|f| < M$ on Ω or f is constant on $\overline{\Omega}$. ■

Theorem 5.20 (**Minimum Principle**). *Let f be analytic on a region Ω such that $f(z) \neq 0$ for all $z \in \Omega$.*

1. *If $|f|$ has a local minimum at some $z_0 \in \Omega$, then f is constant on Ω.*
2. *Let*
$$\beta = \inf_{z\in\Omega} |f(z)|.$$
Then either $|f| > \beta$ on Ω or f is constant on Ω.
3. *Suppose Ω is bounded. If*
$$\liminf |f(z_n)| \geq \beta$$
for every sequence $(z_n) \subseteq \Omega$ that converges to some $z \in \partial\Omega$, then either $|f| > \beta$ on Ω or f is constant on Ω.
4. *Suppose Ω is bounded and f is continuous on $\overline{\Omega}$. If $m = \min\{|f(z)| : z \in \partial\Omega\}$, then either $|f| > m$ on Ω or f is constant on $\overline{\Omega}$. Moreover,*
$$\min_{z\in\overline{\Omega}} |f(z)| = \min_{z\in\partial\Omega} |f(z)|.$$

Proof.
Proof of Part (1). Suppose $|f|$ has a local minimum at some $z_0 \in \Omega$. Since f is nonvanishing on Ω, we have that $1/f$ is analytic on Ω, and moreover $|1/f| = 1/|f|$ has a local maximum at $z_0 \in \Omega$. Thus $1/f$ is constant on Ω by Theorem 5.19(1), and the conclusion follows.

Proof of Part (2). If $\beta = 0$, then $|f| > \beta$ on Ω is immediate. Suppose $\beta > 0$. Then

$$\frac{1}{\beta} = \sup_{z\in\Omega} \left|\frac{1}{f(z)}\right|$$

and by Theorem 5.19(2) it follows that either $1/|f| < 1/\beta$ on Ω or $1/f$ is constant on Ω. Therefore either $|f| > \beta$ on Ω or f is constant on Ω. ■

Lemma 5.21 (**Schwarz's Lemma**). *Let f be analytic on $\mathbb{B}$, with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{B}$. Then*

$$|f(z)| \leq |z| \tag{5.7}$$

for all $z \in \mathbb{B}$, and

$$|f'(0)| \leq 1. \tag{5.8}$$

If equality holds in (5.7) *for some $z \neq 0$, or if equality holds in* (5.8)*, then there exists some $\lambda \in \mathbb{S}$ such that $f(z) = \lambda z$ for all $z \in \mathbb{B}$.*

Proof. Define $g : \mathbb{B} \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z)/z, & z \in \mathbb{B}' \\ f'(0), & z = 0, \end{cases}$$

clearly analytic on $\mathbb{B}'$. Since $f(0) = 0$ by hypothesis, we have $\lim_{z\to 0} f(z)/z = f'(0)$, so that g is continuous on $\mathbb{B}$ and therefore analytic on $\mathbb{B}$ by Corollary 4.22.

Fix $z \in \mathbb{B}'$, and let $|z| < r < 1$. Since g is analytic on $B_r(0)$ and continuous on $\overline{B}_r(0)$, Theorem 5.19(4) implies that

$$|g(z)| \leq \max_{w\in\overline{B}_r(0)} |g(w)| = \max_{w\in\partial B_r(0)} |g(w)| = \max_{w\in\partial B_r(0)} \frac{|f(w)|}{r} = \frac{1}{r} \max_{w\in\partial B_r(0)} |f(w)| \leq \frac{1}{r}.$$

That is, $|g(z)| \leq r^{-1}$ for all $r < 1$ such that $r > |z|$, and hence $|g(z)| \leq 1$ for all $z \in \mathbb{B}'$. The continuity of g on $\mathbb{B}$ then ensures that $|g(z)| \leq 1$ for all $z \in \mathbb{B}$, and thus $|f'(0)| \leq 1$ in particular. Again noting that $f(0) = 0$, $|g| \leq 1$ on $\mathbb{B}$ makes clear that $|f(z)| \leq |z|$ for all $z \in \mathbb{B}$.

Suppose $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{B}'$. Then $|g(z_0)| = 1$, and since $|g| \leq 1$ on $\mathbb{B}$, we find that $|g|$ has a local maximum at z_0 and hence $g \equiv \lambda$ for some constant $\lambda \in \mathbb{C}$ by Theorem 5.19(1). This gives $f(z) = \lambda z$ for all $z \in \mathbb{B}$, with $|z_0| = |f(z_0)| = |\lambda||z_0|$ for $|z_0| \neq 0$ leading us to conclude that $|\lambda| = 1$. That is, $\lambda \in \mathbb{S}$.

Finally, suppose $|f'(0)| = 1$. Now $|g|$ has a local maximum at 0, so that once again $g \equiv \lambda$ for some $\lambda \in \mathbb{C}$. For any $z \in \mathbb{B}'$,

$$g(z) = \frac{f(z)}{z} = \frac{\lambda z}{z} = \lambda,$$

whence

$$1 = |f'(0)| = \lim_{z\to 0} |g(z)| = |\lambda|,$$

and so $\lambda \in \mathbb{S}$ once more. ■

Exercise 5.22 (**AN2.4.5**). If f is analytic on an open connected set Ω and $|f|$ is constant on Ω, show that f is constant on Ω.

Solution. Suppose f is analytic on an open connected set Ω and $|f|$ is constant on Ω. Then, for any fixed $z_0 \in \Omega$ and $\epsilon > 0$ such that $B_\epsilon(z_0) \subseteq \Omega$, we have $|f(z)| = |f(z_0)|$ for all $z \in B_\epsilon(z_0)$, so f has a local maximum at z_0. Therefore f is constant on Ω by the Maximum Principle. ■

Exercise 5.23 (**AN2.4.8**). Suppose that $K \subseteq \mathbb{C}$ is compact, f is continuous on K, and f is analytic on K°. Show that

$$\max_{z\in K} |f(z)| = \max_{z\in\partial K} |f(z)|. \tag{5.9}$$

Moreover, if $|f(z_0)| = \max_{z\in K} |f(z)|$ for some $z_0 \in K^\circ$, then f is constant on the component of K° that contains z_0.

Solution. Since K is compact and f is continuous on K, there exists some $z_0 \in K$ such that

$$M = \max_{z \in K} |f(z)| = |f(z_0)|.$$

If $z_0 \in \partial K$, then (5.9) follows immediately and we're done.

Suppose that $z_0 \in K^\circ$. Let Ω be the component of K° containing z_0. Thus Ω is an open connected set, f is analytic on Ω, and $|f|$ has a local maximum at $z_0 \in \Omega$. By the Maximum Principle and the continuity of f on $\overline{\Omega} \subseteq K$, we conclude that $f \equiv M$ on $\overline{\Omega}$. Now, $\partial\Omega \neq \varnothing$ since Ω is a bounded set, and thus there exists some $w \in \partial\Omega \subseteq \overline{\Omega}$ such that $f(w) = M$. If $w \notin \partial K$, then $w \in K^\circ$ and so w must lie in the interior of some component of K°, and not on the boundary of some component, which is a contradiction. Therefore $w \in \partial K$, and since $w \in K$ also, we obtain (5.9) once again. ■

Exercise 5.24 (AN2.4.9). Suppose that Ω is a bounded open set, f is continuous on $\overline{\Omega}$ and analytic on Ω. Show that

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$

Solution. The set $\overline{\Omega}$ is compact, and the interior of $\overline{\Omega}$ is Ω. By the previous exercise we obtain

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\overline{\Omega}} |f(z)|,$$

and then the desired result follows by observing that $\partial\overline{\Omega} = \partial\Omega$. ■

Exercise 5.25 (AN2.4.24). Suppose that f is analytic on $\mathbb{B}$, with $f(0) = 0$. For each $n \in \mathbb{N}$ define $f_n : \mathbb{B} \to \mathbb{C}$ by $f_n(z) = f(z^n)$. Prove that $\sum f_n$ is uniformly convergent on compact subsets of $\mathbb{B}$.

Solution. Let $K \subseteq \mathbb{B}$ be a compact set. Let $0 < r < 1$ be such that $K \subseteq B_r(0)$. Since f is continuous on $\overline{B}_r(0)$, there exists some $M \in (0, \infty)$ such that $|f(z)| \leq M$ for all $z \in \overline{B}_r(0)$. Define $\varphi : \mathbb{B} \to \mathbb{C}$ by $\varphi(z) = f(z)/M$, and define $h : \mathbb{B} \to B_r(0)$ by $h(z) = rz$. Since h is analytic on $\mathbb{B}$ and φ is analytic on $h(\mathbb{B}) = B_r(0)$, by the Chain Rule $\varphi \circ h$ is analytic on $\mathbb{B}$. Also

$$(\varphi \circ h)(0) = \varphi(h(0)) = \varphi(0) = \frac{f(0)}{M} = 0,$$

and for any $z \in \mathbb{B}$,

$$|(\varphi \circ h)(z)| = |\varphi(rz)| = \frac{|f(rz)|}{M} \leq 1.$$

Hence by Schwarz's Lemma

$$|(\varphi \circ h)(z)| = |\varphi(rz)| \leq |z|$$

for all $z \in \mathbb{B}$. From this it is immediate that $|\varphi(z)| \leq |z|/r$ for any $z \in B_r(0)$, so that

$$|f(z)| \leq \frac{M|z|}{r}$$

on $B_r(0)$.

Now, for any $n \in \mathbb{N}$ and $z \in B_r(0)$, since

$$|z| < r < 1 \quad \Rightarrow \quad |z^n| < r^n < r \quad \Rightarrow \quad z^n \in B_r(0),$$

we find that
$$|f_n(z)| = |f(z^n)| \le \frac{M|z^n|}{r} < \frac{Mr^n}{r} = Mr^{n-1}.$$
Thus
$$\|f_n\|_K \le Mr^{n-1}$$
for each $n \in \mathbb{N}$, and since $\sum Mr^{n-1}$ is a convergent series, we conclude by the Weierstrass M-Test that $\sum f_n$ converges uniformly on K. ∎

5.4 – The Maximum Principle for Harmonic Functions

Theorem 5.26 (**Identity Theorem for Harmonic Functions**). *Suppose u is harmonic on a region Ω. If u is constant on some nonempty open set $\Omega' \subseteq \Omega$, then u is constant on Ω.*

Proof. Suppose $u \equiv c$ on open $\varnothing \neq \Omega' \subseteq \Omega$. Define

$$A = \big\{a \in \Omega : \exists\rho > 0\big(u|_{B_\rho(a)} \equiv c\big)\big\}.$$

By construction A is open in Ω, and it is nonempty since $A \supseteq \Omega'$. Fix $z \in \Omega \setminus A$. Let $r > 0$ be such that $B := B_r(z) \subseteq \Omega$. By Theorem 3.15 there exists harmonic $v : B \to \mathbb{R}$ such that $f = u + iv$ is analytic on B. Define $g : B \to \mathbb{C}$ by $g = e^f$, which is also analytic on B.

Now, suppose there exists $w_0 \in B$ with $w_0 \in A$. Then there can be found $\rho > 0$ such that $B_\rho(w_0) \subseteq B$ and $u \equiv c$ on $B_\rho(w_0)$. Since $|g| = e^{\operatorname{Re} f} = e^u$ by Theorem 4.42, it is clear that $|g|$ has a local maximum at w_0. Thus g is constant on B by Theorem 5.19(1), which implies that e^u is constant on B. Noting that u is real-valued and the exponential function is injective on $\mathbb{R}$, it follows that u is constant on B. In particular $u|_B \equiv c$, which implies that $z \in A$, a contradiction. Hence $B \cap A = \varnothing$ must be the case, or equivalently $B \subseteq \Omega \setminus A$, and we see that $\Omega \setminus A$ is open in Ω. Since Ω is connected and $A \neq \varnothing$, it follows that $\Omega \setminus A = \varnothing$ and therefore $A = \Omega$. That is, $u \equiv c$ on Ω. ■

Theorem 5.27 (**Maximum Principle for Harmonic Functions**). *Let u be harmonic on a region Ω.*

1. *If u has a local maximum at some $z_0 \in \Omega$, then u is constant on Ω.*
2. *Let*
$$\alpha = \sup_{z\in\Omega} u(z).$$
Then either $u < \alpha$ on Ω or u is constant on Ω.
3. *Suppose Ω is bounded. If*
$$\limsup u(z_n) \leq \alpha$$
for every sequence $(z_n) \subseteq \Omega$ that converges to some $z \in \partial\Omega$, then either $u < \alpha$ on Ω or u is constant on Ω.
4. *Suppose Ω is bounded and u is continuous on $\overline{\Omega}$. If $M = \max\{u(z) : z \in \partial\Omega\}$, then either $u < M$ on Ω or u is constant on $\overline{\Omega}$. Moreover,*
$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in\partial\Omega} u(z).$$

Proof.
Proof of Part (1). Suppose u has a local maximum at $z_0 \in \Omega$. Let $r > 0$ be such that $B := B_r(z_0) \subseteq \Omega$ and $u(z) \leq u(z_0)$ for all $z \in B$. By Theorem 3.15 there exists harmonic $v : B \to \mathbb{R}$ such that $f = u + iv$ is analytic on B. Define $g = e^f$. Since $|g| = e^u$, it is clear from the strictly increasing nature of the exponential function on $\mathbb{R}$ that $|g|$ has a local maximum at z_0. Thus g is constant on B by Theorem 5.19(1), implying that e^u—and hence u—is constant on B. Therefore u is constant on Ω by Theorem 5.26. ■

Theorem 5.28 (Minimum Principle for Harmonic Functions). *Let u be harmonic on a region Ω.*

1. *If u has a local minimum at some $z_0 \in \Omega$, then u is constant on Ω.*
2. *Let*
$$\beta = \inf_{z\in\Omega} u(z).$$
Then either $u > \beta$ on Ω or u is constant on Ω.
3. *Suppose Ω is bounded. If*
$$\liminf u(z_n) \geq \beta$$
for every sequence $(z_n) \subseteq \Omega$ that converges to some $z \in \partial\Omega$, then either $u > \beta$ on Ω or u is constant on Ω.
4. *Suppose Ω is bounded and u is continuous on $\overline{\Omega}$. If $m = \min\{u(z) : z \in \partial\Omega\}$, then either $u > m$ on Ω or u is constant on $\overline{\Omega}$. Moreover,*
$$\min_{z\in\overline{\Omega}} u(z) = \min_{z\in\partial\Omega} u(z).$$

Exercise 5.29 (AN2.4.6). If f is continuous on $\overline{\mathbb{B}}$, analytic on $\mathbb{B}$, and real-valued on $\partial\mathbb{B}$, then f is constant on $\overline{\mathbb{B}}$.

Solution. Suppose $f = u + iv$ is continuous on $\overline{\mathbb{B}}$, analytic on $\mathbb{B}$, and real-valued on $\partial\mathbb{B}$. Then $u, v : \overline{\mathbb{B}} \to \mathbb{R}$ are harmonic on $\mathbb{B}$ by Theorem 4.28, and in particular v is continuous on $\overline{\mathbb{B}}$ with
$$\max_{z\in\partial\mathbb{B}} v(z) = M = 0 = m = \min_{z\in\partial\mathbb{B}} v(z),$$
since $v(z) = 0$ for all $z \in \partial\mathbb{B}$. By Theorem 5.27(4) either $v < 0$ on $\mathbb{B}$ or v is constant on $\mathbb{B}$, and by Theorem 5.28(4) either $v > 0$ on $\mathbb{B}$ or v is constant on $\mathbb{B}$. Since $v < 0$ on $\mathbb{B}$ contradicts both conclusions of Theorem 5.28, we must conclude that v is a constant on $\mathbb{B}$, and hence $v_x \equiv 0$ and $v_y \equiv 0$ on $\mathbb{B}$. Now, since f is complex-differentiable on $\mathbb{B}$, by the Cauchy-Riemann equations
$$f'(z) = u_x(z) + iu_y(z) = v_y(z) - iv_x(z) = 0$$
for all $z \in \mathbb{B}$. That is, $f' \equiv 0$ on $\mathbb{B}$, and therefore f is constant on $\mathbb{B}$ by Theorem 3.26. Since f is continuous on $\overline{\mathbb{B}}$, it follows that f must be constant on $\overline{\mathbb{B}}$. ■

Exercise 5.30 (AN2.4.19). Suppose f and g are analytic on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$. If $\operatorname{Re} f = \operatorname{Re} g$ on $\partial\mathbb{B}$, prove that $f - g$ is constant on $\overline{\mathbb{B}}$.

Solution. Suppose $\operatorname{Re} f = \operatorname{Re} g$ on $\partial\mathbb{B}$. The analyticity of $f = u + iv$ and $g = \hat{u} + i\hat{v}$ on $\mathbb{B}$ implies that u and $\hat{u}$ are harmonic on $\mathbb{B}$, and the continuity of f and g on $\overline{\mathbb{B}}$ implies that u and $\hat{u}$ are continuous on $\overline{\mathbb{B}}$. Thus $\operatorname{Re} f - \operatorname{Re} g = u - \hat{u}$ is harmonic on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$, with
$$\max_{z\in\partial\mathbb{B}}(u - \hat{u})(z) = M = 0 = m = \min_{z\in\partial\mathbb{B}}(u - \hat{u})(z).$$
By Theorem 5.27 either $u - \hat{u} < 0$ on $\mathbb{B}$ or $u - \hat{u}$ is constant on $\mathbb{B}$, and by the Theorem 5.28 either $u - \hat{u} > 0$ on $\mathbb{B}$ or $u - \hat{u}$ is constant on $\mathbb{B}$. Since $u - \hat{u} < 0$ on $\mathbb{B}$ contradicts both conclusions of the Theorem 5.28, we must conclude that $u - \hat{u}$ is a constant on $\mathbb{B}$, and hence $u_x - \hat{u}_x \equiv 0$ and

$u_y - \hat{u}_y \equiv 0$ on $\mathbb{B}$. Now, since $f - g$ is complex-differentiable on $\mathbb{B}$, by the Cauchy-Riemann equations

$$\begin{aligned}(f-g)'(z) = f'(z) - g'(z) &= [u_x(z) - iu_y(z)] - [\hat{u}_x(z) - i\hat{u}_y(z)] \\ &= (u_x - \hat{u}_x)(z) - i(u_y - \hat{u}_y)(z) = 0\end{aligned}$$

for all $z \in \mathbb{B}$. That is, $(f-g)' \equiv 0$ on $\mathbb{B}$, and therefore $f - g$ is constant on $\mathbb{B}$ by Theorem 3.26. Since $f - g$ is continuous on $\overline{\mathbb{B}}$, it follows that $f - g$ must be constant on $\overline{\mathbb{B}}$. ■

Exercise 5.31 (AN2.4.21). Prove that if $u : \mathbb{C} \to \mathbb{R}$ is a nonnegative harmonic function, then u is constant.

Solution. Suppose that u is harmonic such that $u(z) \geq 0$ for all $z \in \mathbb{C}$. By Theorem 3.15 there exists a harmonic function $v : \mathbb{C} \to \mathbb{R}$ such that $f = u + iv$ is an entire function. Now, $if = -v + iu$ is likewise entire, and since $\text{Im}(if) = u \geq 0$ on $\mathbb{C}$, it follows by Exercise 5.5 that if, and hence f itself, is constant. That is, there exist $a, b \in \mathbb{R}$ such that

$$f(z) = u(z) + iv(z) = a + ib$$

for all $z \in \mathbb{C}$, which shows that $u \equiv a$ on $\mathbb{C}$. ■

6

Cauchy's Theorem

6.1 – Logarithms and Arguments

It has been established by Theorem 4.42(9) that, for any $\theta \in \mathbb{R}$, the exponential function restricted to the horizontal strip

$$H_\theta = \mathbb{R} \times [\theta, \theta + 2\pi) = \{x + iy : x \in \mathbb{R} \text{ and } y \in [\theta, \theta + 2\pi)\}$$

is a bijection on $\mathbb{C}_*$, and therefore it has an inverse function $\mathbb{C}_* \to H_\theta$. In what follows we let $\exp_\theta$, the **θ-exponential**, denote the restriction of the exponential function to H_θ; that is, $\exp_\theta = \exp|_{H_\theta}$.

Definition 6.1. *For any $\theta \in \mathbb{R}$, the **θ-logarithm** is the function $\log_\theta = \exp_\theta^{-1}$. That is, $\log_\theta : \mathbb{C}_* \to H_\theta$ is such that*

$$\forall z \in H_\theta\big(\log_\theta(\exp_\theta(z)) = z\big) \quad \text{and} \quad \forall z \in \mathbb{C}_*\big(\exp_\theta(\log_\theta(z)) = z\big).$$

*The **argument** of $\log_\theta$ is the function $\arg_\theta : \mathbb{C}_* \to [\theta, \theta + 2\pi)$ given by $\arg_\theta(z) = \mathrm{Im}(\log_\theta(z))$.*

*The **principal logarithm** is the function $\mathrm{Log} = \log_{-\pi}$, and the **principal argument** is $\mathrm{Arg} = \arg_{-\pi}$.*

Recalling the last two parts of Theorem 4.42, an illustration of the workings of $\exp_\theta$ and $\log_\theta$ is provided by Figure 11.

Typically when working with $\log_\theta$ it is understood that the exponential function is restricted to H_θ, and so we may write

$$\forall z \in H_\theta\big(\log_\theta(\exp(z)) = z\big) \quad \text{and} \quad \forall z \in \mathbb{C}_*\big(\exp(\log_\theta(z)) = z\big).$$

Some basic properties of the θ-logarithm are as follows.

Proposition 6.2. *For all $z \in \mathbb{C}_*$,*

$$\log_\theta(z) = \ln|z| + i\arg_\theta(z),$$

where $\arg_\theta(z)$ is the unique number in the interval $[\theta, \theta + 2\pi)$ for which

$$z = |z|e^{i\arg_\theta(z)}.$$

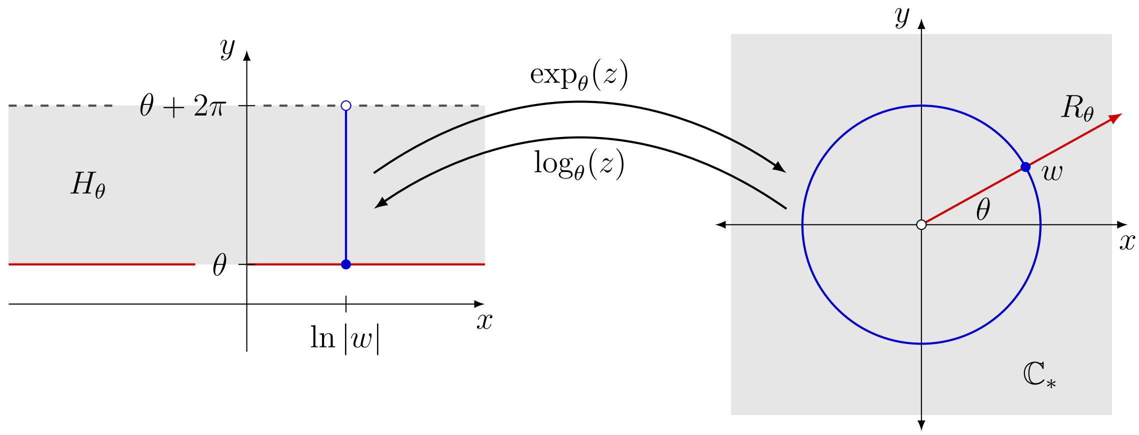

FIGURE 11.

Proof. Let $z \in \mathbb{C}_*$, and let $w = \log_\theta(z)$. By definition w is the unique number in H_θ for which $\exp(w) = z$, while by Theorem 4.42(6) we have $|\exp(w)| = e^{\operatorname{Re} w}$. Hence $e^{\operatorname{Re} w} = |z| > 0$, so that

$$\ln|z| = \ln(e^{\operatorname{Re} w}) = \operatorname{Re} w = \operatorname{Re}(\log_\theta(z)).$$

Since $\arg_\theta(z) = \operatorname{Im}(\log_\theta(z))$, we obtain

$$w = \log_\theta(z) = \ln|z| + i\arg_\theta(z)$$

as desired, which in turn yields

$$z = \exp(w) = \exp\big(\ln|z| + i\arg_\theta(z)\big) = e^{\ln|z|}e^{i\arg_\theta(z)} = |z|e^{i\arg_\theta(z)}$$

as was also to be shown.

Now,

$$w \in H_\theta \;\;\Rightarrow\;\; \operatorname{Im}(w) \in [\theta, \theta + 2\pi) \;\;\Rightarrow\;\; \arg_\theta(z) \in [\theta, \theta + 2\pi);$$

and if $\alpha, \beta \in [\theta, \theta + 2\pi)$ are such that $z = |z|e^{i\alpha}$ and $z = |z|e^{i\beta}$, so that $\exp(i\alpha) = \exp(i\beta)$ obtains, then

$$\exp(i\alpha) = \exp(i\beta) \;\;\Rightarrow\;\; i\alpha = i\beta \;\;\Rightarrow\;\; \alpha = \beta$$

since $i\alpha, i\beta \in H_\theta$ and exp is one-to-one on H_θ. Therefore $\arg_\theta(z)$ is the unique number in $[\theta, \theta + 2\pi)$ for which $z = |z|e^{i\arg_\theta(z)}$. ■

Let $\theta \in \mathbb{R}$, and suppose $z = re^{i\alpha}$ for some $\alpha \in [\theta, \theta + 2\pi)$. By Proposition 6.2,

$$e^{i\arg_\theta(z)} = \frac{z}{|z|} = \frac{re^{i\alpha}}{r} = e^{i\alpha}$$

for $i\alpha, i\arg_\theta(z) \in H_\theta$, and so $\arg_\theta(z) = \alpha$ since the exponential function is injective on H_θ by Theorem 4.42(9). This proves the following.

Proposition 6.3. *Let $\theta \in \mathbb{R}$. If $\alpha \in [\theta, \theta + 2\pi)$, then*

$$\arg_\theta(re^{i\alpha}) = \alpha$$

for any $r > 0$. In particular $\arg_\theta(re^{i\theta}) = \theta$.

In general $\arg_\theta(re^{i\alpha})$ may be said to be an angle that is co-terminal with the angle α between the ray $R_\alpha = \{re^{i\alpha} : r \in [0,\infty)\}$ and the positive real axis.

Proposition 6.4. *Let $R_\theta = \{re^{i\theta} : r \in [0,\infty)\}$. Then $\log_\theta$ and $\arg_\theta$ are continuous at each point on $\mathbb{C} \setminus R_\theta$, and discontinuous at each point on R_θ.*

Proof. Clearly $\arg_\theta$ and $\log_\theta$ are not continuous at 0 since neither function is defined there. Fix $z \in R_\theta \setminus \{0\}$, so $z = re^{i\theta}$ for some $r > 0$. Define $(z_n)_{n\in\mathbb{N}}$ by

$$z_n = re^{i(\theta+2\pi-1/n)},$$

so that $\lim_{n\to\infty} z_n = z$. For each n, $\theta+2\pi-1/n \in [\theta, \theta+2\pi)$, and hence $\arg_\theta(z_n) = \theta+2\pi-1/n$ by Proposition 6.3. Now,

$$\lim_{n\to\infty} \arg_\theta(z_n) = \lim_{n\to\infty}\left(\theta + 2\pi - \frac{1}{n}\right) = \theta + 2\pi \neq \theta = \arg_\theta(z),$$

and so $\arg_\theta$ is not continuous at z by Theorem 2.20. Moreover, by Proposition 6.2,

$$\begin{aligned}\lim_{n\to\infty} \log_\theta(z_n) &= \lim_{n\to\infty}\big[\ln|z_n| + i\arg_\theta(z_n)\big] = \lim_{n\to\infty}\big[\ln r + i\arg_\theta(z_n)\big] \\ &= \ln r + \theta + 2\pi \neq \ln r + \theta = \log_\theta(z),\end{aligned}$$

and so $\log_\theta$ is also not continuous at z.

The continuity of $\arg_\theta$ and $\log_\theta$ on $\mathbb{C} \setminus R_\theta$ is easily shown using Propositions 6.2 and 6.3. ■

Proposition 6.5. *The function $\log_\theta : \mathbb{C}_* \to H_\theta$ is analytic precisely on $\mathbb{C} \setminus R_\theta$, with*

$$\log_\theta'(z) = \frac{1}{z}$$

for all $z \in \mathbb{C} \setminus R_\theta$.

Proof. By Proposition 6.4 the function $\log_\theta : \mathbb{C} \setminus R_\theta \to H_\theta^\circ$ is continuous, and of course $\exp : H_\theta^\circ \to \mathbb{C}$ is analytic. Also $\exp'(z) = \exp(z) \neq 0$ for all $z \in H_\theta^\circ$, and $\exp(\log_\theta(z)) = z$ for all $z \in \mathbb{C} \setminus R_\theta$. Thus $\log_\theta$ is analytic on $\mathbb{C} \setminus R_\theta$ by Theorem 3.6, and by the same theorem

$$\log_\theta'(z) = \frac{1}{\exp'(\log_\theta(z))} = \frac{1}{\exp(\log_\theta(z))} = \frac{1}{z}$$

for all $z \in \mathbb{C} \setminus R_\theta$. ■

For any $\theta \in \mathbb{R}$ we call the restriction of $\log_\theta$ to its domain of analyticity $\mathbb{C} \setminus R_\theta$ a **branch** of the logarithm. What we mean here by "the logarithm" is the so-called "multivalued-function" $\mathbb{C}_* \to \mathbb{C}$ that maps each $z \neq 0$ to *all* $w \in \mathbb{C}$ for which $\exp(w) = z$. The symbol log (no subscript and lowercase "l") will be used either to denote an unspecified branch of the logarithm function, or to serve as a placeholder for different branches of the logarithm function.

In particular the principal logarithm $\operatorname{Log} : \mathbb{C}_* \to \mathbb{R} \times [-\pi, \pi)$ is analytic on the set $\mathbb{C}\setminus(-\infty, 0]$ known as the **slit plane**. Indeed, in light of earlier findings, $\operatorname{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{R} \times (-\pi, \pi)$ is an analytic bijection whose inverse is also analytic, which is to say the principal logarithm restricted to the slit plane is a **diffeomorphism**. Henceforth we call Log restricted to $\mathbb{C}\setminus(-\infty, 0]$ the **principal branch** of the logarithm.

Recalling that the principal logarithm's full domain is $\mathbb{C}_*$, we are in a position to define the **complex exponentiation** w^z of any $w \in \mathbb{C}_*$ by $z \in \mathbb{C}$ as follows:

$$w^z = e^{z \operatorname{Log} w}. \tag{6.1}$$

For fixed $z \in \mathbb{C}$ the function $w \mapsto w^z$ is seen to be analytic on $\mathbb{C} \setminus (-\infty, 0]$ by Proposition 6.5 and Theorem 3.5.

Example 6.6. We find a power series representation for the principal logarithm function Log on $B_1(1)$. First we note that Log is analytic on $B_1(1)$ since, by Proposition 6.5, it is in fact analytic on $\mathbb{C} \setminus R_{-\pi}$. By Theorem 4.12 we have, for all $z \in B_1(1)$,

$$\operatorname{Log}(z) = \sum_{n=0}^{\infty} \frac{\operatorname{Log}^{(n)}(1)}{n!}(z-1)^n.$$

Using Proposition 6.5 for the first derivative,

$$\operatorname{Log}'(z) = \frac{0!}{z}, \quad \operatorname{Log}''(z) = -\frac{1!}{z^2}, \quad \operatorname{Log}'''(z) = \frac{2!}{z^3}, \quad \operatorname{Log}^{(4)}(z) = \frac{3!}{z^4},$$

and in general

$$\operatorname{Log}^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{z^n}$$

for $n \geq 1$ (which can be formally affirmed by induction). Thus $\operatorname{Log}^{(n)}(1) = (-1)^{n-1}(n-1)!$ for $n \geq 1$, and since $\operatorname{Log}(1) = 0$ we obtain

$$\operatorname{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$$

for all $z \in B_1(1)$. ■

Definition 6.7. *Let (X, d) be a metric space, and let $f : X \to \mathbb{C}_*$ be continuous. A continuous function $\lambda : X \to \mathbb{C}$ is a **continuous logarithm** of f on X if $f(x) = e^{\lambda(x)}$ for all $x \in X$. A continuous function $\alpha : X \to \mathbb{R}$ is a **continuous argument** of f on X if $f(x) = |f(x)|e^{i\alpha(x)}$ for all $x \in X$.*

Proposition 6.8. *Let $f : X \to \mathbb{C}$ be continuous.*

1. *If λ is a continuous logarithm of f, then $\operatorname{Im} \lambda$ is a continuous argument of f.*
2. *If α is a continuous argument of f, then $\ln |f| + i\alpha$ is a continuous logarithm of f.*
3. *Suppose X is a connected set. If λ_1 and λ_2 are continuous logarithms of f, then $\lambda_1 - \lambda_2 \equiv 2\pi i k$ for some $k \in \mathbb{Z}$.*
4. *Suppose X is a connected set. If α_1 and α_2 are continuous arguments of f, then $\alpha_1 - \alpha_2 \equiv 2\pi k$ for some $k \in \mathbb{Z}$.*

Proof.
Proof of Part (1). Suppose $\lambda : X \to \mathbb{C}$ is a continuous logarithm of f on X, so $f(x) = e^{\lambda(x)}$ for all $x \in X$. Then

$$|f(x)| = \left|e^{\operatorname{Re} \lambda(x) + i \operatorname{Im} \lambda(x)}\right| = \left|e^{\operatorname{Re} \lambda(x)} e^{i \operatorname{Im} \lambda(x)}\right| = e^{\operatorname{Re} \lambda(x)} \left|e^{i \operatorname{Im} \lambda(x)}\right| = e^{\operatorname{Re} \lambda(x)},$$

and so

$$f(x) = e^{\operatorname{Re}\lambda(x)}e^{i\operatorname{Im}\lambda(x)} = |f(x)|e^{i\operatorname{Im}\lambda(x)},$$

where $\operatorname{Im}\lambda$ is continuous on X by Theorem 2.23. Hence $\operatorname{Im}\lambda$ is a continuous argument of f.

Proof of Part (2). Suppose $\alpha : X \to \mathbb{R}$ is a continuous argument of f, so $f(x) = |f(x)|e^{i\alpha(x)}$ for all $x \in X$. Let $\lambda = \ln|f| + i\alpha$. Then

$$e^{\lambda(x)} = e^{\ln|f(x)|+i\alpha(x)} = e^{\ln|f(x)|}e^{i\alpha(x)} = |f(x)|e^{i\alpha(x)} = f(x)$$

for all $x \in X$, and since λ is continuous on X we find that λ is a continuous argument for f.

Proof of Part (3). Let λ_1 and λ_2 be continuous logarithms of f. For each $x \in X$ we have

$$e^{\lambda_1(x)} = f(x) = e^{\lambda_2(x)},$$

and so $\lambda_1(x) = \lambda_2 + 2\pi ik(x)$ for some $k(x) \in \mathbb{Z}$. That is, $(\lambda_1 - \lambda_2)(x) = 2\pi ik(x)$ for all $x \in X$. Suppose $k : X \to \mathbb{Z}$ is not constant, so that there exist $x, y \in X$ such that $k(x) = m \neq n = k(y)$. Define the sets $A = \{2\pi im\}$ and

$$B = \{2\pi ik(x) : x \in X \text{ and } k(x) \neq m\} = \operatorname{Ran}(\lambda_1 - \lambda_2) \setminus A,$$

so that $\operatorname{Ran}(\lambda_1 - \lambda_2) = A \cup B$. Both A and B are open sets in the discrete topological subspace $A \cup B \subseteq \mathbb{C}$, and since $\lambda = \lambda_1 - \lambda_2$ is continuous, it follows that $X_1 = \lambda^{-1}(A)$ and $X_2 = \lambda^{-1}(B)$ are open sets in X. Now, $A, B \neq \varnothing$ implies $X_1, X_2 \neq \varnothing$ (note that $2\pi in \in B$), and since $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \varnothing$, we see that X_1 and X_2 constitute a separation of X and therefore X is not connected! As this is a contradiction, we conclude that $k : X \to \mathbb{Z}$ must be constant, and so $\lambda_1 - \lambda_2 \equiv 2\pi ik$ on X for some fixed $k \in \mathbb{Z}$.

Proof of Part (4). Let α_1 and α_2 be continuous arguments of f on X. Then $\lambda_1 = \ln|f| + i\alpha_1$ and $\lambda_2 = \ln|f| + i\alpha_2$ are continuous logarithms of f by Part (1), and so $\lambda_1 - \lambda_2 \equiv 2\pi ik$ for some $k \in \mathbb{Z}$ by Part (3). Now,

$$2\pi ik \equiv \lambda_1 - \lambda_2 = (\ln|f| + i\alpha_1) - (\ln|f| + i\alpha_2) = i(\alpha_1 - \alpha_2),$$

and so $\alpha_1 - \alpha_2 \equiv 2\pi k$. ∎

Proposition 6.9. *If $\gamma : [a,b] \to \mathbb{C}_*$ is a curve, then γ has a continuous argument.*

Proof. Suppose $\gamma : [a,b] \to \mathbb{C}_*$ is a curve. Since $[a,b]$ is compact and $|\gamma| : [a,b] \to \mathbb{R}$ is continuous, the Extreme Value Theorem implies there exist some $\tau \in [a,b]$ such that $|\gamma(\tau)| \leq |\gamma(t)|$ for all $t \in [a,b]$. Setting $\epsilon = |\gamma(\tau)|$ and noting that $0 \notin \gamma^*$, we have $\epsilon > 0$.

Now, γ is uniformly continuous on $[a,b]$, which is to say there exists $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < \epsilon$ whenever $s, t \in [a,b]$ are such that $|s - t| < \delta$. Let

$$a = t_0 < t_1 < \cdots < t_n = b$$

be a partition of $[a,b]$ with $\gamma([t_{k-1}, t_k]) \subseteq B_\epsilon(\gamma(t_{k-1}))$ for each $1 \leq k \leq n$. Define $\gamma_k = \gamma|_{[t_{k-1},t_k]}$. Since $B_\epsilon(\gamma(t_{k-1})) \subseteq \mathbb{C} \setminus R_{\theta_k}$ for some ray R_{θ_k}, by Proposition 6.4 we find that $\alpha_k = \arg_{\theta_k} \circ \gamma_k$ is a continuous argument for γ_k.

By Proposition 6.8 the arguments α_1 and α_2 for γ_1 and γ_2 are such that $\alpha_2(t_1)-\alpha_1(t_1)=2\pi j$ for some $j\in\mathbb{Z}$. Defining $\hat{\alpha}_2=\alpha_2-2\pi j$, we find that $\hat{\alpha}_2$ is a continuous argument for γ_2 such that $\alpha_1\cup\hat{\alpha}_2$ is a continuous argument for $\gamma_1\cup\gamma_2$. The arguments $\hat{\alpha}_2$ and α_3 for γ_2 and γ_3 likewise differ by an integer multiple of 2π, and so we may define a new argument $\hat{\alpha}_3$ for γ_3 such that $\alpha_1\cup\hat{\alpha}_2\cup\hat{\alpha}_3$ is a continuous argument for $\gamma_1\cup\gamma_2\cup\gamma_3$. Continuing in this way, we come to construct a continuous argument for $\bigcup_{k=1}^n\gamma_k=\gamma$ and we are done. ∎

Definition 6.10. *Let f be analytic on Ω. Then $g:\Omega\to\mathbb{C}$ is an* ***analytic logarithm*** *of f on Ω if g is analytic and $\exp(g)=f$ on Ω.*

By Theorem 4.42(4) it is clear that for an analytic function $f:\Omega\to\mathbb{C}$ to have an analytic logarithm on Ω, it is necessary that $f(z)\neq 0$ for all $z\in\Omega$; that is, f must be nonvanishing on Ω. Is this sufficient? The next theorem makes clear that it is not, unless for instance Ω happens to be a starlike region.

Theorem 6.11. *Let f be analytic and nonvanishing on Ω. Then the following statements are equivalent.*

1. *f has an analytic logarithm on Ω.*
2. *f'/f has a primitive on Ω.*
3. *$\oint_\gamma f'/f=0$ for every closed path γ in Ω.*

Corollary 6.12. *If Ω is an open set such that*

$$\oint_\gamma g=0$$

for every analytic function $g:\Omega\to\mathbb{C}$ and every closed path γ in Ω, then every nonvanishing analytic function $\Omega\to\mathbb{C}$ has an analytic logarithm on Ω.

Proof. Let $f:\Omega\to\mathbb{C}$ be a nonvanishing analytic function. Then f'/f is analytic on Ω, so that

$$\oint_\gamma f'/f=0$$

for every closed path γ in Ω. Therefore f has an analytic logarithm on Ω by Theorem 6.11. ∎

Exercise 6.13 (AN3.2.1a). Let $\Omega\subseteq\mathbb{C}_*$ such that $\Omega\cap R_\theta\neq\varnothing$ for all $\theta\in\mathbb{R}$. Show that $\log_\theta$ is not analytic on Ω for any θ.

Solution. Fix $\theta\in\mathbb{R}$. There exists some $z_0\in\Omega$ such that $z_0\in R_\theta$, and since $\log_\theta$ is not continuous at z_0 by Proposition 6.4, we conclude that it is also not analytic there. Hence $\log_\theta$ is not analytic on Ω. ∎

Exercise 6.14 (AN3.2.1b). Let f be the identity function $f(z)=z$. Show there exists a region $\Omega\subseteq\mathbb{C}_*$ such that $\Omega\cap R_\theta\neq\varnothing$ for all $\theta\in\mathbb{R}$, and yet f has an analytic logarithm on Ω.

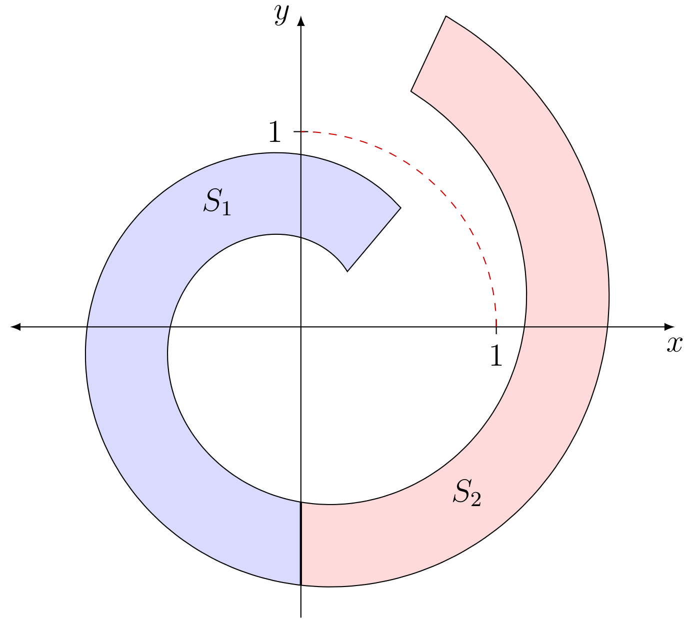

FIGURE 12. The region Ω.

Solution. Consider the region $\Omega \subseteq \mathbb{C}$ shown in Figure 12. For the sake of interest, Ω as pictured is the open set of points that lie between the paths given by

$$\gamma_1(t) = \left(\tfrac{3}{20}t^{3/2} - 1\right)e^{it} \quad \text{and} \quad \gamma_2(t) = \left(\tfrac{3}{20}t^{3/2} + 1\right)e^{it}$$

for $7 < t < 14$. More precisely

$$\Omega = \bigcup_{t\in(7,14)} \big(\gamma_1(t), \gamma_2(t)\big),$$

where $(\gamma_1(t), \gamma_2(t))$ is the open line segment with endpoints $\gamma_1(t)$ and $\gamma_2(t)$. Let $S_1 \subseteq \Omega$ be the set

$$S_1 = \{z \in \Omega : \mathrm{Re}(z) \leq 0\} \cup \{z \in \Omega : \mathrm{Re}(z), \mathrm{Im}(z) \geq 0 \text{ and } |z| < 1\},$$

and let $S_2 \subseteq \Omega$ be

$$S_2 = \{z \in \Omega : \mathrm{Re}(z) \geq 0 \text{ and } z \notin S_1\},$$

as shown in Figure 12. Define $\lambda : \Omega \to \mathbb{C}$ by

$$\lambda(z) = \begin{cases} \log_0(z), & z \in S_1 \\ \log_\pi(z), & z \in S_2 \end{cases}$$

Note that $S = S_1 \cap S_2$ is nonempty, and so we must verify that λ is well-defined on S. If $z \in S$, then $z = re^{3\pi i/2}$ for some $r > 0$, and since $3\pi/2 \in [0, 2\pi) \cap [\pi, 3\pi)$, by Proposition 6.3 we obtain $\arg_0(z) = \arg_\pi(z) = 3\pi/2$. Hence

$$\log_0(z) = \ln|z| + i\arg_0(z) = \ln|z| + i\arg_\pi(z) = \log_\pi(z)$$

by Proposition 6.2, and so λ is a well-defined (and continuous) function throughout its domain.

For any $z \in S_1^\circ = S_1 \setminus S$, we have $z = re^{i\theta}$ for some $\theta \in (0, 2\pi)$, and so

$$e^{\lambda(z)} = \exp(\log_0(z)) = z = f(z).$$

Similarly, for any $z \in S_2^\circ = S_2 \setminus S$, we have $z = re^{i\theta}$ for some $\theta \in (\pi, 3\pi)$, and so

$$e^{\lambda(z)} = \exp(\log_\pi(z)) = z = f(z).$$

Finally, if $z \in S$, then $\lambda(z) = \log_0(z) = \log_\pi(z)$ as observed earlier, so that once again $e^{\lambda(z)} = z = f(z)$. Therefore λ is a continuous logarithm of f.

It remains to verify that λ is analytic on Ω. Certainly it is analytic on S_1°, since $\lambda|_{S_1^\circ} \equiv \log_0$, $S_1^\circ \subseteq \mathbb{C} \setminus R_0$ and $\log_0$ is analytic on $\mathbb{C} \setminus R_0$ by Proposition 6.5. Similarly, λ is analytic on S_2°, since $\lambda|_{S_2^\circ} \equiv \log_\pi$, $S_2^\circ \subseteq \mathbb{C} \setminus R_\pi$ and $\log_\pi$ is analytic on $\mathbb{C} \setminus R_\pi$. Fix $w \in S$. There exists some $\epsilon > 0$ sufficiently small that $B = B_\epsilon(w) \subseteq \Omega$, and $z \in B$ implies that $z = re^{i\theta}$ for some $\theta \in (\pi, 2\pi) \subseteq [0, 2\pi) \cap [\pi, 3\pi)$. As a consequence, $z \in B$ implies that $\arg_\pi(z) = \theta = \arg_0(z)$, so that $\log_\pi(z) = \log_0(z)$ and hence $\lambda|_B \equiv \log_0$. This shows that λ is analytic on B, and therefore analytic at w. Since $w \in S$ is arbitrary, we conclude that λ is analytic on $S \cup S_1^\circ \cup S_2^\circ = \Omega$. ■

Exercise 6.15 (AN3.2.5). Let $f(z) = z$ for all z. Show that f does not have a continuous argument on $\mathbb{S}$.

Solution. Suppose that f does have a continuous argument on $\mathbb{S}$. Thus there exists some continuous $\alpha : \mathbb{S} \to \mathbb{R}$ such that, for all $z \in \mathbb{S}$, we have $f(z) = |f(z)|e^{i\alpha(z)}$, and thus $z = e^{i\alpha(z)}$. Define $\gamma : [0, 2\pi] \to \mathbb{S}$ by $\gamma(t) = e^{it}$. Since $|\gamma(t)| = 1$, we obtain $\gamma(t) = |\gamma(t)|e^{it}$ for all $t \in [0, 2\pi]$, which shows that $\beta : [0, 2\pi] \to \mathbb{R}$ given by $\beta(t) = t$ is a continuous argument of γ. On the other hand we have

$$\gamma(t) = e^{i\alpha(\gamma(t))} = |\gamma(t)|e^{i(\alpha\circ\gamma)(t)}$$

for all $t \in [0, 2\pi]$, where the continuity of γ on $[0, 2\pi]$ and the continuity of α on $\mathbb{S}$ imply the continuity of $\alpha \circ \gamma$ on $[0, 2\pi]$, and so $\alpha \circ \gamma$ is a continuous argument of γ. Since β and $\alpha \circ \gamma$ are continuous arguments of γ, it follows that $\alpha \circ \gamma - \beta \equiv 2\pi k$ for some $k \in \mathbb{Z}$. From this we obtain

$$2\pi k = \alpha(\gamma(0)) - \beta(0) = \alpha(1) \quad \text{and} \quad 2\pi k = \alpha(\gamma(2\pi)) - \beta(2\pi) = \alpha(1) - 2\pi,$$

so that $\alpha(1) = 2\pi k$ and $\alpha(1) = 2\pi(k+1)$, a contradiction. Therefore f must have no continuous argument on $\mathbb{S}$. ■

Exercise 6.16 (AN3.2.6). Let

$$S = \big\{z : \mathrm{Re}(z) \in [a, b] \text{ and } \mathrm{Im}(z) \in [c, d]\big\},$$

and suppose $f : S \to \mathbb{C}_*$ is continuous. Show that f has a continuous logarithm on S.

Solution. Since f is continuous and S is a closed set, the set $f(S) \subseteq \mathbb{C}_*$ is likewise closed and thus $\epsilon = \min\{|f(z)| : z \in S\}$ is greater than 0. Because f is uniformly continuous on S, there exists some $\delta > 0$ such that, for any $z, w \in S$,

$$|z - w| \leq \delta \;\;\Rightarrow\;\; |f(z) - f(w)| < \epsilon.$$

Let

$$X = \{x_j : 0 \leq j \leq m,\, x_{j-1} < x_j,\, x_0 = a,\, x_m = b\}$$

be a partition of $[a,b]$ with mesh $\|X\| < \delta/2$, and let

$$Y = \{y_k : 0 \leq k \leq n,\ y_{k-1} < y_k,\ y_0 = c,\ y_n = d\}$$

be a partition of $[c,d]$ with mesh $\|Y\| < \delta/2$. Then $Z = X + iY$, where

$$X + iY = \{x + iy : x \in X \text{ and } y \in Y\},$$

is a partition of S such that $\|Z\| < \delta$. The partition Z subdivides S into rectangular subsets

$$S_{jk} = \big\{z : \mathrm{Re}(z) \in [x_{j-1}, x_j] \text{ and } \mathrm{Im}(z) \in [y_{k-1}, y_k]\big\}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$, where $\mathrm{diam}(S_{jk}) \leq \|Z\| < \delta$. Define $z_{jk} \in S_{jk}$ to be the point given by $z_{jk} = x_j + iy_k$. Now,

$$z \in S_{jk} \ \Rightarrow\ |z - z_{jk}| < \delta \ \Rightarrow\ |f(z) - f(z_{jk})| < \epsilon,$$

so that $f(S_{jk}) \subseteq B_\epsilon(z_{jk})$. If f_{jk} denotes f restricted to S_{jk}, then $f_{jk} : S_{jk} \to B_\epsilon(z_{jk})$. Observing that $0 \notin B_\epsilon(z_{jk})$, there exists some $\alpha_{jk} \in \mathbb{R}$ such that $B_\epsilon(z_{jk}) \subseteq \mathbb{C} \setminus R_{\alpha_{jk}}$; and since $\arg_{\alpha_{jk}}$ is continuous on $\mathbb{C} \setminus R_{\alpha_{jk}}$ by Proposition 6.4, and

$$f(z) = |f(z)|e^{i\arg_{\alpha_{jk}}(f(z))}$$

for all $z \in S_{jk}$ by Proposition 6.2, we conclude that $\theta_{jk} := \arg_{\alpha_{jk}} \circ f$ is a continuous argument of f_{jk}.

Fix $1 \leq j \leq m$. For each $1 \leq k \leq n-1$ we find that θ_{jk} and $\theta_{j(k+1)}$ are continuous arguments for f restricted to the connected set $S_{jk} \cap S_{j(k+1)}$, and thus θ_{jk} and $\theta_{j(k+1)}$ must differ by an integer multiple of 2π on this set. Thus $\theta_{j1} - \theta_{j2} = 2\pi\ell_1$ for some $\ell_1 \in \mathbb{Z}$, but if we define

$$\varphi_j(z) = \begin{cases} \theta_{j1}(z), & z \in S_{j1} \\ \theta_{j2}(z) + 2\pi\ell_1, & z \in S_{j2} \end{cases}$$

it is easy to verify that φ_j is a continuous argument of f on $S_{j1} \cup S_{j2}$. Adjusting θ_{j3} by an appropriate constant $2\pi\ell_2$, we may extend φ_j to obtain a continuous argument of f on $S_{j1} \cup S_{j2} \cup S_{j3}$, and so on until we have constructed a continuous argument φ_j for f on

$$S_j = \bigcup_{k=1}^{n} S_{jk}.$$

Now, observing that φ_1 and φ_2 are continuous arguments for f on the line segment $S_1 \cap S_2$, we may adjust φ_2 by an integer multiple of 2π so as to obtain a continuous argument of f on $S_1 \cup S_2$. Continuing in this fashion, we finally obtain a continuous argument of f on

$$S = \bigcup_{j=1}^{m} S_j.$$

Now Proposition 6.8(2) implies that f has a continuous logarithm on S. $\blacksquare$

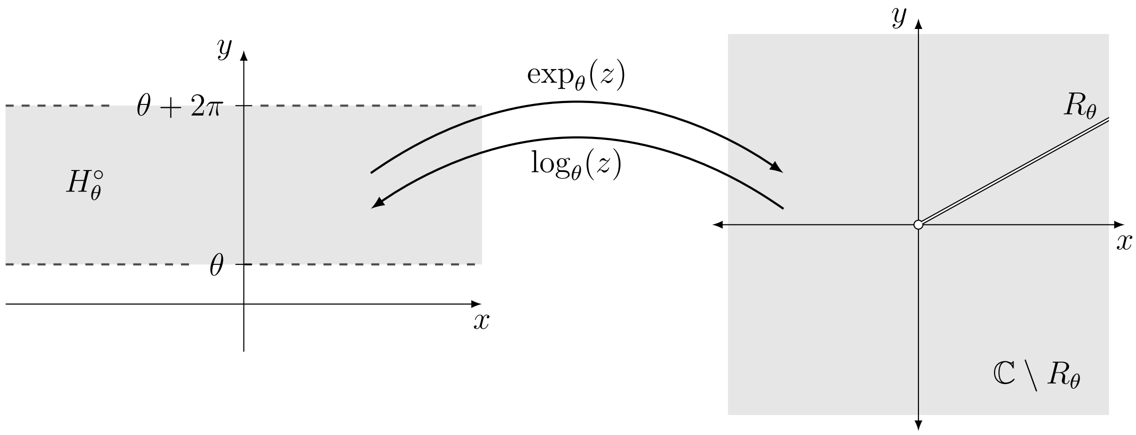

FIGURE 13.

Remark. Consider the rectangle $R = [a, b] \times [c, d] \subseteq \mathbb{R}^2$, and suppose $f : R \to \mathbb{C}_*$ is continuous. Let S be as in Exercise 6.16. Now, since $\varphi : S \to R$ given by $\varphi(x + iy) = (x, y)$ is a homeomorphism, it follows that $f \circ \varphi : S \to \mathbb{C}_*$ is continuous and so has a continuous logarithm $\lambda : S \to \mathbb{C}$ by Exercise 6.16. Hence

$$f(x, y) = (f \circ \varphi)(x + iy) = e^{\lambda(x+iy)} = e^{(\lambda\circ\varphi^{-1})(x,y)}$$

for all $(x, y) \in R$, which shows that $\lambda \circ \varphi^{-1} : R \to \mathbb{C}$ is a continuous argument of f.

Exercise 6.17 (AN3.2.7). Let f be analytic and zero-free on Ω, and suppose that λ is a continuous logarithm of f on Ω. Show that λ is analytic on Ω.

Solution. Fix $z_0 \in \Omega$. Let $\theta \in \mathbb{R}$ be such that $\lambda(z_0) \in H_\theta^\circ$. Since λ is continuous at z_0, and Ω and H_θ° are open, there exists some $\delta > 0$ such that $B = B_\delta(z_0) \subseteq \Omega$ and $z \in B$ implies $\lambda(z) \in H_\theta^\circ$. Thus $\lambda(B) \subseteq H_\theta^\circ$, whence $\exp(\lambda(B)) \subseteq \mathbb{C} \setminus R_\theta$ obtains (see Figure 13), and so $z \in B$ implies $f(z) = \exp(\lambda(z)) \in \mathbb{C} \setminus R_\theta$ and we conclude that $f(B) \subseteq \mathbb{C} \setminus R_\theta$.

By Proposition 6.5 the function $\log_\theta : \mathbb{C} \setminus R_\theta \to H_\theta^\circ$ is analytic, and $\exp : H_\theta^\circ \to \mathbb{C} \setminus R_\theta$ is its inverse. For any $z \in B$ we have

$$(\log_\theta \circ f)(z) = \log_\theta(f(z)) = \log_\theta(\exp(\lambda(z))) = \lambda(z),$$

since $\lambda(z) \in H_\theta^\circ$. Hence $\lambda = \log_\theta \circ f$ on B, and since f is analytic on B and $\log_\theta$ is analytic on $f(B)$, by the Chain Rule we conclude that λ is analytic on B. In particular λ is analytic at z_0, and since $z_0 \in \Omega$ is arbitrary it follows that λ is analytic on Ω. ■

6.2 – Winding Numbers

Proposition 6.18. *Suppose that $\gamma : [a,b] \to \mathbb{C}$ is a closed curve, with $z \notin \gamma^*$ and $\alpha : [a,b] \to \mathbb{R}$ a continuous argument of $\gamma - z$.*

1. *$\alpha(b) - \alpha(a) = 2\pi n$ for some $n \in \mathbb{Z}$.*
2. *If $\hat{\alpha} : [a,b] \to \mathbb{R}$ is another continuous argument of $\gamma - z$, then $\hat{\alpha}(b) - \hat{\alpha}(a) = \alpha(b) - \alpha(a)$.*

Proof.
Proof of Part (1). First note that $\gamma_z := \gamma - z$ is a closed curve, and since $z \notin \gamma^*$ implies that $0 \notin \gamma_z^*$, Proposition 6.9 ensures γ_z does indeed have a continuous argument α. That is,

$$\gamma_z(t) = |\gamma_z(t)|e^{i\alpha(t)}$$

for all $t \in [a,b]$, and from $\gamma_z(b) = \gamma_z(a)$ it follows that $e^{i\alpha(b)} = e^{i\alpha(a)}$. Therefore $\alpha(b) - \alpha(a) = 2\pi n$ for some $n \in \mathbb{Z}$ by Theorem 4.42(7).

Proof of Part (2). Let $\hat{\alpha}$ be another continuous argument of γ_z. Then $\hat{\alpha} - \alpha \equiv 2\pi k$ for some $k \in \mathbb{Z}$ by Proposition 6.8(4), and so

$$\hat{\alpha}(b) - \hat{\alpha}(a) = [\alpha(b) + 2\pi k] - [\alpha(a) + 2\pi k] = \alpha(b) - \alpha(a)$$

as desired. ■

Definition 6.19. *Suppose that $\gamma : [a,b] \to \mathbb{C}$ is a closed curve with $z \notin \gamma^*$, and let $\alpha : [a,b] \to \mathbb{R}$ be any continuous argument of $\gamma - z$. The **winding number** of z with respect to γ is*

$$\operatorname{wn}(\gamma, z) = \frac{\alpha(b) - \alpha(a)}{2\pi}.$$

The first part of Proposition 6.18 makes clear that the winding number $\operatorname{wn}(\gamma, z)$ is always an integer, and the second part guarantees that the winding number is independent of the continuous argument α chosen for $\gamma - z$.

Proposition 6.20. *Let $\gamma, \gamma_1, \gamma_2 : [a,b] \to \mathbb{C}$ be closed curves.*

1. *If $z \notin \gamma^*$, then $\operatorname{wn}(\gamma, z) = \operatorname{wn}(\gamma + w, z + w)$ for all $w \in \mathbb{C}$.*
2. *If $z \notin \gamma^*$, then $\operatorname{wn}(\gamma, z) = \operatorname{wn}(\gamma - z, 0)$.*
3. *If $0 \notin \gamma_1^* \cup \gamma_2^*$, then*
$$\operatorname{wn}(\gamma_1\gamma_2, 0) = \operatorname{wn}(\gamma_1, 0) + \operatorname{wn}(\gamma_2, 0) \quad \textit{and} \quad \operatorname{wn}(\gamma_1/\gamma_2, 0) = \operatorname{wn}(\gamma_1, 0) - \operatorname{wn}(\gamma_2, 0).$$
4. *If $\gamma^* \subseteq B_r(z_0)$, then $\operatorname{wn}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus B_r(z_0)$.*
5. *If $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all $t \in [a,b]$, then $0 \notin \gamma_1^* \cup \gamma_2^*$ and $\operatorname{wn}(\gamma_1, 0) = \operatorname{wn}(\gamma_2, 0)$*

Proof.
Proof of Part (1). Suppose $z \notin \gamma^*$. Fix $w \in \mathbb{C}$. Let $\alpha : [a,b] \to \mathbb{R}$ be a continuous argument of $\gamma - z$. Since $\gamma + w : [a,b] \to \mathbb{C}$ is a closed curve with $z + w \notin (\gamma + w)^*$, and since $(\gamma + w) - (z + w) = \gamma - z$, it follows that α is also a continuous argument of $(\gamma + w) - (z + w)$. Therefore

$$\operatorname{wn}(\gamma + w, z + w) = \frac{\alpha(b) - \alpha(a)}{2\pi} = \operatorname{wn}(\gamma, z)$$

by Definition 6.19.

Proof of Part (2). This follows from the preceding part simply by choosing $w = -z$.

Proof of Part (3). Suppose $0 \notin \gamma_1^* \cup \gamma_2^*$, so $\gamma_1(t) \neq 0$ and $\gamma_2(t) \neq 0$ for all $t \in [a,b]$, which shows that $(\gamma_1\gamma_2)(t) \neq 0$ for $t \in [a,b]$, and hence $\gamma_1\gamma_2$ is a closed curve with $0 \neq (\gamma_1\gamma_2)^*$. Let $\alpha_1, \alpha_2 : [a,b] \to \mathbb{R}$ be continuous arguments for γ_1, γ_2, so

$$\gamma_1(t) = |\gamma_1(t)|e^{i\alpha_1(t)} \quad \text{and} \quad \gamma_2(t) = |\gamma_2(t)|e^{i\alpha_2(t)}$$

for $t \in [a,b]$. Then

$$(\gamma_1\gamma_2)(t) = |\gamma_1(t)||\gamma_2(t)|e^{i\alpha_1(t)}e^{i\alpha_2(t)} = |(\gamma_1\gamma_2)(t)|e^{i(\alpha_1+\alpha_2)(t)},$$

and so $\alpha_1 + \alpha_2$ is a continuous argument of $\gamma_1\gamma_2$ on $[a,b]$. By Definition 6.19,

$$\begin{aligned} \operatorname{wn}(\gamma_1\gamma_2, 0) &= \frac{(\alpha_1+\alpha_2)(b) - (\alpha_1+\alpha_2)(a)}{2\pi} = \frac{\alpha_1(b) - \alpha_1(a)}{2\pi} + \frac{\alpha_2(b) - \alpha_2(a)}{2\pi} \\ &= \operatorname{wn}(\gamma_1, 0) + \operatorname{wn}(\gamma_2, 0). \end{aligned}$$

Also

$$(\gamma_1/\gamma_2)(t) = \frac{|\gamma_1(t)|e^{i\alpha_1(t)}}{|\gamma_2(t)|e^{i\alpha_2(t)}} = |(\gamma_1/\gamma_2)(t)|e^{i(\alpha_1-\alpha_2)(t)}$$

shows $\alpha_1 - \alpha_2$ to be a continuous argument of γ_1/γ_2 on $[a,b]$, and so

$$\begin{aligned} \operatorname{wn}(\gamma_1/\gamma_2, 0) &= \frac{(\alpha_1-\alpha_2)(b) - (\alpha_1-\alpha_2)(a)}{2\pi} = \frac{\alpha_1(b) - \alpha_1(a)}{2\pi} - \frac{\alpha_2(b) - \alpha_2(a)}{2\pi} \\ &= \operatorname{wn}(\gamma_1, 0) - \operatorname{wn}(\gamma_2, 0). \end{aligned}$$

Proof of Part (4). Suppose $\gamma^* \subseteq B := B_r(z_0)$, and fix $z \in \mathbb{C} \setminus B$. Define $f : B \to \mathbb{C}$ by $f(w) = w - z$ for all $w \in B$. Clearly f is analytic and nonvanishing on B. Since B is a starlike region, by Theorem 3.40 any analytic function $g : B \to \mathbb{C}$ has a primitive on B, and hence $\oint_\Gamma g = 0$ for every closed path Γ in B by Theorem 3.24. Thus f has an analytic logarithm λ on B by Corollary 6.12, and by Proposition 6.8(1) we have $\theta = \operatorname{Im}\lambda$ as a continuous argument of f on B; that is, $f(w) = |f(w)|e^{i\theta(w)}$ for all $w \in B$, and therefore

$$f(\gamma(t)) = |f(\gamma(t))|e^{i\theta(\gamma(t))}$$

for all $t \in [a,b]$. Setting $\alpha = \theta \circ \gamma$, we have

$$(\gamma - z)(t) = \gamma(t) - z = f(\gamma(t)) = |f(\gamma(t))|e^{i\theta(\gamma(t))} = |(\gamma - z)(t)|e^{i\alpha(t)},$$

and so α is a continuous argument of $\gamma - z$. Since $z \notin \gamma^*$ and $\gamma(a) = \gamma(b)$, we have

$$\operatorname{wn}(\gamma, z) = \frac{\alpha(b) - \alpha(a)}{2\pi} = \frac{\theta(\gamma(b)) - \theta(\gamma(a))}{2\pi} = 0$$

as was to be shown.

Proof of Part (5). Suppose $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all $t \in [a,b]$. If $0 \in \gamma_1^*$, so that $\gamma_1(\tau) = 0$ for some $\tau \in [a,b]$, then we obtain the contradiction $|\gamma_2(\tau)| < 0$. If $0 \in \gamma_2^*$, so that $\gamma_2(\tau) = 0$ for some $\tau \in [a,b]$, then we obtain the contradiction $|\gamma_1(\tau)| < |\gamma_1(\tau)|$. Hence $0 \notin \gamma_1^* \cup \gamma_2^*$ must be the case. Now,

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad \Rightarrow \quad |(\gamma_2/\gamma_1)(t) - 1| < 1$$

for all $t \in [a,b]$, and so $(\gamma_2/\gamma_1)^* \subseteq B_1(1)$. Since $0 \in \mathbb{C} \setminus B_1(1)$, by part (4) we obtain $\text{wn}(\gamma_2/\gamma_1, 0) = 0$. But $\text{wn}(\gamma_2/\gamma_1, 0) = \text{wn}(\gamma_2, 0) - \text{wn}(\gamma_1, 0)$ by part (3), and therefore $\text{wn}(\gamma_1, 0) = \text{wn}(\gamma_2, 0)$. ■

Proposition 6.21. *Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve, and let $\mathfrak{n} : \mathbb{C} \setminus \gamma^* \to \mathbb{Z}$ be given by $\mathfrak{n}(z) = \text{wn}(\gamma, z)$.*

1. *$\mathfrak{n}$ is constant on each component of $\mathbb{C} \setminus \gamma^*$.*
2. *$\mathbb{C} \setminus \gamma^*$ has precisely one unbounded component U, with $\mathfrak{n}|_U \equiv 0$.*

Proof.
Proof of Part (1). Since γ is continuous and $[a,b]$ is compact, by Theorems 2.41 and 2.40 the trace γ^* is closed and bounded, and so $\mathbb{C} \setminus \gamma^*$ is open. Let Ω be a component of $\mathbb{C} \setminus \gamma^*$, and for some $z_0 \in \Omega$ define

$$A = \{w \in \Omega : \mathfrak{n}(w) = \mathfrak{n}(z_0)\}.$$

Fix $w_0 \in A$. By Proposition 2.35 the set Ω is open in $\mathbb{C}$, so choose some $r > 0$ such that $B_r(w_0) \subseteq \Omega$, and suppose $w \in B_r(w_0)$. Since $w \notin \gamma^*$, Proposition 6.20 implies that

$$\begin{aligned} \text{wn}(\gamma, w) - \text{wn}(\gamma, w_0) &= \text{wn}(\gamma - w, 0) - \text{wn}(\gamma - w_0, 0) \\ &= \text{wn}\left(\frac{\gamma - w}{\gamma - w_0}, 0\right) = \text{wn}(1 + \Gamma, 0), \end{aligned} \tag{6.2}$$

where $\Gamma : [a,b] \to \mathbb{C}$ is the curve given by

$$\Gamma(t) = \frac{w_0 - w}{\gamma(t) - w_0}.$$

Now, for any $t \in [a,b]$ we have $\gamma(t) \in \mathbb{C} \setminus B_r(w_0)$, so $|\gamma(t) - w_0| \geq r$ and hence

$$|\Gamma(t)| < \frac{r}{|\gamma(t) - w_0|} \leq 1,$$

which shows that $\Gamma^* \subseteq B_1(0)$, and finally $(\Gamma + 1)^* \subseteq B_1(1)$. Noting that $0 \notin B_1(1)$, Proposition 6.20(4) yields $\text{wn}(\Gamma + 1, 0) = 0$, and thus $\text{wn}(\gamma, w) = \text{wn}(\gamma, w_0)$ by (6.2). That is, $\mathfrak{n}(w) = \mathfrak{n}(w_0) = \mathfrak{n}(z_0)$ for all $w \in B_r(w_0)$, so $B_r(w_0) \subseteq A$ and we see that A is an open set.

Next suppose that $w_1 \in \Omega \setminus A$, and let $r > 0$ be such that $B_r(w_1) \subseteq \Omega$. The same argument as before will show that $\mathfrak{n}(w) = \mathfrak{n}(w_1)$ for all $w \in B_r(w_1)$, and since $\mathfrak{n}(w_1) \neq \mathfrak{n}(z_0)$, it follows that $B_r(w_1) \subseteq \Omega \setminus A$ and hence $\Omega \setminus A$ is an open set. Since Ω is connected and $A \neq \varnothing$, we conclude that $\Omega \setminus A = \varnothing$. That is, $A = \Omega$, or equivalently $\mathfrak{n} \equiv \mathfrak{n}(z_0)$ on Ω.

Proof of Part (2). As noted already, γ^* is closed and bounded, so in particular there exists some $r > 0$ such that $\gamma^* \subseteq B_r(0)$, and thus $S := \mathbb{C} \setminus B_r(0) \subseteq \mathbb{C} \setminus \gamma^* := X$. Since S is connected, there must exist some component U of X such that $U \supseteq S$, and it is clear that this component is unbounded. This proves the existence of an unbounded component U.

Now suppose U_1 and U_2 are unbounded components of X. Then $U_1 \cap S \neq \varnothing$ and $U_2 \cap S \neq \varnothing$, and since S is connected, by Theorem 2.34 it follows that $S \subseteq U_1$ and $S \subseteq U_2$. This implies that

$U_1 \cap U_2 \neq \varnothing$, so by two more applications of Theorem 2.34 we obtain $U_1 \subseteq U_2$ and $U_2 \subseteq U_1$, and therefore $U_1 = U_2$. This proves the uniqueness of the unbounded component U that we found.

Finally, let $z \in U$ be such that $|z| > r$. Then $\operatorname{wn}(\gamma, z) = 0$ by Proposition 6.21(4), and since $\mathfrak{n}$ is constant on U by part (1), we conclude that $\mathfrak{n}|_U \equiv 0$. ∎

Theorem 6.22. *Let γ be a closed path with $z_0 \notin \gamma^*$. Then*

$$\operatorname{wn}(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0} dz.$$

Corollary 6.23. *Let γ be a closed path, and let f be analytic on an open set Ω containing γ^*. If $z_0 \notin (f \circ \gamma)^*$, then*

$$\operatorname{wn}(f \circ \gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - z_0} dz.$$

The following exercise could be completed making use of Theorem 6.22, but Cauchy's Integral Formula for a Circle from §4.2 will be used instead.

Exercise 6.24 (AN3.2.2). For any $a, b \in \mathbb{C}$ with $a \neq b$, let $f(z) = (z - a)(z - b)$ for all $z \in \Omega = \mathbb{C} \setminus [a, b]$. Show that f has an analytic square root on Ω, but not an analytic logarithm.

Solution. Let φ be the angle between $[a, b]$ and the positive real axis, as in Figure 14. Fix $z \in \Omega$, so that $z - a \notin [0, b - a]$ and $z - b \notin [a - b, 0] = [0, a - b]$. For definiteness we may suppose $[0, b - a] \subseteq R_\varphi$, so that $[0, a - b] \subseteq R_{\varphi+\pi}$. Then $z - a = r_1 e^{i\theta_1}$ for some $\theta_1 \in [\varphi, \varphi + 2\pi)$, where either $\theta_1 \neq \varphi$, or $\theta_1 = \varphi$ and $r_1 > |b - a|$; and $z - b = r_2 e^{i\theta_2}$ for some $\theta_2 \in [\varphi, \varphi + 2\pi)$, where either $\theta_2 \neq \varphi + \pi$, or $\theta_2 = \varphi + \pi$ and $r_2 > |a - b|$. Now, define

$$h(z) = \sqrt{|f(z)|} \exp\left(\frac{i}{2}\big(\arg_\varphi(z - a) + \arg_\varphi(z - b)\big)\right),$$

which with an application of Proposition 6.3 becomes

$$h(z) = \sqrt{|f(z)|} \exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) = \sqrt{r_1 r_2} e^{i\theta_1/2} e^{i\theta_2/2},$$

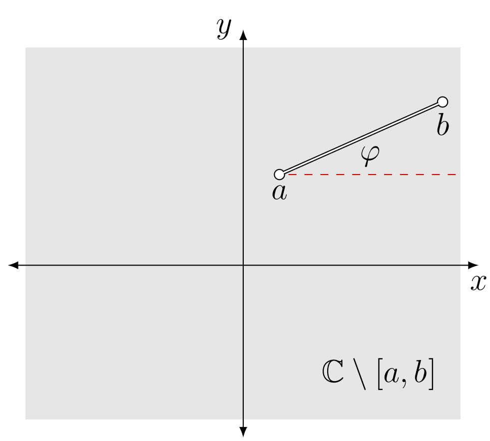

FIGURE 14.

and thus

$$h^2(z) = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = (z-a)(z-b) = f(z).$$

This shows that $h : \Omega \to \mathbb{C}$ is an analytic square root for f on Ω.

Next, for any $z \in \Omega$,

$$(f'/f)(z) = \frac{2z-(a+b)}{(z-a)(z-b)} = \frac{1}{z-a} + \frac{1}{z-b},$$

and so for any $r > 0$ such that $[a, b] \subseteq B_r(0)$,

$$\oint_{C_r(0)} f'/f = \oint_{C_r(0)} \left(\frac{1}{z-a} + \frac{1}{z-b} \right) dz = \oint_{C_r(0)} \frac{1}{z-a} dz + \oint_{C_r(0)} \frac{1}{z-b} dz.$$

By Cauchy's Integral Formula for a Circle, since $a, b \in B_r(0)$ and the constant function 1 is analytic on $\mathbb{C}$, we have

$$\frac{1}{2\pi i} \oint_{C_r(0)} \frac{1}{z-a} dz = 1 \quad \text{and} \quad \frac{1}{2\pi i} \oint_{C_r(0)} \frac{1}{z-b} dz = 1,$$

and hence

$$\oint_{C_r(0)} f'/f = 2\pi i + 2\pi i = 4\pi i \neq 0$$

for the closed path $C_r(0)$ in Ω. Therefore f does not have an analytic logarithm on Ω by Theorem 6.11. ■

Exercise 6.25 (AN3.2.3). Let f be a nonvanishing analytic function on Ω. Show that the following are equivalent.

1. f has an analytic logarithm on Ω.
2. f has an analytic nth root on Ω for all $n \in \mathbb{N}$.
3. f has an analytic nth root on Ω for infinitely many $n \in \mathbb{N}$.

Solution.
(1) ⇒ (2): Suppose that f has an analytic logarithm on Ω, so there exists some $\lambda : \Omega \to \mathbb{C}$ such that $f = e^\lambda$. Fix $n \in \mathbb{N}$, and define $h : \Omega \to \mathbb{C}$ by $h(z) = e^{\lambda(z)/n}$. Thus $h = \exp \circ \lambda/n$, where λ/n is analytic on Ω and exp is analytic on the range of λ/n, and so h is analytic on Ω by the Chain Rule. Moreover, for any $z \in \Omega$,

$$h^n(z) = \prod_{k=1}^{n} \exp\left(\frac{\lambda(z)}{n}\right) = \exp\left(\sum_{k=1}^{n} \frac{\lambda(z)}{n}\right) = \exp\big(\lambda(z)\big) = e^{\lambda(z)} = f(z),$$

using the property $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$. Therefore $h^n = f$, and we conclude that h is an analytic nth root of f on Ω.

(2) ⇒ (3): Suppose f has an analytic nth root on Ω for all $n \in \mathbb{N}$. Then it follows trivially that f has an analytic nth root on Ω for infinitely many $n \in \mathbb{N}$.

(3) ⇒ (1): Suppose f has an analytic nth root h_n on Ω for infinitely many $n \in \mathbb{N}$. For all $z \in \Omega$ we have $f(z) = h_n^n(z)$, and so

$$f'(z) = n h_n^{n-1}(z) h_n'(z)$$

by the Chain Rule. Now, for any closed path γ in Ω,

$$\oint_\gamma \frac{f'(z)}{f(z)}dz = \oint_\gamma \frac{nh_n^{n-1}(z)h_n'(z)}{h_n^n(z)}dz = n\oint_\gamma \frac{h_n'(z)}{h_n(z)}dz,$$

and since $0 \notin (h_n \circ \gamma)^*$ on account of h_n being zero-free on Ω, it follows by Corollary 6.23 that

$$\oint_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i n \operatorname{wn}(h_n \circ \gamma, 0),$$

and hence

$$2\pi i \operatorname{wn}(h_n \circ \gamma, 0) = \frac{1}{n}\oint_\gamma \frac{f'(z)}{f(z)}dz.$$

Let $(h_{n_k})_{k=1}^\infty$ be a sequence of functions such that h_{n_k} is an n_kth root for f on Ω. Then

$$|\operatorname{wn}(h_{n_k} \circ \gamma, 0)| = \frac{1}{2\pi n_k}\left|\oint_\gamma \frac{f'(z)}{f(z)}dz\right|$$

for all $k \geq 1$. For sufficiently large k we obtain $0 \leq |\operatorname{wn}(h_{n_k}\circ\gamma, 0)| < 1$, and thus $\operatorname{wn}(h_{n_k}\circ\gamma, 0) = 0$ since a winding number must be an integer. This immediately implies that

$$\oint_\gamma \frac{f'(z)}{f(z)}dz = 0,$$

and since γ is an arbitrary closed path in Ω, we conclude by Theorem 6.11 that f has an analytic logarithm on Ω. ∎

6.3 – Cauchy's Theorem

Lemma 6.26. *Let $f : \Omega \to \mathbb{C}$ be analytic, and define $g : \Omega \times \Omega \to \mathbb{C}$ by*

$$g(w,z) = \begin{cases} \dfrac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(z), & w = z \end{cases} \tag{6.3}$$

The function g is continuous, and for each $w \in \Omega$ the function $\Omega \to \mathbb{C}$ given by $z \mapsto g(w,z)$ is analytic on Ω.

Proof. The proof in [AN] is for the most part clear except for the argument that g is continuous at (w,z) in the case when $w = z$. Let $(z,z) \in \Omega \times \Omega$, and let $(w_n, z_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega \times \Omega$ that converges to (z,z). Fix $\epsilon > 0$. Since f' is continuous at z, there exists some $\delta > 0$ such that, for all $\zeta \in \Omega$,

$$|\zeta - z| < \delta \quad \Rightarrow \quad |f'(\zeta) - f'(z)| < \epsilon. \tag{6.4}$$

Since $w_n, z_n \to z$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that $w_n, z_n \in B_\delta(z)$ for all $n \geq N$. Fix $n \geq N$. If $w_n = z_n$, then $g(w_n, z_n) = f'(z_n)$, and since $|z_n - z| < \delta$, by (6.4) we find that

$$|g(w_n, z_n) - f'(z)| = |f'(z_n) - f'(z)| < \epsilon.$$

Suppose $w_n \neq z_n$. Then

$$\begin{aligned} g(w_n, z_n) = \frac{f(w_n) - f(z_n)}{w_n - z_n} &= \frac{1}{w_n - z_n} \int_{[z_n, w_n]} f'(\zeta)d\zeta \\ &= \frac{1}{w_n - z_n} \int_0^1 f'((1-t)z_n + tw_n)(w_n - z_n)\,dt \\ &= \int_0^1 f'((1-t)z_n + tw_n)\,dt. \end{aligned}$$

Now,

$$w_n, z_n \in B_\delta(z) \quad \Rightarrow \quad [w_n, z_n] \subseteq B_\delta(z) \quad \Rightarrow \quad \forall t \in [0,1]\big((1-t)z_n + tw_n \in B_\delta(z)\big),$$

and hence

$$\big|f'((1-t)z_n + tw_n) - f'(z)\big| < \epsilon$$

for all $0 \leq t \leq 1$ by (6.4). Therefore,

$$\begin{aligned} |g(w_n, z_n) - f'(z)| &= \left| \int_0^1 f'((1-t)z_n + tw_n)\,dt - f'(z) \right| \\ &= \left| \int_0^1 \big[f'((1-t)z_n + tw_n) - f'(z)\big]\,dt \right| \\ &\leq \int_0^1 \big|f'((1-t)z_n + tw_n) - f'(z)\big|\,dt \\ &\leq \int_0^1 \epsilon\,dt = \epsilon. \end{aligned}$$

We see that for every sequence $(w_n, z_n)_{n\in\mathbb{N}}$ in $\Omega\times\Omega$ that converges to (z,z), the sequence $(g(w_n, z_n))_{n\in\mathbb{N}}$ converges to $g(z,z)$. Therefore g is continuous at (z,z). ■

Remark. Notice that $g(w,z) = g(z,w)$ for all $(w,z)\in\Omega\times\Omega$, and so Lemma 6.26 implies an additional, symmetric conclusion: for each $z\in\Omega$ the function $\Omega\to\mathbb{C}$ given by $w\mapsto g(w,z)$ is analytic on Ω.

Lemma 6.27. *Let $[a,b]\subseteq\mathbb{R}$, and let $\varphi : \Omega\times[a,b]\to\mathbb{C}$ be continuous. Suppose $\varphi(\cdot,t)$ is analytic on Ω for each $t\in[a,b]$. If $F:\Omega\to\mathbb{C}$ is given by*

$$F(z) = \int_a^b \varphi(z,t)\,dt,$$

then F is analytic on Ω, and

$$F'(z) = \int_a^b \frac{\partial\varphi}{\partial z}(z,t)\,dt$$

for all $z\in\Omega$

Proof. Fix $z_0\in\Omega$, let $r>0$ be such that $\overline{B}_r(z_0)\subseteq\Omega$, and let $\gamma(s) = z_0 + re^{is}$ for $s\in[0,2\pi]$. For each $t\in[a,b]$ Cauchy's Integral Formula for a Circle gives

$$\varphi(z,t) = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{\varphi(w,t)}{w-z}\,dw$$

for all $z\in B_r(z_0)$, and so by Fubini's Theorem in [MT][3] we have

$$\begin{aligned}
F(z) &= \int_a^b \left(\frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{\varphi(w,t)}{w-z}\,dw\right)dt = \frac{1}{2\pi i}\int_a^b\int_0^{2\pi} \frac{\varphi(\gamma(s),t)\gamma'(s)}{\gamma(s)-z}\,ds\,dt\\
&= \frac{1}{2\pi i}\int_0^{2\pi}\int_a^b \frac{\varphi(\gamma(s),t)\gamma'(s)}{\gamma(s)-z}\,dt\,ds = \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{\gamma'(s)}{\gamma(s)-z}\int_a^b \varphi(\gamma(s),t)\,dt\right)ds\\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{F(\gamma(s))\gamma'(s)}{\gamma(s)-z}\,ds = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{F(w)}{w-z}\,dw := G(z) \qquad (6.5)
\end{aligned}$$

for all $z\in B_r(z_0)$. Since F is continuous on $C_r(z_0)$, the function G defined in (6.5) is analytic on $\mathbb{C}\setminus C_r(z_0)$ by Theorem 4.19, and hence F is analytic on $B_r(z_0)$ since $F=G$ there. Now, by Corollary 4.20,

$$\frac{\partial\varphi}{\partial z}(z,t) = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{\varphi(w,t)}{(w-z)^2}\,dw$$

for all $t\in[a,b]$ and $z\in B_r(z_0)$, and so by another application of Corollary 4.20 and Fubini's Theorem we obtain

$$\begin{aligned}
F'(z) &= \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{F(w)}{(w-z)^2}\,dw = \frac{1}{2\pi i}\oint_{C_r(z_0)}\int_a^b \frac{\varphi(w,t)}{(w-z)^2}\,dt\,dw\\
&= \frac{1}{2\pi i}\int_a^b\oint_{C_r(z_0)} \frac{\varphi(w,t)}{(w-z)^2}\,dw\,dt = \int_a^b \frac{\partial\varphi}{\partial z}(z,t)\,dt
\end{aligned}$$

for all $z\in B_r(z_0)$. ■

[3] In particular refer to Example 3.18.

Theorem 6.28 (**Cauchy's Integral Formula**). *Let γ be a closed path in Ω such that* $\text{wn}(\gamma, z) = 0$ *for all* $z \notin \Omega$. *If f is analytic on Ω and $z \in \Omega \setminus \gamma^*$, then*

$$\text{wn}(\gamma, z) f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} \, dw.$$

Proof. Suppose that $f : \Omega \to \mathbb{C}$ is analytic. Let $g : \Omega \times \Omega \to \mathbb{C}$ be given by (6.3), and define $F : \Omega \to \mathbb{C}$ by

$$F(z) = \oint_\gamma g(w, z) \, dw = \int_a^b g(\gamma(t), z)\gamma'(t) \, dt.$$

Define $\varphi : \Omega \times [a, b] \to \mathbb{C}$ by $\varphi(z, t) = g(\gamma(t), z)\gamma'(t)$, and for each $t \in [a, b]$ define $\psi_t : \Omega \to \mathbb{C}$ by $\psi_t(z) = \varphi(z, t)$.

By Lemma 6.26 the function g is continuous on $\Omega \times \Omega$, and since γ and γ' are continuous on $[a, b]$, it follows that φ is continuous on $\Omega \times [a, b]$. Also by Lemma 6.26, $\xi_w(z) = g(w, z)$ is analytic on Ω for all $w \in \Omega$, and thus $\xi_{\gamma(t)}(z) = g(\gamma(t), z)$ is analytic on Ω for all $t \in [a, b]$. From this fact it easily follows that ψ_t is analytic on Ω for each fixed $t \in [a, b]$, since

$$\psi_t(z) = \varphi(z, t) = g(\gamma(t), z)\gamma'(t) = \xi_{\gamma(t)}(z)\gamma'(t).$$

Therefore F is analytic on Ω by Lemma 6.27.

Define $\Omega' = \{z \in \mathbb{C} \setminus \gamma^* : \text{wn}(\gamma, z) = 0\}$, which is an open set by Proposition 6.21. Clearly $\mathbb{C} \setminus \Omega \subseteq \Omega'$ and $\Omega \cup \Omega' = \mathbb{C}$. Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = \begin{cases} \oint_\gamma g(w, z) \, dw, & z \in \Omega \\ \oint_\gamma \frac{f(w)}{w - z} \, dw, & z \in \Omega' \end{cases}$$

If $z \in \Omega \cap \Omega'$, then

$$g(w, z) = \frac{f(w) - f(z)}{w - z}$$

for all $w \in \gamma^*$ since $z \notin \gamma^*$, and so

$$\oint_\gamma g(w, z) \, dw = \oint_\gamma \frac{f(w)}{w - z} \, dw - \oint_\gamma \frac{f(z)}{w - z} \, dw = \oint_\gamma \frac{f(w)}{w - z} \, dw - 2\pi i \, \text{wn}(\gamma, z) f(z) = \oint_\gamma \frac{f(w)}{w - z} \, dw$$

by Theorem 6.22 and the fact that $\text{wn}(\gamma, z) = 0$. Hence h is a well-defined function. In fact h is an entire function, since it is analytic on Ω' by Theorem 4.19, and it is analytic on Ω since $h|_\Omega = F$.

Let $M = \max\{|f(w)| : w \in \gamma^*\}$, and let $R > 0$ be such that $\gamma^* \subseteq B_R(0)$. Then $\text{wn}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus B_R(0)$ by Proposition 6.20(4), and so $\mathbb{C} \setminus B_R(0) \subseteq \Omega'$. Thus, for any $z \in \mathbb{C} \setminus B_R(0)$,

$$|h(z)| = \left| \oint_\gamma \frac{f(w)}{w - z} \, dw \right| \leq \oint_\gamma \left| \frac{f(w)}{w - z} \right| dw \leq \oint_\gamma \frac{M}{|w - z|} \, dw \leq \oint_\gamma \frac{M}{|z| - |w|} \, dw,$$

where $|z| \geq R > |w|$ for all $w \in \gamma^*$. This shows that $|h(z)| \to 0$ as $|z| \to \infty$, and since h is bounded on any closed disc $\overline{B}_r(0)$, we conclude that h is a bounded entire function. By Liouville's Theorem h must be constant on $\mathbb{C}$, and in fact $h \equiv 0$ since, again, $|h(z)| \to 0$ as $|z| \to \infty$.

Finally, fix $z \in \Omega \setminus \gamma^*$. Then

$$0 = h(z) = \oint_\gamma g(w,z)\,dw = \oint_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i\,\mathrm{wn}(\gamma,z)f(z),$$

which yields

$$\mathrm{wn}(\gamma,z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw$$

as desired. ∎

Theorem 6.29 (**Cauchy's Theorem**). *Let γ be a closed path in Ω. Then $\mathrm{wn}(\gamma,z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$ if and only if $\oint_\gamma f = 0$ for every function f that is analytic on Ω.*

Proof. Suppose that $\mathrm{wn}(\gamma,z) = 0$ for all $z \notin \Omega$. Let $f : \Omega \to \mathbb{C}$ be analytic. Fix $z \in \Omega \setminus \gamma^*$, and define the analytic function $\varphi : \Omega \to \mathbb{C}$ by $\varphi(w) = (w-z)f(w)$. By Cauchy's Integral Formula,

$$\mathrm{wn}(\gamma,z)\varphi(z) = \frac{1}{2\pi i}\oint_\gamma \frac{\varphi(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_\gamma f(w)\,dw,$$

and since $\varphi(z) = 0$ it follows that $\oint_\gamma f = 0$.

Next, suppose that $\oint_\gamma f = 0$ for every function f that is analytic on Ω. Fix $z \notin \Omega$, and define the analytic function $f : \Omega \to \mathbb{C}$ by

$$f(w) = \frac{1}{w-z}.$$

Applying Theorem 6.22, we obtain

$$\mathrm{wn}(\gamma,z) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{w-z}\,dw = \frac{1}{2\pi i}\oint_\gamma f = 0$$

as desired. ∎

Proposition 6.30 (**Cauchy's Integral Formula for Cycles**). *Let γ be a cycle in Ω such that $\mathrm{wn}(\gamma,z) = 0$ for all $z \notin \Omega$. If f is analytic on Ω and $z \in \Omega \setminus \gamma^*$, then*

$$\mathrm{wn}(\gamma,z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw.$$

Proof. We have $\gamma = k_1\gamma_1 + \cdots + k_m\gamma_m$, where $k_1,\ldots,k_m \in \mathbb{Z}$ and each $\gamma_j : [a,b] \to \Omega$ is a closed path. Suppose that $f : \Omega \to \mathbb{C}$ is analytic. Let $g : \Omega \times \Omega \to \mathbb{C}$ be given by (6.3), and for each $1 \le j \le m$ define $F_j : \Omega \to \mathbb{C}$ by

$$F_j(z) = \oint_{\gamma_j} g(w,z)\,dw = \int_a^b g(\gamma_j(t),z)\gamma_j'(t)dt.$$

Define $\varphi_j : \Omega \times [a,b] \to \mathbb{C}$ by $\varphi_j(z,t) = g(\gamma_j(t),z)\gamma_j'(t)$, and for each $t \in [a,b]$ define $\psi_{j,t} : \Omega \to \mathbb{C}$ by $\psi_{j,t}(z) = \varphi_j(z,t)$.

By Lemma 6.26 the function g is continuous on $\Omega \times \Omega$, and since γ_j and γ_j' are continuous on $[a,b]$, it follows that φ_j is continuous on $\Omega \times [a,b]$. Also by Lemma 6.26, $\xi_w(z) = g(w,z)$ is analytic on Ω for all $w \in \Omega$, and thus $\xi_{\gamma_j(t)}(z) = g(\gamma_j(t), z)$ is analytic on Ω for all $t \in [a, b]$. From this fact it easily follows that $\psi_{j,t}$ is analytic on Ω for each fixed $t \in [a, b]$, since

$$\psi_{j,t}(z) = \varphi_j(z, t) = g(\gamma_j(t), z)\gamma_j'(t) = \xi_{\gamma_j(t)}(z)\gamma_j'(t).$$

Therefore F_j is analytic on Ω for each $1 \leq j \leq m$ by Lemma 6.27. Defining $F : \Omega \to \mathbb{C}$ by

$$F(z) = \oint_\gamma g(w, z)\,dw = \sum_{j=1}^m k_j \oint_{\gamma_j} g(w, z)\,dw = \sum_{j=1}^m k_j F_j(z),$$

it is clear that F is also analytic on Ω.

Define $\Omega' = \{z \in \mathbb{C} \setminus \gamma^* : \mathrm{wn}(\gamma, z) = 0\}$, which is an open set by Proposition 6.21. Clearly $\mathbb{C} \setminus \Omega \subseteq \Omega'$ and $\Omega \cup \Omega' = \mathbb{C}$. Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = \begin{cases} \oint_\gamma g(w, z)\,dw, & z \in \Omega \\ \oint_\gamma \frac{f(w)}{w-z}\,dw, & z \in \Omega' \end{cases}$$

If $z \in \Omega \cap \Omega'$, then

$$g(w, z) = \frac{f(w) - f(z)}{w - z}$$

for all $w \in \gamma^*$ since $z \notin \gamma^*$, and so

$$\oint_\gamma g(w, z)\,dw = \oint_\gamma \frac{f(w)}{w - z}\,dw - \oint_\gamma \frac{f(z)}{w - z}\,dw = \oint_\gamma \frac{f(w)}{w - z}\,dw - 2\pi i\,\mathrm{wn}(\gamma, z)f(z) = \oint_\gamma \frac{f(w)}{w - z}\,dw$$

by Theorem 6.22 (which easily adapts to cycles) and the fact that $\mathrm{wn}(\gamma, z) = 0$. Hence h is a well-defined function. In fact h is an entire function, since it is analytic on Ω' by Theorem 4.19, and it is analytic on Ω since $h|_\Omega = F$.

Let

$$M = \max\{|k_j f(w)| : w \in \gamma^* \text{ and } 1 \leq j \leq m\},$$

and for each $1 \leq j \leq m$ let $R_j > 0$ be such that $\gamma_j^* \subseteq B_{R_j}(0)$. Then $\mathrm{wn}(\gamma_j, z) = 0$ for all $z \in \mathbb{C} \setminus B_{R_j}(0)$ by Proposition 6.20(4), and if we let $R = \max\{R_j : 1 \leq j \leq m\}$, then $\mathrm{wn}(\gamma, z) = 0$ on $\mathbb{C} \setminus B_R(0)$ and so $\mathbb{C} \setminus B_R(0) \subseteq \Omega'$. Thus, for any $z \in \mathbb{C} \setminus B_R(0)$,

$$\begin{aligned} |h(z)| = \left|\oint_\gamma \frac{f(w)}{w - z}\,dw\right| &= \left|\sum_{j=1}^m k_j \oint_{\gamma_j} \frac{f(w)}{w - z}\,dw\right| \leq \sum_{j=1}^m \left|\oint_{\gamma_j} \frac{k_j f(w)}{w - z}\,dw\right| \\ &\leq \sum_{j=1}^m \oint_{\gamma_j} \left|\frac{k_j f(w)}{w - z}\right| dw \leq \sum_{j=1}^m \oint_{\gamma_j} \frac{M}{|w - z|}\,dw \leq \sum_{j=1}^m \oint_{\gamma_j} \frac{M}{|z| - |w|}\,dw, \end{aligned}$$

where $|z| \geq R > |w|$ for all $w \in \gamma^*$. This shows that $|h(z)| \to 0$ as $|z| \to \infty$, and since h is bounded on any closed disc $\overline{B}_r(0)$, we conclude that h is a bounded entire function. By Liouville's Theorem h must be constant on $\mathbb{C}$, and in fact $h \equiv 0$ since, again, $|h(z)| \to 0$ as $|z| \to \infty$.

Finally, fix $z \in \Omega \setminus \gamma^*$. Then

$$0 = h(z) = \oint_\gamma g(w, z)\,dw = \oint_\gamma \frac{f(w)}{w - z}\,dw - 2\pi i\,\mathrm{wn}(\gamma, z)f(z),$$

which yields

$$\operatorname{wn}(\gamma, z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw$$

as desired. ■

Proposition 6.31 (**Cauchy's Theorem for Cycles**). *Let γ be a cycle in Ω. Then $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$ if and only if $\oint_\gamma f = 0$ for every function f that is analytic on Ω.*

Proof. Let $\gamma = k_1\gamma_1 + \cdots + k_m\gamma_m$ be a cycle in Ω. Suppose that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$, and let f be analytic on Ω. Fix $z \in \mathbb{C} \setminus \gamma^*$, and let $\varphi : \Omega \to \mathbb{C}$ be the analytic function given by $\varphi(w) = (w - z)f(w)$. By Cauchy's Integral Formula for Cycles, since $\varphi(z) = 0$,

$$\oint_\gamma f = \oint_\gamma \frac{(w-z)f(w)}{w-z}\,dw = \oint_\gamma \frac{\varphi(w)}{w-z}\,dw = 2\pi i\operatorname{wn}(\gamma, z)\varphi(z) = 0.$$

For the converse, suppose that $\oint_\gamma f = 0$ for every analytic $f : \Omega \to \mathbb{C}$, and fix $z \notin \Omega$. Define f by

$$f(w) = \frac{1}{w-z},$$

which is analytic on Ω. Now, since each γ_j is a closed path such that $z \notin \gamma_j^*$, by Theorem 6.22 we obtain

$$0 = \oint_\gamma f = \sum_{j=1}^m k_j \oint_{\gamma_j} \frac{1}{w-z}\,dw = 2\pi i \sum_{j=1}^m k_j \operatorname{wn}(\gamma_j, z) = 2\pi i\operatorname{wn}(\gamma, z),$$

and hence $\operatorname{wn}(\gamma, z) = 0$. ■

Corollary 6.32. *Let γ_1 and γ_2 be cycles in Ω. Then $\operatorname{wn}(\gamma_1, z) = \operatorname{wn}(\gamma_2, z)$ for all $z \notin \Omega$ if and only if*

$$\oint_{\gamma_1} f = \oint_{\gamma_2} f$$

for every function f that is analytic on Ω.

Proof. Consider the cycle $\gamma_1 - \gamma_2$: by Cauchy's Theorem for Cycles $\operatorname{wn}(\gamma_1 - \gamma_2, z) = 0$ for all $z \notin \Omega$ if and only if $\oint_{\gamma_1-\gamma_2} f = 0$ for every function f that is analytic on Ω. By definition,

$$\operatorname{wn}(\gamma_1 - \gamma_2, z) = 0 \quad\Leftrightarrow\quad \operatorname{wn}(\gamma_1, z) - \operatorname{wn}(\gamma_2, z) = 0 \quad\Leftrightarrow\quad \operatorname{wn}(\gamma_1, z) = \operatorname{wn}(\gamma_2, z)$$

and

$$\oint_{\gamma_1-\gamma_2} f = 0 \quad\Leftrightarrow\quad \oint_{\gamma_1} f - \oint_{\gamma_2} f = 0 \quad\Leftrightarrow\quad \oint_{\gamma_1} f = \oint_{\gamma_2} f,$$

and the desired conclusion readily follows. ■

Theorem 6.33 (**Generalized Cauchy's Integral Formula**). *Let γ be a closed path in Ω such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$, and suppose $f : \Omega \to \mathbb{C}$ is analytic. Show that*

$$\operatorname{wn}(\gamma, z)f^{(k)}(z) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z)^{k+1}}\,dw$$

for all $z \in \Omega \setminus \gamma^*$ and $k \geq 0$.

Proof. By Cauchy's Integral Formula

$$2\pi i \operatorname{wn}(\gamma, z) f(z) = \oint_\gamma \frac{f(w)}{w - z} \, dw$$

for all $z \in \Omega \setminus \gamma^*$, and since

$$F(z) = \oint_\gamma \frac{f(w)}{w - z} \, dw$$

is analytic on $\mathbb{C} \setminus \gamma^*$ by Theorem 4.19, it follows that $F|_{\Omega \setminus \gamma^*}(z) = 2\pi i \operatorname{wn}(\gamma, z) f(z)$ is analytic on $\Omega \setminus \gamma^*$. Since $\operatorname{wn}(\gamma, z)$ is constant on each component of $\mathbb{C} \setminus \gamma^*$ by Proposition 6.21, we have

$$F|_{\Omega \setminus \gamma^*}^{(k)}(z) = 2\pi i \operatorname{wn}(\gamma, z) f^{(k)}(z)$$

for each $z \in \Omega \setminus \gamma^*$ and $k \geq 0$, whereas Theorem 4.19 gives

$$F^{(k)}(z) = k! \oint_\gamma \frac{f(w)}{(w - z)^{k+1}} \, dw$$

for $z \in \mathbb{C} \setminus \gamma^*$ and $k \geq 0$. Therefore, for $z \in \Omega \setminus \gamma^*$ and $k \geq 0$,

$$2\pi i \operatorname{wn}(\gamma, z) f^{(k)}(z) = k! \oint_\gamma \frac{f(w)}{(w - z)^{k+1}} \, dw,$$

from which comes the desired result. ∎

Exercise 6.34 (AN3.3.1). Let γ be a closed path in Ω. Suppose that $\oint_\gamma f = 0$ for every analytic $f : \Omega \to \mathbb{C}$. Without using Cauchy's Integral Formula or Cauchy's Theorem, show that for all analytic $f : \Omega \to \mathbb{C}$ and $z \in \Omega \setminus \gamma^*$,

$$\operatorname{wn}(\gamma, z) f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} \, dw.$$

Solution. Let $f : \Omega \to \mathbb{C}$ be analytic, and fix $z \in \Omega \setminus \gamma^*$. Define $g : \Omega \times \Omega \to \mathbb{C}$ by (6.3). By the remark following the proof of Lemma 6.26 the function $\varphi : \Omega \to \mathbb{C}$ given by $\varphi(w) = g(w, z)$ is analytic on Ω, and so $\oint_\gamma \varphi = 0$. Since $w \in \gamma^*$ implies $w \neq z$,

$$0 = \oint_\gamma \varphi = \oint_\gamma g(w, z) \, dw = \oint_\gamma \frac{f(w) - f(z)}{w - z} \, dw = \oint_\gamma \frac{f(w)}{w - z} \, dw - \oint_\gamma \frac{f(z)}{w - z} \, dw,$$

whence

$$\oint_\gamma \frac{f(w)}{w - z} \, dw = \oint_\gamma \frac{f(z)}{w - z} \, dw.$$

Now, by Theorem 6.22,

$$\operatorname{wn}(\gamma, z) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{w - z} \, dw,$$

and therefore

$$\operatorname{wn}(\gamma, z) f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{w - z} \, dw = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} \, dw$$

as was to be shown. ∎

Exercise 6.35 (AN3.3.4). Evaluate

$$\oint_{C_2(0)} \frac{1}{z^2-1}dz$$

Solution. By partial fraction decomposition we obtain

$$\frac{1}{z^2-1} = \frac{1/2}{z-1} - \frac{1/2}{z+1},$$

and so

$$\oint_{C_2(0)} \frac{1}{z^2-1}dz = \frac{1}{2}\oint_{C_2(0)} \frac{1}{z-1}dz - \frac{1}{2}\oint_{C_2(0)} \frac{1}{z+1}dz. \tag{6.6}$$

The path $C_2(0)$ may be parameterized by $\gamma(t) = 2e^{it}$, $t \in [0, 2\pi]$. Let $\Gamma = \gamma - 1$, which is to say Γ is a parameterization of the circle $C_2(-1)$. We have

$$\Gamma(t) = |\Gamma(t)|e^{i\arg_0(\Gamma(t))}$$

for all $t \in [0, 2\pi]$, and so $\alpha : [0, 2\pi] \to \mathbb{R}$ given by $\alpha = \arg_0 \circ \Gamma$ is a continuous argument of $\Gamma : [0, 2\pi] \to C_2(-1)$. In particular[4]

$$\alpha(0) = \lim_{t\to 0^+} \alpha(t) = 0 \quad \text{and} \quad \alpha(2\pi) = \lim_{t\to 2\pi^-} \alpha(t) = 2\pi,$$

and hence

$$\operatorname{wn}(\gamma, 1) = \frac{\alpha(2\pi) - \alpha(0)}{2\pi} = 1.$$

Now, by Theorem 6.22,

$$\oint_{C_2(0)} \frac{1}{z-1}dz = 2\pi i \operatorname{wn}(\gamma, 1) = 2\pi i,$$

and by a similar argument

$$\oint_{C_2(0)} \frac{1}{z+1}dz = 2\pi i \operatorname{wn}(\gamma, -1) = 2\pi i.$$

Therefore we obtain

$$\oint_{C_2(0)} \frac{1}{z^2-1}dz = 0$$

by (6.6) above. ■

Exercise 6.36 (AN3.3.5). Evaluate

$$\oint_{\gamma_j} \frac{e^z + \cos z}{z^4}dz,$$

for $j = 1, 2$, where γ_1 and γ_2 are the closed paths shown in Figure 15.

[4]One way to approach the construction of α is to define $\alpha(t) = \arg_0(\Gamma(t))$ for $t \in (0, 2\pi)$, and then observe that we must *define* $\alpha(0) = 0$ and $\alpha(2\pi) = 2\pi$ in order to secure continuity on $[0, 2\pi]$.

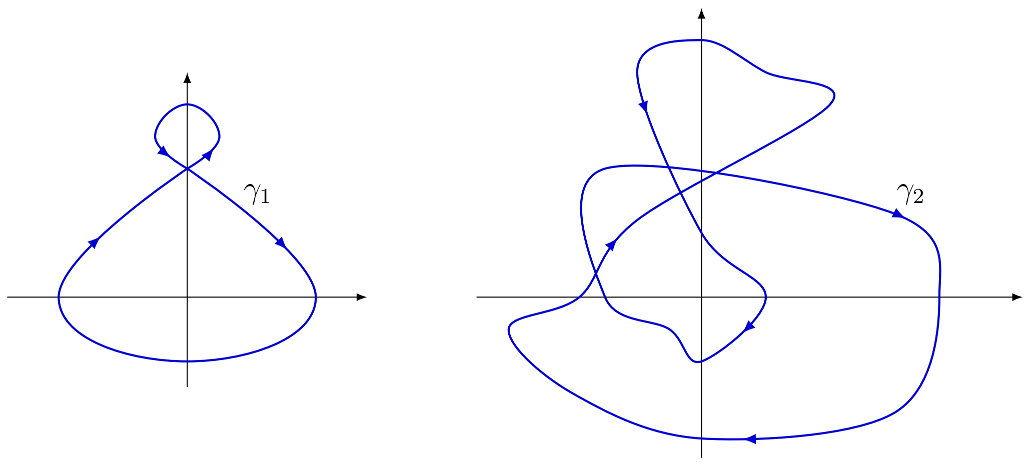

FIGURE 15.

Solution. Let $r > 0$ be sufficiently large so that $\Omega = B_r(0)$ contains each γ_j^*. Clearly $\text{wn}(\gamma_j, z) = 0$ for all $z \notin \Omega$, and clearly $f : \Omega \to \mathbb{C}$ given by $f(z) = e^z + \cos z$ is analytic on Ω. Since $0 \notin \Omega \setminus (\gamma_1^* \cup \gamma_2^*)$, by Theorem 6.33 we have

$$\text{wn}(\gamma_j, 0) f^{(3)}(0) = \frac{3!}{2\pi i} \oint_{\gamma_j} \frac{f(z)}{z^4} dz = \frac{3}{\pi i} \oint_{\gamma_j} \frac{e^z + \cos z}{z^4} dz$$

for each $j = 1, 2$. Now, $f^{(3)}(z) = e^z + \sin z$, so that $f^{(3)}(0) = 1$ and

$$\oint_{\gamma_j} \frac{e^z + \cos z}{z^4} dz = \frac{\pi i}{3} \text{wn}(\gamma_j, 0).$$

By inspection we have $\text{wn}(\gamma_1, 0) = -1$ and $\text{wn}(\gamma_2, 0) = -2$, and therefore

$$\oint_{\gamma_1} \frac{e^z + \cos z}{z^4} dz = -\frac{\pi i}{3} \quad \text{and} \quad \oint_{\gamma_2} \frac{e^z + \cos z}{z^4} dz = -\frac{2\pi i}{3}.$$

■

Exercise 6.37 (AN3.3.6). Consider $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = a \cos t + ib \sin t$ for some $a, b \in \mathbb{R}_*$. Evaluate $\oint_\gamma dz/z$, and use the result to show that

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 t + b^2 \sin^2 t} dt = \frac{2\pi}{ab}.$$

Solution. The path γ is a positively oriented ellipse centered at the origin, and thus $0 \notin \gamma^*$ and $\text{wn}(\gamma, 0) = 1$. By Theorem 6.22

$$\oint_\gamma \frac{1}{z} dz = 2\pi i \, \text{wn}(\gamma, 0) = 2\pi i, \tag{6.7}$$

whereas by definition

$$\oint_\gamma \frac{1}{z} dz = \oint_\gamma \frac{\overline{z}}{|z|^2} dz = \int_0^{2\pi} \frac{\overline{\gamma(t)}}{|\gamma(t)|^2} \gamma'(t) dt = \int_0^{2\pi} \frac{(a \cos t - ib \sin t)(ib \cos t - a \sin t)}{a^2 \cos^2 t + b^2 \sin^2 t} dt$$

$$= \int_0^{2\pi} \frac{(b^2 - a^2)\cos t \sin t}{a^2 \cos^2 t + b^2 \sin^2 t} dt + i \int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt \tag{6.8}$$

Equating the imaginary parts of (6.7) and (6.8) yields

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi,$$

which readily delivers the desired result. ■

6.4 – The Extended Complex Plane

Let $\|\cdot\|$ be the usual euclidean norm in $\mathbb{R}^3$, and let $|\cdot|$ be the usual norm in $\mathbb{C}$ (the modulus). The norms $\|\cdot\|$ and $|\cdot|$ induce **euclidean metrics** ρ and d in the customary way: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $z, w \in \mathbb{C}$ we have

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \quad \text{and} \quad d(z, w) = |z - w|.$$

With these metrics we construct the metric spaces $(\mathbb{R}^3, \rho)$ and $(\mathbb{C}, d)$, and give each the customary topology induced by the metric.

Define the **Riemann sphere** S_R to be the sphere in $\mathbb{R}^3$ with center located at $(0, 0, 1/2)$ and radius $1/2$:

$$S_R = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\}.$$

Let $S'_R = S_R \setminus \{(0, 0, 1)\}$, and define $h : \mathbb{R}^2 \times \{0\} \to S'_R$ by

$$h(x, y, 0) = \left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, \frac{x^2 + y^2}{1 + x^2 + y^2} \right).$$

As shown in [AN], h is a bijection such that

$$h^{-1}(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0 \right),$$

and therefore h is a homeomorphism between the metric spaces $(\mathbb{R}^2 \times \{0\}, \rho)$ and (S'_R, ρ).

Next, define $k : \mathbb{C} \to \mathbb{R}^2 \times \{0\}$ to be the natural identification

$$k(x + iy) = (x, y, 0),$$

which is clearly a bijection. For $x + iy$ and $u + iv$ in $\mathbb{C}$ we have

$$\begin{aligned} d\big((x + iy), (u + iv)\big) &= \big|(x + iy) - (u + iv)\big| = \sqrt{(x - u)^2 + (y - v)^2} \\ &= \|(x, y, 0) - (u, v, 0)\| = \rho\big((x, y, 0), (u, v, 0)\big) \\ &= \rho\big(k(x + iy), k(u + iv)\big), \end{aligned}$$

which shows that k, in addition to being a bijection, is also an isometry (i.e. it is a distance-preserving map) between the metric spaces $(\mathbb{C}, d)$ and $(\mathbb{R}^2 \times \{0\}, \rho)$. If

$$B_{\epsilon,\rho}(x, y, 0) = \big\{(u, v, 0) \in \mathbb{R}^2 \times \{0\} : \rho\big((u, v, 0), (x, y, 0)\big) < \epsilon\big\}$$

and

$$B_{\epsilon,d}(x + iy) = \big\{u + iv \in \mathbb{C} : d\big(u + iv, x + iy\big) < \epsilon\big\},$$

we find that

$$k^{-1}\big(B_{\epsilon,\rho}(x, y, 0)\big) = B_{\epsilon,d}(x + iy) \quad \text{and} \quad k\big(B_{\epsilon,d}(x + iy)\big) = B_{\epsilon,\rho}(x, y, 0),$$

so that k is a continuous open map, and therefore k is a homeomorphism between the metric spaces $(\mathbb{C}, d)$ and $(\mathbb{R}^2 \times \{0\}, \rho)$.

Now, the map $h \circ k : \mathbb{C} \to S_R'$ is a homeomorphism between $(\mathbb{C}, d)$ and (S_R', ρ). Let ∞ denote an object not belonging to $\mathbb{C}$, and define $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $g : \overline{\mathbb{C}} \to S_R$ be given by

$$g(z) = \begin{cases} h(k(z)), & z \in \mathbb{C} \\ (0,0,1), & z = \infty \end{cases} \tag{6.9}$$

Clearly g is a bijection. If we define $\bar{d} : \overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \mathbb{R}$ by

$$\bar{d}(z, w) = \rho\big(g(z), g(w)\big) \tag{6.10}$$

for all $z, w \in \overline{\mathbb{C}}$, then $(\overline{\mathbb{C}}, \bar{d}\,)$ is in fact a metric space called the **extended complex plane**, and the metric $\bar{d}$, called the **chordal metric**, induces a topology on $\overline{\mathbb{C}}$ that we'll call the **chordal topology**. From (6.10) it is immediate that g is an isometry, and therefore a homeomorphism, between $(\overline{\mathbb{C}}, \bar{d}\,)$ and (S_R, ρ). It is a simple matter of algebra to show that

$$\bar{d}(z, w) = \begin{cases} \frac{|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}, & z, w \in \mathbb{C} \\ \frac{1}{\sqrt{1+|z|^2}}, & z \in \mathbb{C}, w = \infty \\ \frac{1}{\sqrt{1+|w|^2}}, & w \in \mathbb{C}, z = \infty \end{cases} \tag{6.11}$$

Theorem 6.38. *The metric space* $(\overline{\mathbb{C}}, \bar{d}\,)$ *is compact, connected, and complete.*

Proof. The function g given by (6.9) is a homeomorphism between the metric spaces $(\overline{\mathbb{C}}, \bar{d}\,)$ and (S_R, ρ), and since (S_R, ρ) is compact, connected, and complete, it follows that $(\overline{\mathbb{C}}, \bar{d}\,)$ also has these properties. ■

Proposition 6.39. *The identity map* $\mathbb{1} : (\mathbb{C}, d) \to (\mathbb{C}, \bar{d}\,)$ *is a homeomorphism. Thus a set* $S \subseteq \mathbb{C}$ *is open, closed, connected, or compact in* $(\mathbb{C}, d)$ *if and only if* S *is open, closed, connected, or compact in* $(\mathbb{C}, \bar{d}\,)$

Proof. Certainly $\mathbb{1}$ is a bijection. That $\mathbb{1}$ and $\mathbb{1}^{-1}$ are each open maps can be shown directly using (6.11). ■

This proposition relieves us of having to make a distinction between a set $S \subseteq \mathbb{C}$ being connected or compact with respect to the metric d versus the metric $\bar{d}$, since in fact $S \subseteq \mathbb{C}$ is connected (resp. compact) in $(\mathbb{C}, d)$ iff it is connected (resp. compact) in $(\mathbb{C}, \bar{d}\,)$ iff it is connected (resp. compact) in $(\overline{\mathbb{C}}, \bar{d}\,)$! The second "iff" obtains from the fact that the topology on $\mathbb{C}$ induced by the metric $\bar{d}$ is the same as the subspace topology $\mathbb{C}$ inherits from $(\overline{\mathbb{C}}, \bar{d}\,)$.

Going forward, for any $r > 0$ and $z \in \mathbb{C}$, we adhere to the convention that $B_r(z)$ represents an open ball with center $z \in \mathbb{C}$ and radius r with respect to the *euclidean metric* $d = |\cdot|$ on $\mathbb{C}$, even if $B_r(z)$ is regarded as being in the metric space $(\overline{\mathbb{C}}, \bar{d}\,)$. The reason is that d is a simpler metric to work with than $\bar{d}$, and anyway Proposition 6.39 shows that "d-balls" in $\mathbb{C}$ are homeomorphic to "$\bar{d}$-balls" in $\mathbb{C}$.

In the statement of the next proposition care should be taken to distinguish $\infty \in \overline{\mathbb{C}}$ from $+\infty \in \overline{\mathbb{R}}$.

Proposition 6.40. *Let f be a function, and let $z_0 \in \mathbb{C}$ be a limit point of* $\text{Dom}(f)$*. Then*

$$\lim_{z \to z_0} f(z) = \infty$$

if and only if

$$\lim_{z \to z_0} |f(z)| = +\infty.$$

Proof. Suppose that $f(z) \to \infty$ as $z \to z_0$. Fix $\alpha > 0$. Choose $\epsilon > 0$ such that

$$\sqrt{1/\epsilon^2 - 1} > \alpha.$$

There is some $\delta > 0$ such that, for all $z \in \text{Dom}(f)$,

$$0 < |z - z_0| < \delta \;\;\Rightarrow\;\; \bar{d}(f(z), \infty) < \epsilon.$$

Now,

$$\bar{d}(f(z), \infty) < \epsilon \;\;\Leftrightarrow\;\; \frac{1}{\sqrt{1 + |f(z)|^2}} < \epsilon \;\;\Leftrightarrow\;\; |f(z)| > \sqrt{1/\epsilon^2 - 1} > \alpha, \tag{6.12}$$

and so we see that for every $\alpha > 0$ there exists some $\delta > 0$ such that $z \in B'_\delta(z_0) \cap \text{Dom}(f)$ implies $|f(z)| > \alpha$, which is to say $|f(z)| \to +\infty$ as $z \to z_0$.

Next, suppose that $|f(z)| \to +\infty$ as $z \to z_0$. Fix $\epsilon > 0$. We can assume ϵ is sufficiently small so that $1/\epsilon^2 > 1$. Let $\alpha = \sqrt{1/\epsilon^2 - 1}$. There is some $\delta > 0$ such that, for all $z \in \text{Dom}(f)$,

$$0 < |z - z_0| < \delta \;\;\Rightarrow\;\; |f(z)| > \alpha,$$

Now, $|f(z)| > \alpha$ and (6.12) imply that $\bar{d}(f(z), \infty) < \epsilon$. Therefore $f(z) \to \infty$ as $z \to z_0$. ∎

Remark. If $(z_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{C}$, then in fact

$$\lim_{n \to \infty} z_n = \infty \in \overline{\mathbb{C}} \;\;\Leftrightarrow\;\; \lim_{n \to \infty} |z_n| = +\infty \in \overline{\mathbb{R}}.$$

The proof may be done by applying an argument similar to that used in the proof of Proposition 6.40.

Proposition 6.41. *If S is closed in $(\overline{\mathbb{C}}, \bar{d})$ and $\infty \notin S$, then S is compact in $(\mathbb{C}, d)$.*

Proof. Suppose S is closed in $(\overline{\mathbb{C}}, \bar{d})$, with $\infty \notin S$. Then S is closed in $(\mathbb{C}, d)$ by Proposition 6.39. Suppose S is unbounded in $(\mathbb{C}, d)$. Then $S \notin B_n(0)$ for all $n \in \mathbb{N}$, and so we can construct a sequence $(z_n)_{n \in \mathbb{N}}$ in S such that $|z_n| > n$ for each n. Thus $|z_n| \to +\infty$ as $n \to \infty$, and by the above remark we have $\lim_{n \to \infty} z_n = \infty \in \overline{\mathbb{C}}$. This shows that ∞ is a limit point of S, and since S is closed in $(\overline{\mathbb{C}}, \bar{d})$, we must conclude that $\infty \in S$—a contradiction. Hence S is bounded as well as closed in $(\mathbb{C}, d)$, and therefore is compact in $(\mathbb{C}, d)$ by Theorem 2.40. ∎

Let $(X, \mathcal{T})$ be a locally compact Hausdorff topological space, and let ∞ be some object not belonging to X. Define $X_\infty = X \cup \{\infty\}$. The **one-point compactification** of $(X, \mathcal{T})$ is the topological space $(X_\infty, \mathcal{T}_\infty)$ with topology

$$\mathcal{T}_\infty = \mathcal{T} \cup \{U \subseteq X_\infty : X_\infty \setminus U \text{ is a compact subset of } X\}.$$

For the next exercise a couple facts about compact sets must be recalled. First, if X is a topological space, $S \subseteq X$ has the subspace topology, and $K \subseteq S$ is compact relative to S, then K is compact relative to X. Second, if X and Y are topological spaces, $f : X \to Y$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Exercise 6.42 (AN3.4.4). Let $\mathcal{T}$ be the topology on $\mathbb{C}$ induced by the euclidean metric d, and let $\overline{\mathcal{T}}$ be the topology on $\overline{\mathbb{C}}$ induced by the chordal metric $\bar{d}$. Show that the topological space $(\overline{\mathbb{C}}, \overline{\mathcal{T}})$ is homeomorphic to $(\mathbb{C}_\infty, \mathcal{T}_\infty)$, which is the one-point compactification of $(\mathbb{C}, \mathcal{T})$.

Solution. It is natural to regard the object ∞ in $\overline{\mathbb{C}}$ as being the same as that in $\mathbb{C}_\infty$, and so $\mathbb{C}_\infty = \overline{\mathbb{C}}$ as sets. Let $\bar{\mathbb{1}} : (\overline{\mathbb{C}}, \overline{\mathcal{T}}) \to (\overline{\mathbb{C}}, \mathcal{T}_\infty)$ be the identity map, so that $\bar{\mathbb{1}}(z) = z$ for all $z \in \mathbb{C}$ and $\bar{\mathbb{1}}(\infty) = \infty$. Clearly $\bar{\mathbb{1}}$ is a bijection.

Letting

$$\mathcal{T}' = \{U \subseteq \overline{\mathbb{C}} : \overline{\mathbb{C}} \setminus U \text{ is a compact subset of } (\mathbb{C}, d)\},$$

then by definition $\mathcal{T}_\infty = \mathcal{T} \cup \mathcal{T}'$. Suppose that $U \in \mathcal{T}_\infty$. If $U \in \mathcal{T}$, then U is open in $(\mathbb{C}, d)$, and since the identity map $\mathbb{1} : (\mathbb{C}, d) \to (\mathbb{C}, \bar{d})$ is a homeomorphism by Proposition 6.39, it follows that $\mathbb{1}(U) = U$ is open in $(\mathbb{C}, \bar{d})$, and hence in $(\overline{\mathbb{C}}, \bar{d})$,[5] and therefore $U \in \overline{\mathcal{T}}$. If instead $U \in \mathcal{T}'$, then $K = \overline{\mathbb{C}} \setminus U$ is a compact subset of $\mathbb{C}$. By Proposition 6.39 it follows that $\mathbb{1}(K) = K$ is compact in $(\mathbb{C}, \bar{d})$, and hence in $(\overline{\mathbb{C}}, \bar{d})$. Since $K = \overline{\mathbb{C}} \setminus U$ and $\overline{\mathbb{C}} \setminus U$ is closed in $(\overline{\mathbb{C}}, \bar{d})$, we conclude that U is open in $(\overline{\mathbb{C}}, \bar{d})$ and therefore $U \in \overline{\mathcal{T}}$. We have now shown that $\mathcal{T}_\infty \subseteq \overline{\mathcal{T}}$.

Next, suppose that $U \in \overline{\mathcal{T}}$. If $\infty \notin U$, then U is open in $(\mathbb{C}, \bar{d})$, and so is open in $(\mathbb{C}, d)$ by Proposition 6.39, and thus $U \in \mathcal{T} \subseteq \mathcal{T}_\infty$. Suppose that $\infty \in U$. Then U is open in $(\overline{\mathbb{C}}, \bar{d})$, so that $K = \overline{\mathbb{C}} \setminus U$ is closed in $(\overline{\mathbb{C}}, \bar{d})$. However, $(\overline{\mathbb{C}}, \bar{d})$ is compact by Theorem 6.38, and since closed subsets of compact sets are compact, we have that K is compact in $(\overline{\mathbb{C}}, \bar{d})$. In fact, since $\infty \notin K$, we have $K \subseteq \mathbb{C} \subseteq \overline{\mathbb{C}}$, and so K is compact in $(\mathbb{C}, \bar{d})$, and then by Proposition 6.39 we find that $K = \mathbb{1}^{-1}(K)$ is compact in $(\mathbb{C}, d)$. That is $K = \overline{\mathbb{C}} \setminus U$ is a compact subset of $(\mathbb{C}, d)$, so that $U \in \mathcal{T}' \subseteq \mathcal{T}_\infty$. We have now shown that $\overline{\mathcal{T}} \subseteq \mathcal{T}_\infty$.

We now have $\overline{\mathcal{T}} = \mathcal{T}_\infty$, which is to say the open subsets of $(\overline{\mathbb{C}}, \overline{\mathcal{T}})$ are precisely the open subsets of $(\overline{\mathbb{C}}, \mathcal{T}_\infty)$. Thus, for any open set U in $(\overline{\mathbb{C}}, \overline{\mathcal{T}})$ we have $\bar{\mathbb{1}}(U) = U$ is open in $(\overline{\mathbb{C}}, \mathcal{T}_\infty)$, and for any open set U in $(\overline{\mathbb{C}}, \mathcal{T}_\infty)$ we have $\bar{\mathbb{1}}^{-1}(U) = U$ is open in $(\overline{\mathbb{C}}, \overline{\mathcal{T}})$. Therefore $\bar{\mathbb{1}}$ and $\bar{\mathbb{1}}^{-1}$ are continuous and we conclude that $\bar{\mathbb{1}}$ is a homeomorphism. ■

Theorem 6.43 (L'Hôpital's Rule). *Suppose f, g are analytic at $z_0 \in \mathbb{C}$, and not identically zero on any neighborhood of z_0. If*

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = 0,$$

then $\lim_{z \to z_0} f(z)/g(z)$ exists in $\overline{\mathbb{C}}$, and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}. \tag{6.13}$$

Proof. Since f and g are analytic at z_0, there exists some $r > 0$ such that both functions are analytic on $B_r(z_0)$. By Theorem 4.29 there exist sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ such that

$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^\infty b_n (z - z_0)^n$$

[5] Observe that $\mathbb{C}$ is itself an open set in $(\overline{\mathbb{C}}, \bar{d})$.

for all $z \in B_r(z_0)$. Since f and g are not identically zero on $B_r(z_0)$,

$$k = \min\{n : a_n \neq 0\} \quad \text{and} \quad m = \min\{n : b_n \neq 0\}$$

both exist in $\mathbb{W}$. Indeed, $f(z), g(z) \to 0$ as $z \to z_0$ implies that $f(z_0) = a_0 = 0$ and $g(z_0) = b_0 = 0$, and so $k, m \geq 1$. By Exercise 5.10 there exist analytic functions $\varphi, \psi : B_r(z_0) \to \mathbb{C}$ such that $\varphi(z_0), \psi(z_0) \neq 0$, and

$$f(z) = (z - z_0)^k \varphi(z) \quad \text{and} \quad g(z) = (z - z_0)^m \psi(z)$$

for all $z \in B_r(z_0)$. We consider three cases: $k = m$, $k > m$, and $k < m$.

Suppose that $k = m$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m \varphi(z)}{(z - z_0)^m \psi(z)} = \lim_{z \to z_0} \frac{\varphi(z)}{\psi(z)} = \frac{\varphi(z_0)}{\psi(z_0)} \in \mathbb{C},$$

while

$$\begin{aligned} \lim_{z \to z_0} \frac{f'(z)}{g'(z)} &= \lim_{z \to z_0} \frac{(z - z_0)^m \varphi'(z) + m(z - z_0)^{m-1} \varphi(z)}{(z - z_0)^m \psi'(z) + m(z - z_0)^{m-1} \psi(z)} = \lim_{z \to z_0} \frac{(z - z_0)\varphi'(z) + m\varphi(z)}{(z - z_0)\psi'(z) + m\psi(z)}, \\ &= \frac{(z_0 - z_0)\varphi'(z_0) + m\varphi(z_0)}{(z_0 - z_0)\psi'(z_0) + m\psi(z_0)} = \frac{\varphi(z_0)}{\psi(z_0)}, \end{aligned}$$

thereby affirming (6.13).

Suppose that $k > m$, so that $k = m + \ell$ for some $\ell > 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(z - z_0)^{m+\ell} \varphi(z)}{(z - z_0)^m \psi(z)} = \lim_{z \to z_0} \frac{(z - z_0)^\ell \varphi(z)}{\psi(z)} = 0 \in \mathbb{C},$$

while

$$\begin{aligned} \lim_{z \to z_0} \frac{f'(z)}{g'(z)} &= \lim_{z \to z_0} \frac{(z - z_0)^{m+\ell} \varphi'(z) + (m + \ell)(z - z_0)^{m+\ell-1} \varphi(z)}{(z - z_0)^m \psi'(z) + m(z - z_0)^{m-1} \psi(z)} \\ &= \lim_{z \to z_0} \frac{(z - z_0)^{\ell+1} \varphi'(z) + (m + \ell)(z - z_0)^\ell \varphi(z)}{(z - z_0)\psi'(z) + m\psi(z)} = \frac{0}{m\psi(z_0)} = 0, \end{aligned}$$

again affirming (6.13).

Suppose that $k < m$, so that $m = k + \ell$ for some $\ell > 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(z - z_0)^k \varphi(z)}{(z - z_0)^{k+\ell} \psi(z)} = \lim_{z \to z_0} \frac{\varphi(z)}{(z - z_0)^\ell \psi(z)} = \infty \in \overline{\mathbb{C}}$$

since

$$\left| \frac{\varphi(z)}{(z - z_0)^\ell \psi(z)} \right| \to +\infty$$

as $z \to z_0$, while

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \lim_{z \to z_0} \frac{(z - z_0)\varphi'(z) + k\varphi(z)}{(z - z_0)^{\ell+1} \psi'(z) + (k + \ell)(z - z_0)^\ell \psi(z)} = \infty,$$

which affirms (6.13) once more.

We now have shown that, in all cases, $\lim_{z \to z_0} f(z)/g(z)$ exists in $\overline{\mathbb{C}}$ and (6.13) holds. ■

Exercise 6.44. Suppose f, g are analytic on a deleted neighborhood of $z_0 \in \mathbb{C}$. Show that if

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = \infty,$$

then $\lim_{z \to z_0} f(z)/g(z)$ exists in $\overline{\mathbb{C}}$.

Solution. By Proposition 6.40 we have $|f(z)|, |g(z)| \to +\infty$ as $z \to z_0$, so there exists some $\epsilon > 0$ such that f, g are analytic and nonvanishing on $B'_\epsilon(z_0)$, and thus $1/f, 1/g$ are analytic on $B'_\epsilon(z_0)$. Since $1/f, 1/g \to 0$ as $z \to z_0$, if we define $(1/f)(z_0) = (1/g)(z_0) = 0$, then by Corollary 4.22 both functions are analytic on $B_\epsilon(z_0)$. Now, because $1/f, 1/g$ are analytic at z_0 and not identically zero on any neighborhood of z_0, by L'Hôpital's Rule

$$\lim_{z \to z_0} \frac{(1/g)(z)}{(1/f)(z)}$$

exists in $\overline{\mathbb{C}}$. But f, g are nonvanishing on $B'_\epsilon(z_0)$, so that

$$\frac{(1/g)(z)}{(1/f)(z)} = \frac{1/g(z)}{1/f(z)} = \frac{f(z)}{g(z)}$$

for all $z \in B'_\epsilon(z_0)$, and hence

$$\lim_{z \to z_0} \frac{(1/g)(z)}{(1/f)(z)} = \lim_{z \to z_0} \frac{f(z)}{g(z)}.$$

Therefore $\lim_{z \to z_0} f(z)/g(z)$ exists in $\overline{\mathbb{C}}$. ■

6.5 – The Hexagon Lemma

Lemma 6.45 (**Hexagon Lemma**). *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $K \subseteq \Omega$ be a nonempty compact set. Then there exist closed polygonal paths $\gamma_1, \ldots, \gamma_m$ in $\Omega \setminus K$ such that*

$$\sum_{j=1}^{m} \mathrm{wn}(\gamma_j, z) = \begin{cases} 1, & z \in K \\ 0, & z \notin \Omega. \end{cases}$$

Proof. For arbitrary $n \in \mathbb{N}$ place a regular hexagon H with sides of length $1/n$ in the plane $\mathbb{C}$ so that the interval $[0, 1/n]$ forms the bottom side of H. The placement of H uniquely determines a partition $\mathcal{P}_n$ of $\mathbb{C}$ into hexagonal tiles, each tile congruent to H, as in Figure 16. Since K is compact and $\mathbb{C} \setminus \Omega$ is closed, we have

$$\mathrm{dist}(K, \mathbb{C} \setminus \Omega) = \inf\{|z - w| : z \in K,\ w \in \mathbb{C} \setminus \Omega\} = \epsilon > 0$$

by Theorem 2.45. Let $n \in \mathbb{N}$ be such that $1/n < \epsilon/4$, and let

$$\mathcal{Q} = \{H \in \mathcal{P}_n : H \cap K \neq \varnothing\}.$$

Then $K \subseteq \bigcup \mathcal{Q} \subseteq \Omega$... ■

Definition 6.46. *An open set $\Omega \subseteq \mathbb{C}$ is **homologically simply connected** if $\mathrm{wn}(\gamma, z) = 0$ for every closed path γ in Ω and every $z \in \mathbb{C} \setminus \Omega$.*

That the complex plane $\mathbb{C}$ is homologically simply connected obtains from Definition 6.46 as a vacuous truth.

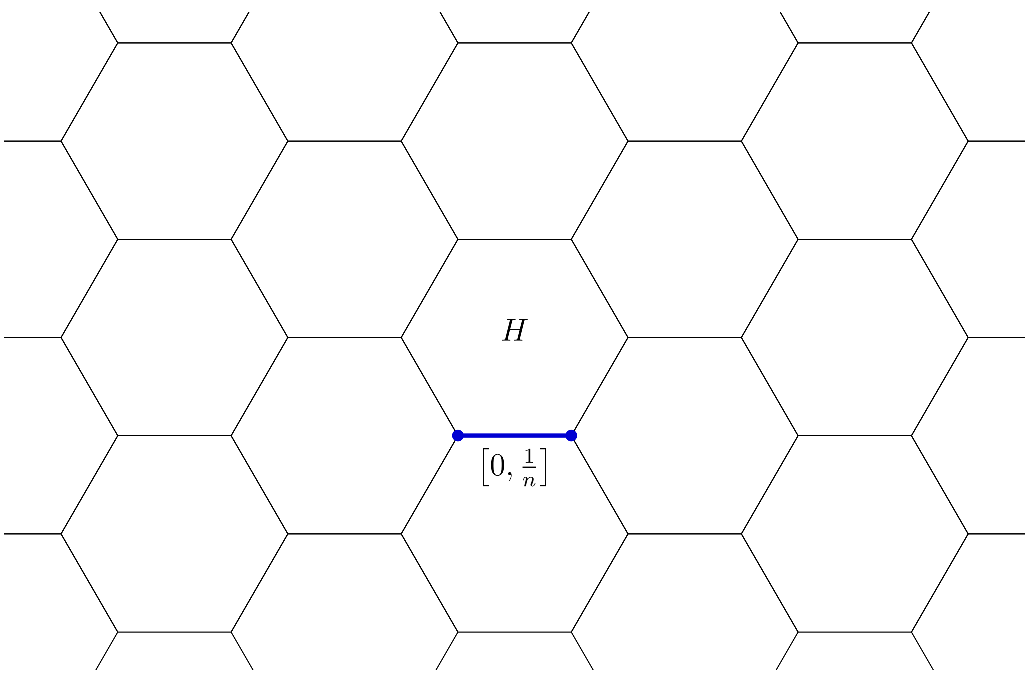

Figure 16.

Equivalent to the statement of Definition 6.46 is the statement in which "closed path" is replaced with "cycle," as is easily verified. Two closed curves γ_0 and γ_1 in an open set Ω are **Ω-homologous** if $\mathrm{wn}(\gamma_0, z) = \mathrm{wn}(\gamma_1, z)$ for all $z \in \mathbb{C} \setminus \Omega$. If γ is a closed curve in Ω that is Ω-homologous to a point, which is a constant curve $[a, b] \to \{z_0\}$ for some $z_0 \in \Omega$, then we say that γ is **Ω-homologous to zero**. It should be clear that a closed curve γ in Ω is Ω-homologous to zero if and only if $\mathrm{wn}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$. Thus an open set Ω is homologically simply connected if and only if every closed path in Ω is Ω-homologous to zero.

Proposition 6.47. *Let $\Omega \subseteq \mathbb{C}$ be an open set.*

1. *If $\overline{\mathbb{C}} \setminus \Omega$ is connected, then every closed curve in Ω is Ω-homologous to zero.*
2. *If every closed path in Ω is Ω-homologous to zero, then $\overline{\mathbb{C}} \setminus \Omega$ is connected.*

Proof.
Proof of Part (1). Suppose $\overline{\mathbb{C}} \setminus \Omega$ is connected, and let γ be a closed curve in Ω. Define $\mathrm{wn}(\gamma, \infty) = 0$, and also define

$$A = \{z \in \overline{\mathbb{C}} \setminus \Omega : \mathrm{wn}(\gamma, z) = 0\} \quad \text{and} \quad B = \{z \in \overline{\mathbb{C}} \setminus \Omega : \mathrm{wn}(\gamma, z) \neq 0\},$$

so $\infty \in A$. Since γ^* is compact there exists some $r \in \mathbb{R}_+$ such that $\gamma^* \subseteq B_r(0)$. Then $\mathrm{wn}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus B_r(0)$ by Proposition 6.20(4), so that $U_\infty := \overline{\mathbb{C}} \setminus \overline{B}_r(0) \subseteq \overline{\mathbb{C}} \setminus B_r(0) \subseteq A$ with $U_\infty \cap B = \varnothing$. Note that U_∞ is open in $(\overline{\mathbb{C}}, \bar{d}\,)$.

If $z \in A \setminus \{\infty\}$, then $z \in \mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus \gamma^*$ with $\mathrm{wn}(\gamma, z) = 0$, and so z lies in some component U_z of $\mathbb{C} \setminus \gamma^*$ on which $\mathrm{wn}(\gamma, \cdot)$ is identically zero by Proposition 6.21. If $z \in B$, then $z \neq \infty$ with $\mathrm{wn}(\gamma, z) \neq 0$, and so z lies in some component V_z of $\mathbb{C} \setminus \gamma^*$ on which $\mathrm{wn}(\gamma, \cdot)$ is nonvanishing by Proposition 6.21. Clearly the collection of components U_z is disjoint from the collection of components V_z. Moreover each U_z and V_z is open in $(\mathbb{C}, d)$ by Proposition 2.35, hence open in $(\overline{\mathbb{C}}, \bar{d}\,)$ by Proposition 6.39.

We now see that

$$A \subseteq U := \bigcup_{z \in A} U_z \quad \text{and} \quad B \subseteq V := \bigcup_{z \in B} V_z,$$

with U and V open in $(\overline{\mathbb{C}}, \bar{d}\,)$, and $U \cap V = \varnothing$. Thus $A = \overline{\mathbb{C}} \setminus \Omega \cap U$ and $B = \overline{\mathbb{C}} \setminus \Omega \cap V$, which is to say A and B are both open in $\overline{\mathbb{C}} \setminus \Omega$ such that $A \cap B = \varnothing$ and $A \cup B = \overline{\mathbb{C}} \setminus \Omega$. Since $\overline{\mathbb{C}} \setminus \Omega$ is connected and A is nonempty (it contains all points outside $B_r(0)$), we conclude that $B = \varnothing$. Hence $A = \overline{\mathbb{C}} \setminus \Omega$, so that $\mathrm{wn}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$. Therefore Ω is Ω-homologous to zero.

Proof of Part (2). Suppose $\overline{\mathbb{C}} \setminus \Omega$ is not connected, so there exist disjoint nonempty sets $K, L \subseteq \overline{\mathbb{C}} \setminus \Omega$, closed in $(\overline{\mathbb{C}}, \bar{d}\,)$, such that $K \cup L = \overline{\mathbb{C}} \setminus \Omega$. Assume $\infty \in L$, so that $\infty \notin K$ and we have $K \subseteq \mathbb{C} \setminus \Omega$. Then K is compact in $(\mathbb{C}, d)$ by Proposition 6.41.

Now, L closed in $(\overline{\mathbb{C}}, \bar{d}\,)$ implies $\overline{\mathbb{C}} \setminus L$ is open in $(\overline{\mathbb{C}}, \bar{d}\,)$, and since $\overline{\mathbb{C}} \setminus L = \mathbb{C} \cap (\overline{\mathbb{C}} \setminus L)$, it follows that $\overline{\mathbb{C}} \setminus L$ is open in $(\mathbb{C}, \bar{d}\,)$ as well, and therefore is open in $(\mathbb{C}, d)$ by Proposition 6.39. Noting that $\mathbb{C} \setminus L = \overline{\mathbb{C}} \setminus L$, we see $\Omega' := \mathbb{C} \setminus L$ is open in $(\mathbb{C}, d)$, and moreover $K \subseteq \Omega'$. By the Hexagon Lemma there exist closed paths $\gamma_1, \ldots, \gamma_m$ in $\Omega' \setminus K$ such that $\sum_{j=1}^m \mathrm{wn}(\gamma_j, z) = 1$ if $z_0 \in K$. From the chain of equivalencies

$$w \in \Omega' \setminus K \;\Leftrightarrow\; w \in \mathbb{C} \setminus L \;\&\; w \notin K \;\Leftrightarrow\; w \notin K \cup L \;\Leftrightarrow\; w \notin \overline{\mathbb{C}} \setminus \Omega \;\Leftrightarrow\; w \in \Omega$$

we find that $\Omega' \setminus K = \Omega$, and so $\gamma_1, \ldots, \gamma_m$ are closed paths in Ω. Recalling that $\mathbb{C} \setminus \Omega = K \neq \varnothing$, let $z_0 \in \mathbb{C} \setminus \Omega$. Then $\sum_{j=1}^m \operatorname{wn}(\gamma_j, z_0) = 1$, and so there exists some $1 \leq k \leq m$ such that $\operatorname{wn}(\gamma_k, z_0) \neq 0$. Thus γ_k is a closed path in Ω that is not Ω-homologous to zero. ■

We now collect many of our results into a single theorem, which will be expanded considerably in §11.3.

Theorem 6.48 (**Second Cauchy Theorem**). *Let $\Omega \subseteq \mathbb{C}$ be an open set. Then the following statements are equivalent.*

1. *$\overline{\mathbb{C}} \setminus \Omega$ is connected.*
2. *Every closed curve in Ω is Ω-homologous to zero.*
3. *Ω is homologically simply connected.*
4. *For every closed path γ in Ω and every analytic function f on Ω,*

$$\oint_\gamma f = 0.$$

5. *Every analytic function on Ω has a primitive on Ω.*
6. *Every nonvanishing analytic function on Ω has an analytic logarithm on Ω.*
7. *Every nonvanishing analytic function on Ω has an analytic nth root for all $n \in \mathbb{N}$.*

Proof. That (1) implies (2) is given by Proposition 6.47(1). That (2) implies (3) follows from the remarks after Definition 6.46 and the observation that every closed path is a closed curve. That (3) implies (1) follows from the remarks after Definition 6.46 and Proposition 6.47(2). The equivalency of (3) and (4) follows from Cauchy's Theorem. The equivalency of (4) and (5) follows from Theorem 3.41 and the Fundamental Theorem of Path Integrals. That (4) implies (6) is a consequence of Corollary 6.12. Exercise 6.25 gives the equivalency of (6) and (7).

To complete the proof we show that (6) implies (3). Suppose every nonvanishing analytic function on Ω has an analytic logarithm on Ω. Let γ be any closed path in Ω, and let $z \in \mathbb{C} \setminus \Omega$ be arbitrary. Define $f : \Omega \to \mathbb{C}$ by $f(w) = w - z$. Then f is a nonvanishing analytic function on Ω, and so has an analytic function on Ω. By Theorem 6.11 it follows that $\oint_\gamma f'/f = 0$. Since $f \circ \gamma = \gamma - z$ and $0 \notin (f \circ \gamma)^*$, by Corollary 6.23 we obtain

$$\operatorname{wn}(\gamma - z, 0) = \operatorname{wn}(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} = 0,$$

and hence $\operatorname{wn}(\gamma, z) = 0$ by Proposition 6.20(2). Therefore Ω is homologically simply connected. ■

Exercise 6.49 (AN3.4.1a). Let $\Omega \subseteq \mathbb{C}$ be the open annulus

$$\Omega = \{z \in \mathbb{C} : 1/2 < |z| < 2\},$$

which clearly is an open connected set. Now, $\overline{\mathbb{C}} \setminus \Omega$ is the set in the metric space $(\overline{\mathbb{C}}, \bar{d}\,)$ given by

$$\overline{\mathbb{C}} \setminus \Omega = \overline{B}_{1/2}(0) \cup (\overline{\mathbb{C}} \setminus B_2(0)).$$

To show that Ω is not homologically simply connected, let $\gamma : [0, 2\pi] \to \Omega$ be the closed path in Ω given by $\gamma(t) = e^{it}$, and let $f : \Omega \to \mathbb{C}$ be given by $f(z) = 1/z$. Since f is analytic on Ω, $0 \notin \gamma^*$, and $\mathrm{wn}(\gamma, 0) = 1$, we have

$$\oint_\gamma f = \oint_\gamma \frac{1}{z} dz = 2\pi i \,\mathrm{wn}(\gamma, 0) = 2\pi i$$

by Theorem 6.22. Thus $\oint_\gamma f \neq 0$, and the Second Cauchy Theorem implies that Ω is not homologically simply connected. ■

Exercise 6.50 (AN3.4.1b). Let $\Omega = B_1(-2) \cup B_1(2)$, which is a disjoint union of two open balls of radius 1 in $\mathbb{C}$. Then Ω is a homologically simply connected subset of $(\mathbb{C}, d)$ since $\overline{\mathbb{C}} \setminus \Omega$ is connected in $(\overline{\mathbb{C}}, \bar{d})$. However, it is clear that Ω is not connected in $(\mathbb{C}, d)$ (or equivalently in $(\mathbb{C}, \bar{d})$, owing to Proposition 6.39). ■

7

Residue Theory

7.1 – Laurent Series

Given a two-tailed sequence $(z_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$, the associated infinite series is the ordered formal sum

$$\sum_{n\in\mathbb{Z}} z_n = \cdots + z_{-3} + z_{-2} + z_{-1} + z_0 + z_1 + z_2 + z_3 + \cdots , \tag{7.1}$$

here denoted by $\Sigma_{\mathbb{Z}}$ for brevity. We say the series $\Sigma_{\mathbb{Z}}$ is **convergent** if the series

$$\Sigma_+ := \sum_{n=0}^{\infty} z_n \quad \text{and} \quad \Sigma_- := \sum_{n=1}^{\infty} z_{-n}$$

both converge, in which case we define $\Sigma_{\mathbb{Z}} := \Sigma_- + \Sigma_+$; that is,

$$\sum_{n\in\mathbb{Z}} z_n := \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n.$$

Otherwise $\Sigma_{\mathbb{Z}}$ is **divergent**. The series Σ_- is known as the **principal part** of $\Sigma_{\mathbb{Z}}$.

We define $\Sigma_{\mathbb{Z}}$ to be **absolutely convergent** (resp. **uniformly convergent**) if the series Σ_+ and Σ_- are both absolutely convergent (resp. uniformly convergent).

Definition 7.1. *Given $c \in \mathbb{C}$ and sequence $(a_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$, the **Laurent series** with **center** c and **coefficients** a_n is the infinite series*

$$\sum_{n\in\mathbb{Z}} a_n(z-c)^n.$$

For any $0 \le s_1 < s_2 \le \infty$ we define

$$A_{s_1,s_2}(z_0) = \{z \in \mathbb{C} : s_1 < |z - z_0| < s_2\},$$

the **open annulus** with center z_0, and inner and outer radii s_1, s_2, respectively.

Proposition 7.2. *Let f be analytic on Ω, and let $0 < r_1 < r_2 < \infty$ be such that $\overline{A}_{r_1,r_2}(z_0) \subseteq \Omega$. If γ_1 and γ_2 are paths such that $\gamma_j^* = C_{r_j}(z_0)$ and $\mathrm{wn}(\gamma_j, z_0) = 1$, then*

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

for all $z \in A_{r_1,r_2}(z_0)$.

Unless otherwise stated, we assume that any path $C_r(z_0)$ has the positive (i.e. counter-clockwise) orientation, and completes precisely one circuit. That is, $\mathrm{wn}(C_r(z_0), z) = 1$ for any $z \in B_r(z_0)$.

Theorem 7.3. *If f is analytic on $\Omega = A_{s_1,s_2}(z_0)$, then there exists a unique sequence $(a_n)_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n \tag{7.2}$$

for all $z \in \Omega$. Moreover, the series converges absolutely on Ω and uniformly on compact subsets of Ω, and for any $r \in (s_1, s_2)$ the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw \tag{7.3}$$

for all $n \in \mathbb{Z}$.

Proof. Suppose that f is analytic on $\Omega = A_{s_1,s_2}(z_0)$. Fix $r_1, r_2 \in (s_1, s_2)$ with $r_1 < r_2$. Define $\Phi_2 : B_{r_2}(z_0) \to \mathbb{C}$ by

$$\Phi_2(z) = \frac{1}{2\pi i}\oint_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\,dw.$$

For fixed $z \in B_{r_2}(z_0)$ and varying $w \in C_{r_2}(z_0)$ we have

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)-(z-z_0)} = \frac{f(w)}{w-z_0}\cdot\frac{1}{1-\dfrac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}},$$

which the Weierstrass M-Test easily shows is a uniformly convergent series on $C_{r_2}(z_0)$, and so by Corollary 3.38,

$$\begin{aligned}\Phi_2(z) &= \frac{1}{2\pi i}\oint_{C_{r_2}(z_0)}\left[\sum_{n=0}^{\infty}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\right]dw = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left[\oint_{C_{r_2}(z_0)}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\,dw\right]\\ &= \sum_{n=0}^{\infty}\left[\left(\frac{1}{2\pi i}\oint_{C_{r_2}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n\right] = \sum_{n=0}^{\infty}a_n(z-z_0)^n,\end{aligned} \tag{7.4}$$

where for $n \geq 0$ we define

$$a_n = \frac{1}{2\pi i}\oint_{C_{r_2}(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

For arbitrary $z \in B_{r_2}(z_0)$ we have $|z-z_0| = \rho < r_2$. Letting

$$M = \max_{w\in C_{r_2}(z_0)}|f(w)|,$$

Theorem 3.23 gives

$$|a_n(z-z_0)^n| \le \frac{\rho^n}{2\pi}\left|\oint_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right| = \frac{\rho^n}{2\pi}\left(\frac{M}{r_2^{n+1}}\cdot 2\pi r_2\right) = M\left(\frac{\rho}{r_2}\right)^n,$$

and since the series $\sum M(\rho/r_2)^n$ converges in $\mathbb{R}$, by the Direct Comparison Test we conclude that the series $\sum a_n(z-z_0)^n$ converges absolutely on $z \in B_{r_2}(z_0)$.

Now suppose $K \subseteq B_{r_2}(z_0)$ is compact, and let

$$\rho = \max_{z\in K}|z-z_0|.$$

Then $\rho < r_2$, and by Theorem 3.23,

$$\|a_n(z-z_0)^n\|_K = \frac{\rho^n}{2\pi}\sup_{z\in K}\left|\oint_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right| \le M\left(\frac{\rho}{r_2}\right)^n.$$

It follows by the Weierstrass M-test that $\sum a_n(z-z_0)^n$ converges uniformly on K. To summarize, $\sum a_n(z-z_0)^n$ converges absolutely on $B_{r_2}(z_0)$ and uniformly on compact subsets of $B_{r_2}(z_0)$.

Now, define $\Phi_1 : A_{r_1,\infty}(z_0) \to \mathbb{C}$ by

$$\Phi_1(z) = -\frac{1}{2\pi i}\oint_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\,dw.$$

For fixed $z \in A_{r_1,\infty}(z_0)$ and varying $w \in C_{r_1}(z_0)$ we have

$$-\frac{f(w)}{w-z} = \frac{f(w)}{(z-z_0)\left(1-\dfrac{w-z_0}{z-z_0}\right)} = \sum_{n=1}^{\infty}\frac{f(w)(w-z_0)^{n-1}}{(z-z_0)^n},$$

which is a uniformly convergent series on $C_{r_1}(z_0)$, and so

$$\begin{aligned}\Phi_1(z) &= \frac{1}{2\pi i}\oint_{C_{r_1}(z_0)}\left[\sum_{n=1}^{\infty}\frac{f(w)(w-z_0)^{n-1}}{(z-z_0)^n}\right]dw = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\left[\oint_{C_{r_1}(z_0)}\frac{f(w)(w-z_0)^{n-1}}{(z-z_0)^n}\,dw\right]\\ &= \sum_{n=1}^{\infty}\left[\left(\frac{1}{2\pi i}\oint_{C_{r_1}(z_0)}\frac{f(w)}{(w-z_0)^{-n+1}}\,dw\right)(z-z_0)^{-n}\right] = \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}, \qquad (7.5)\end{aligned}$$

where for $n \ge 1$ we define

$$b_n = \frac{1}{2\pi i}\oint_{C_{r_1}(z_0)}\frac{f(w)}{(w-z_0)^{-n+1}}\,dw.$$

It is straightforward to show that $\sum b_n(z-z_0)^{-n}$ converges absolutely on $A_{r_1,\infty}(z_0)$, so the details are omitted. Let $K \subseteq A_{r_1,\infty}(z_0)$ be compact, and let

$$\frac{1}{\rho} = \max_{z\in K}\frac{1}{|z-z_0|} \quad \text{and} \quad M = \max_{w\in C_{r_1}(z_0)}|f(w)|.$$

Then $1/\rho < 1/r_1$, and by Theorem 3.23,

$$\|b_n(z-z_0)^{-n}\|_K = \frac{1}{2\pi\rho^n}\sup_{z\in K}\left|\oint_{C_{r_1}(z_0)}\frac{f(w)}{(w-z_0)^{-n+1}}\,dw\right|$$

$$\leq \frac{1}{2\pi\rho^n}\left(\frac{M}{r_1^{-n+1}} \cdot 2\pi r_1\right)^n = M\left(\frac{r_1}{\rho}\right)^n,$$

where $r_1/\rho < 1$. By the Weierstrass M-Test $\sum b_n(z-z_0)^{-n}$ converges uniformly on K. To summarize, $\sum b_n(z-z_0)^{-n}$ converges absolutely on $A_{r_1,\infty}(z_0)$ and uniformly on compact subsets of $A_{r_1,\infty}(z_0)$.

Let $a_{-n} = b_n$ for $n \geq 1$. Fix $r \in (s_1, s_2)$. By Corollary 6.32

$$a_n = \frac{1}{2\pi i}\oint_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

for all $n \geq 0$, and

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_{r_1}(z_0)} \frac{f(w)}{(w-z_0)^{-n+1}}\,dw = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{-n+1}}\,dw$$

for all $n \geq 1$. Therefore

$$a_n = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

for all $n \in \mathbb{Z}$.

Define the Laurent series $\Sigma_{\mathbb{Z}}$ by

$$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}.$$

Both series on the right converge absolutely on $A_{r_1,r_2}(z_0) = B_{r_2}(z_0) \cap A_{r_1,\infty}(z_0)$ and uniformly on compact subsets of $A_{r_1,r_2}(z_0)$, and therefore so too does $\Sigma_{\mathbb{Z}}$. Since $r_1, r_2 \in (s_1, s_2)$ are arbitrary, we conclude that $\Sigma_{\mathbb{Z}}$ converges absolutely on Ω and uniformly on compact subsets of Ω.

Next, by Proposition 7.2,

$$f(z) = \frac{1}{2\pi i}\oint_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\,dw = \Phi_2(z) + \Phi_1(z)$$

for all $z \in A_{r_1,r_2}(z_0)$. Recalling (7.4) and (7.5),

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n \tag{7.6}$$

for all $z \in A_{r_1,r_2}(z_0)$. Again, $r_1, r_2 \in (s_1, s_2)$ are arbitrary, and so (7.6) holds for all $z \in \Omega$. All aspects of the theorem have now been proved except for the uniqueness of $(a_n)_{n\in\mathbb{Z}}$.

Suppose that $(\hat{a}_n)_{n\in\mathbb{Z}}$ is such that

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{a}_n(z-z_0)^n$$

for all $z \in \Omega$. The existence part of Theorem 7.3, which has just been proven, implies that the Laurent series $\Sigma_{\mathbb{Z}}$ converges absolutely on Ω and uniformly on compact subsets of Ω, and so for any $s_1 < r < s_2$ it converges uniformly on $C_r(z_0)$. Now, for any $k \in \mathbb{Z}$,

$$a_k = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\,dw = \frac{1}{2\pi i}\oint_{C_r(z_0)} \left[\frac{1}{(w-z_0)^{k+1}} \sum_{n\in\mathbb{Z}} \hat{a}_n(w-z_0)^n\right] dw$$

$$= \frac{1}{2\pi i}\oint_{C_r(z_0)}\left[\sum_{n\in\mathbb{Z}}\hat{a}_n(w-z_0)^{n-k-1}\right]dw = \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}\left[\oint_{C_r(z_0)}\hat{a}_n(w-z_0)^{n-k-1}\,dw\right],$$

where the last equality is obtained by applying Corollary 3.38 to the Σ_+ and Σ_- components of $\Sigma_{\mathbb{Z}}$. For $n-k-1 \neq -1$ (i.e. $n \neq k$), the function $f_n(z) = \hat{a}_n(z-z_0)^{n-k-1}$ has a primitive on Ω, and so the Fundamental Theorem of Path Integrals implies that

$$\oint_{C_r(z_0)}\hat{a}_n(w-z_0)^{n-k-1}\,dw = 0,$$

and so by Theorem 6.22

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}\left[\oint_{C_r(z_0)}\hat{a}_n(w-z_0)^{n-k-1}\,dw\right] = \frac{1}{2\pi i}\oint_{C_r(z_0)}\hat{a}_k(w-z_0)^{-1}\,dw \\ &= \frac{\hat{a}_k}{2\pi i}\oint_{C_r(z_0)}\frac{1}{w-z_0}\,dw = \hat{a}_k\,\mathrm{wn}(C_r(z_0),z_0) = \hat{a}_k. \end{aligned}$$

Since $k \in \mathbb{Z}$ is arbitrary we conclude that $(\hat{a}_n)_{n\in\mathbb{Z}} = (a_n)_{n\in\mathbb{Z}}$, proving uniqueness. ■

The series at right in (7.2) is called the **Laurent series representation** of f on $A_{s_1,s_2}(z_0)$, which Theorem 7.3 makes clear is unique on a given annulus $A_{s_1,s_2}(z_0)$ of analyticity. However, Exercise 7.19 in the next section shows how the Laurent series representation of a given function f may vary on different annuli of analyticity.

Theorem 7.4. *Suppose that f is analytic on $\Omega = A_{s_1,s_2}(z_0)$, with Laurent series representation*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$$

for all $z \in A_{s_1,s_2}(z_0)$. Then f has derivatives of all orders on Ω, with

$$f^{(k)}(z) = \sum_{n\in\mathbb{Z}}\big[n(n-1)(n-2)\cdots(n-k+1)a_n\big](z-z_0)^{n-k}$$

for all $k \in \mathbb{N}$ and $z \in \Omega$.

Proof. That f has derivatives of all orders on Ω is assured by Corollary 4.20. We have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n},$$

where Theorem 7.3 and the remarks following Definition 7.1 imply that the two series at right are each absolutely convergent on Ω, and also each converges uniformly on compact subsets of Ω. Let $\sigma, \tau : \Omega \to \mathbb{C}$ be given by

$$\sigma(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \tau(z) = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n},$$

so that $f = \sigma + \tau$, and thus $f^{(k)} = \sigma^{(k)} + \tau^{(k)}$ for all $k \in \mathbb{N}$. Define the sequences $(\sigma_n)_{n=0}^\infty$ and $(\tau_n)_{n=1}^\infty$ of analytic functions $\Omega \to \mathbb{C}$ by

$$\sigma_n(z) = \sum_{k=0}^{n} a_n(z - z_0)^n \quad \text{and} \quad \tau_n(z) = \sum_{k=1}^{n} a_{-n}(z - z_0)^{-n}.$$

Then $(\sigma_n)_{n=0}^\infty$ and $(\tau_n)_{n=1}^\infty$ converge uniformly to σ and τ on compact subsets of Ω, respectively, and hence by Theorem 4.30 both σ and τ are analytic on Ω (and hence have derivatives of all orders on Ω), and moreover

$$\sigma_n^{(k)} \xrightarrow{\mathrm{u}} \sigma^{(k)} \quad \text{and} \quad \tau_n^{(k)} \xrightarrow{\mathrm{u}} \tau^{(k)}$$

on compact subsets of Ω. Thus for any $r_1, r_2 > 0$ such that $s_1 < r_1 < r_2 < s_2$, we have

$$\sigma^{(k)}(z) = \sum_{n=0}^{\infty} [n(n-1)(n-2)\cdots(n-k+1)a_n](z - z_0)^{n-k}$$

and

$$\tau^{(k)}(z) = \sum_{n=1}^{\infty} [-n(-n-1)(-n-2)\cdots(-n-k+1)a_{-n}](z - z_0)^{-n-k}$$

for all $k \in \mathbb{N}$ and $z \in \overline{A}_{r_1,r_2}(z_0)$. That is,

$$f^{(k)}(z) = \sigma^{(k)}(z) + \tau^{(k)}(z) = \sum_{n \in \mathbb{Z}} [n(n-1)(n-2)\cdots(n-k+1)a_n](z - z_0)^{n-k} \tag{7.7}$$

for all $k \in \mathbb{N}$ and $z \in \overline{A}_{r_1,r_2}(z_0)$, and since $r_1, r_2 \in (s_1, s_2)$ are arbitrary, we may finally conclude that (7.7) holds for all $z \in \Omega$. ■

7.2 – Singularities

A function f has a **singularity** at a point $z_0 \in \mathbb{C}$ if f is not analytic at z_0. Strictly speaking this means that any point not in the domain of f is a singularity of f, but in practice the only singularities of f that are of interest are those that are limit points of the domain of f. In the definition that follows recall that $B'_\epsilon(z_0) = B_\epsilon(z_0) \setminus \{z_0\}$ is a deleted neighborhood of z_0.

Definition 7.5. *A function f has an* ***isolated singularity*** *at $z_0 \in \mathbb{C}$ if $f'(z_0)$ does not exist and f is analytic on $B'_\epsilon(z_0)$ for some $\epsilon > 0$. We say that f has an isolated singularity at ∞ if f is analytic on $A_{r,\infty}(0)$ for some $r > 0$.*

A function f has a ***nonisolated singularity*** *at $z_0 \in \mathbb{C}$ if $f'(z_0)$ does not exist and f is not analytic on $B'_\epsilon(z_0)$ for any $\epsilon > 0$. We say that f has a nonisolated singularity at ∞ if f is not analytic on $A_{r,\infty}(0)$ for any $r > 0$.*

Proposition 7.6. *Let f be a function, and define $g(z) = f(1/z)$. Then f has an isolated (resp. nonisolated) singularity at ∞ if and only if g has an isolated (resp. nonisolated) singularity at 0.*

Proof. Suppose that f has an isolated singularity at ∞, so there exists some $r > 0$ such that f is analytic on $A_{r,\infty}(0)$. Define $h : B'_{1/r}(0) \to A_{r,\infty}(0)$ by $h(z) = 1/z$. Since h is analytic on $B'_{1/r}(0)$ and f is analytic on $h(B'_{1/r}(0)) = A_{r,\infty}(0)$, by the Chain Rule it follows that $g = f \circ h$ is analytic on $B'_{1/r}(0)$, and so g has an isolated singularity at 0.

Now suppose that g has an isolated singularity at 0, so there exists some $\epsilon > 0$ such that g is analytic on $B'_\epsilon(0)$. Define $h : A_{1/\epsilon,\infty}(0) \to B'_\epsilon(0)$ by $h(z) = 1/z$. Since h is analytic on $A_{1/\epsilon,\infty}(0)$ and g is analytic on $B'_\epsilon(0)$, by the Chain Rule it follows that $f = g \circ h$ is analytic on $A_{1/\epsilon,\infty}(0)$.

The proof of the proposition's parallel statement concerning nonisolated singularities is much the same. ■

If f has an isolated singularity at $z_0 \in \mathbb{C}$, so that f is given to be analytic on some deleted neighborhood $B'_r(z_0)$, then Theorem 7.3 implies that f has a Laurent series representation

$$\sum_{n \in \mathbb{Z}} a_n (z - z_0)^n \tag{7.8}$$

on $B'_r(z_0)$. We classify isolated singularities based on particular attributes of this representation.

Definition 7.7. *Suppose that f has an isolated singularity at $z_0 \in \mathbb{C}$, and let (7.8) be the Laurent series representation of f on some deleted neighborhood of z_0.*

1. *If $a_n = 0$ for all $n \leq -1$, then f has a* ***removable singularity*** *at z_0.*
2. *If $m \in \mathbb{N}$ is such that $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$, then f has a* ***pole of order m*** *at z_0. A pole of order 1 is also called a* ***simple pole****.*
3. *If $a_n \neq 0$ for infinitely many $n \leq -1$, then f has an* ***essential singularity*** *at z_0.*

Definition 7.8. *Suppose f has an isolated singularity at ∞, and define $g(z) = f(1/z)$. We say f has a* ***removable singularity****,* ***pole of order m****, or* ***essential singularity*** *at ∞ if g has a removable singularity, pole of order m, or essential singularity at 0, respectively.*

It should be clear that any isolated singularity of a function f at some point $z_0 \in \overline{\mathbb{C}}$ must be precisely one of the three types of isolated singularity defined above. Thus, if an isolated singularity is found to not be two of the three types given in Definition 7.7, then it must be the third type.

Theorem 7.9. *Suppose f has an isolated singularity at $z_0 \in \mathbb{C}$. Then the following statements are equivalent.*

1. *f has a removable singularity at z_0.*
2. *There exists some $a \in \mathbb{C}$ such that $f(z) \to a$ as $z \to z_0$.*
3. *There exists some $\delta > 0$ such that f is bounded on $B'_\delta(z_0)$.*

Proof.
(1) → (2). Suppose f has a removable singularity at z_0. There exists some $r > 0$ such that f is analytic on $B' = B'_r(z_0)$, and so by Theorem 7.3 there exists a unique sequence $(a_n)_{n\in\mathbb{Z}}$ such that

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n \tag{7.9}$$

for all $z \in B'$. However, $a_n = 0$ for all $n \leq -1$, and so

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

on B'. Define φ on $B = B_r(z_0)$ by

$$\varphi(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in B$. Then φ is analytic (and hence continuous) on B by Proposition 4.31, and since $\varphi|_{B'} = f$ we have

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \varphi(z) = \varphi(z_0) = a_0 \in \mathbb{C}$$

as desired.

(2) → (3). Suppose that $f(z) \to a \in \mathbb{C}$ as $z \to z_0$. Then there exists some $\delta \in (0, r)$ such that $z \in B'_\delta(z_0)$ implies that $|f(z) - a| < 1$, and hence $|f(z)| < |a| + 1$ for all $z \in B'_\delta(z_0)$; that is, f is bounded on $B'_\delta(z_0)$.

(3) → (1). Suppose that f is bounded on $B'_\delta(z_0)$, where we can assume that $0 < \delta < r$. Define $\varphi : B \to \mathbb{C}$ by

$$\varphi(z) = \begin{cases} (z - z_0)f(z), & z \in B' \\ 0, & z = z_0 \end{cases} \tag{7.10}$$

Certainly φ is analytic on B', and since f is bounded on a deleted neighborhood of z_0 the Squeeze Theorem can readily be used to obtain

$$\lim_{z\to z_0} \varphi(z) = \lim_{z\to z_0} (z - z_0)f(z) = 0 = \varphi(z_0),$$

which shows that φ is continuous at z_0, and hence φ is analytic on B by Corollary 4.22. By Theorem 4.29 there exists a sequence $(b_n)_{n=0}^{\infty}$ in $\mathbb{C}$ such that

$$\varphi(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n \tag{7.11}$$

for all $z \in B$.

On the other hand, for all $z \in B'$ we have

$$f(z) = \frac{\varphi(z)}{z-z_0} = \frac{1}{z-z_0}\sum_{n=0}^{\infty} b_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^{n-1},$$

and since the sequence $(a_n)_{n\in\mathbb{Z}}$ for which (7.9) holds is unique, we conclude that $a_n = 0$ for all $n \leq -2$, and $a_{-1} = b_0$. However, from (7.10) and (7.11) we find that $b_0 = \varphi(z_0) = 0$, and so $a_n = 0$ for $n \leq -1$. Therefore f has a removable singularity at z_0. ■

Theorem 7.10. *Suppose f has an isolated singularity at $z_0 \in \mathbb{C}$. Then*

1. *For all $m \in \mathbb{N}$, f has a pole of order m at z_0 if and only if there exists some $a \in \mathbb{C}_*$ such that*

$$\lim_{z\to z_0} (z-z_0)^m f(z) = a.$$

2. *f has a pole at z_0 if and only if $|f(z)| \to +\infty$ as $z \to z_0$.*
3. *f has an essential singularity at z_0 if and only if $\lim_{z\to z_0} f(z)$ does not exist in $\overline{\mathbb{C}}$.*

Proof.
Proof of Part (1): Suppose f has a pole of order m at z_0. Then there exists some $r > 0$ such that f is analytic on $B' = B'_r(z_0)$, and

$$f(z) = \sum_{n=1}^{m} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n := \sum_{n=-m}^{\infty} a_n(z-z_0)^n,$$

for all $z \in B'$, where $a_{-m} \neq 0$. The series converges absolutely on B', so

$$g(z) := (z-z_0)^m f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^{m+n} = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \tag{7.12}$$

for all $z \in B'$. Define φ on $B = B_r(z_0)$ by

$$\varphi(z) = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$$

for all $z \in B$. Then φ is analytic (and hence continuous) on B by Proposition 4.31, and since $\varphi|_{B'} = g$ we have

$$\lim_{z\to z_0}(z-z_0)^m f(z) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0}\varphi(z) = \varphi(z_0) = a_{-m} \in \mathbb{C}_*$$

as desired.

For the converse, suppose that

$$\lim_{z\to z_0}(z-z_0)^m f(z) = a \in \mathbb{C}_*.$$

There exists a unique sequence $(a_n)_{n=-\infty}^{\infty}$ and $r > 0$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{7.13}$$

for all $z \in B' = B'_r(z_0)$. Define φ on $B = B_r(z_0)$ by

$$\varphi(z) = \begin{cases} (z - z_0)^m f(z), & z \neq z_0 \\ a, & z = z_0 \end{cases}$$

Then φ is analytic on B' and continuous on B, so by Corollary 4.22 φ is analytic on B. By Theorem 4.29 there exists a sequence $(b_n)_{n=0}^{\infty}$ such that

$$\varphi(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

on B, and so

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

on B', which implies that

$$f(z) = \frac{1}{(z - z_0)^m} \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^{n-m} \tag{7.14}$$

on B'. Comparing (7.13) with (7.14) informs us that $a_n = 0$ for all $n < -m$, and $a_{-m} = b_0$. However, $\varphi(z_0) = a$ by the definition of φ, and $\varphi(z_0) = b_0$ by the series representation of φ, whence $a_{-m} = b_0 = a \in \mathbb{C}_*$. Therefore $a_{-m} \neq 0$ and we conclude that f has a pole of order m at z_0.

Proof of Part (2). Suppose that f has a pole at z_0, so there exists some $m \geq 1$ and $a \in \mathbb{C}_*$ such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = a,$$

and hence

$$\lim_{z \to z_0} |z - z_0|^m |f(z)| = |a| > 0.$$

Since

$$\lim_{z \to z_0} \frac{1}{|z - z_0|^m} = +\infty,$$

it follows readily that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left(\frac{1}{|z - z_0|^m} \cdot |z - z_0|^m |f(z)| \right) = +\infty \cdot |a| = +\infty.$$

Conversely, suppose that

$$\lim_{z \to z_0} |f(z)| = +\infty.$$

By Theorem 7.9 it is clear that f cannot have a removable singularity at z_0, and by the Casorati-Weierstrass Theorem below it's seen that f also cannot have an essential singularity at z_0 since, for instance, there exists some $\delta > 0$ such that $|f(z)| > 1$ for all $z \in B'_\delta(z_0)$.

Proof of Part (3). Suppose that $\lim_{z\to z_0} f(z) = a$ for some $a \in \overline{\mathbb{C}}$. If $a \in \mathbb{C}$, then f has a removable singularity at z_0 by Theorem 7.9; and if $a = \infty$, then $|f(z)| \to +\infty$ as $z \to z_0$ and so f has a pole at z_0 by Part (2). Hence f does not have an essential singularity at z_0.

Conversely, suppose that $\lim_{z\to z_0} f(z)$ does not exist in $\overline{\mathbb{C}}$. Then f does not have a removable singularity at z_0 by Theorem 7.9, and f does not have a pole at z_0 by Part (2). Hence f must have an essential singularity at z_0. ■

The manipulations in (7.12) are based on the understanding that an absolutely convergent Laurent series may be multiplied by a polynomial and reindexed in the same fashion as an absolutely convergent power series. In this case the Laurent series is a finite sum added to a power series:

$$\begin{aligned}(z-z_0)^m f(z) &= (z-z_0)^m\left(\sum_{n=1}^{m} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n\right)\\ &= \sum_{n=1}^{m} a_{-n}(z-z_0)^{-n+m} + \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m}\\ &= a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} a_{n-m}(z-z_0)^n\\ &= \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n.\end{aligned}$$

Lemma 7.11. *Let f have an essential singularity at $z_0 \in \mathbb{C}$. If f is nonvanishing on some deleted neighborhood of z_0, then $1/f$ also has an essential singularity at $z_0 \in \mathbb{C}$.*

Proof. Suppose f is nonvanishing on some deleted neighborhood of z_0. Then there exists some $r > 0$ such that f is both analytic and nonvanishing on $B_r'(z_0)$, and hence $1/f$ is analytic on $B_r'(z_0)$.

Suppose $1/f$ is analytic at z_0. Then $1/f$ is continuous on $B_\delta(z_0)$, implying that $\lim_{z\to z_0}(1/f)(z)$ exists in $\mathbb{C}$ and hence $\lim_{z\to z_0} f(z) \in \overline{\mathbb{C}} \setminus \{0\}$. But $\lim_{z\to z_0} f(z) \notin \overline{\mathbb{C}}$ by Theorem 7.10(3), which leaves us with a contradiction. Hence $1/f$ cannot be analytic at z_0, and so $1/f$ has an isolated singularity there.

Now, since $\lim_{z\to z_0} f(z) \notin \overline{\mathbb{C}}$, it follows (recall Proposition 6.40) that $\lim_{z\to z_0}(1/f)(z) \notin \overline{\mathbb{C}}$ also. Therefore $1/f$ has an essential singularity at z_0 by Theorem 7.10(3). ■

Theorem 7.12 (Casorati-Weierstrass Theorem). *Let f have an essential singularity at $z_0 \in \mathbb{C}$, and let $r > 0$ such that f is analytic on $B_r'(z_0)$. Then $f(B_\delta'(z_0))$ is dense in $\mathbb{C}$ for all $0 < \delta \le r$.*

Proof. Fix $\delta \in (0, r]$. Suppose $f(B_\delta'(z_0))$ is not dense in $\mathbb{C}$. Thus there exists some $w \in \mathbb{C}$ and $\epsilon > 0$ such that $B_\epsilon(w) \cap f(B_\delta'(z_0)) = \varnothing$, which is to say $f(z) \notin B_\epsilon(w)$ for all $z \in B_\delta'(z_0)$, and hence $f - w$ is a nonvanishing analytic function on $B_\delta'(z_0)$ with essential singularity at z_0. Define $g = 1/(f - w)$. Then g is analytic on $B_\delta'(z_0)$, and also g has an essential singularity at z_0 by Lemma 7.11. On the other hand $|f(z) - w| \ge \epsilon$ for all $z \in B_\delta'(z_0)$, so g is bounded on

$B'_\delta(z_0)$, and by Theorem 7.9 it follows that g has a removable singularity at z_0. Having arrived at a contradiction, we conclude that $f(B'_\delta(z_0))$ is dense in $\mathbb{C}$. ■

Definition 7.13. *A function f is* ***analytic at infinity*** *if f has a removable singularity at ∞.*

Definition 7.5 certainly makes clear that in order for f to have a removable singularity at ∞, it must first be analytic on an annulus $A_{r,\infty}(0)$ for some $r > 0$. In fact, by Proposition 7.6 and Definition 7.7, f has a removable singularity at ∞ iff $g(z) = f(1/z)$ has a removable singularity at 0. Now, by Theorem 7.9, g has a removable singularity at 0 iff there exists some $a \in \mathbb{C}$ such that defining $g(0) = a$ results in g being continuous at 0, in which case g is continuous on $B_\epsilon(0)$ and analytic on $B'_\epsilon(0)$ for some $\epsilon > 0$, and thus g is analytic on $B_\epsilon(0)$ by Corollary 4.22. Assuming that f is analytic at infinity, we may now meaningfully define

$$f(\infty) = \lim_{z\to 0} f(1/z),$$

where the limit is guaranteed to exist in $\mathbb{C}$.

An entire function that is analytic at infinity is said to be analytic on $\overline{\mathbb{C}}$, the extended complex plane.

Exercise 7.14 (AN4.1.2(a)). Consider the function

$$f(z) = \frac{z}{\sin z}.$$

For any $n \in \mathbb{Z} \setminus \{0\}$ we have, by L'Hôpital's Rule (Theorem 6.43),

$$\lim_{z\to n\pi}(z - n\pi)\frac{z}{\sin z} = \lim_{z\to n\pi}\frac{2z - n\pi}{\cos z} = \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n} = n\pi(-1)^n,$$

and so by Theorem 7.10(1) f has a pole of order 1 at $n\pi$ for each integer $n \neq 0$. As for $n = 0$, since

$$\lim_{z\to 0}\frac{z}{\sin z} = \lim_{z\to 0}\frac{1}{\cos z} = \frac{1}{\cos 0} = 1,$$

we conclude that f has a removable singularity at 0 by Theorem 7.9.

Finally, define

$$g(z) = f(1/z) = \frac{1/z}{\sin(1/z)}.$$

At each $z = 1/n\pi$ the function g fails to be analytic, and since 0 is a limit point of $(1/n\pi)_{n\in\mathbb{N}}$, we conclude that g has a nonisolated singularity at 0, and therefore f has a nonisolated singularity at ∞ by Definition 7.7. ■

Exercise 7.15 (AN4.1.2(b)). Consider the function $f(z) = \exp(1/z)$. Let $\epsilon > 0$ be arbitrary. For sufficiently large $n \in \mathbb{N}$ we have $z_n = -i/n\pi \in B'_\epsilon(0)$, where

$$e^{1/z_n} = e^{-n\pi/i} = e^{n\pi i} = \cos(n\pi) + i\sin(n\pi) = (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

shows that $\lim_{z\to 0} f(z)$ cannot exist in $\overline{\mathbb{C}}$, and so by Theorem 7.10(3) we conclude that f has an essential singularity at 0 by Theorem 7.10(3).

Next, define $g(z) = f(1/z) = \exp(z)$. Since

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \exp(z) = e^0 = 1,$$

g has a removable singularity at 0, and therefore f has a removable singularity at ∞. ∎

Exercise 7.16 (AN4.1.2(c)). Consider the function $f(z) = z\cos(1/z)$. Let $\epsilon > 0$ be arbitrary. For sufficiently large $n \in \mathbb{N}$ we have

$$z_n = \frac{2}{(2n+1)\pi} \in B'_\epsilon(0),$$

where

$$f(z_n) = \frac{2}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi}{2}\right) = 0.$$

On the other hand, for n sufficiently large we have $e^n/2n > 1$ and $w_n = i/n \in B'_\epsilon(0)$, and then

$$f(w_n) = w_n \cos(1/w_n) = w_n\left(\frac{e^{i/w_n} + e^{-i/w_n}}{2}\right) = \frac{i}{2n}\left(e^n + e^{-n}\right)$$

implies that

$$|f(w_n)| = \frac{e^n + e^{-n}}{2n} > \frac{e^n}{2n} > 1.$$

This makes it clear that $\lim_{z\to 0} f(z)$ cannot exist in $\overline{\mathbb{C}}$, and so by Theorem 7.10(3) we conclude that f has an essential singularity at 0.

Next, define $g(z) = f(1/z)$ for all $z \in \mathbb{C}_*$. Since

$$\lim_{z\to 0} zg(z) = \lim_{z\to 0} zf(1/z) = \lim_{z\to 0}\left(z \cdot \frac{1}{z}\cos z\right) = \lim_{z\to 0}\cos z = \cos(0) = 1,$$

g has a pole of order 1 at 0, and therefore f has a pole of order 1 at ∞. ∎

Exercise 7.17 (AN4.1.2(d)). Consider the function

$$f(z) = \frac{1}{z(e^z - 1)},$$

which is not analytic at $2n\pi i$ for any $n \in \mathbb{Z}$. Using L'Hôpital's Rule,

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0}\frac{z}{e^z - 1} = \lim_{z\to 0}\frac{1}{e^z} = 1,$$

so f has a pole of order 2 at 0. For $n \neq 0$ we have, again using L'Hôpital's Rule,

$$\begin{aligned}\lim_{z\to 2n\pi i}(z - 2n\pi i)f(z) &= \lim_{z\to 2n\pi i}\frac{z - 2n\pi i}{z(e^z - 1)} = \lim_{z\to 2n\pi i}\frac{1}{(z+1)e^z - 1}\\ &= \frac{1}{e^{2n\pi i}(2n\pi i + 1) - 1} = \frac{1}{2n\pi i} \neq 0,\end{aligned}$$

and thus f has a pole of order 1 at $2n\pi i$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Next, define

$$g(z) = f(1/z) = \frac{z}{e^{1/z} - 1}.$$

For each $n \in \mathbb{N}$ let $z_n = -i/2n\pi$. Since

$$e^{1/z_n} - 1 = e^{2n\pi i} - 1 = 1 - 1 = 0,$$

g is not analytic on $Z = \{z_n : n \in \mathbb{N}\}$; and since 0 is a limit point of Z, it follows that g has a nonisolated singularity at 0. Therefore f has a nonisolated singularity at ∞. ∎

Exercise 7.18 (AN4.1.2(e)). Consider the function

$$f(z) = \cot z := \frac{\cos z}{\sin z},$$

which is not analytic at $n\pi$ for any $n \in \mathbb{Z}$. Using L'Hôpital's Rule,

$$\lim_{z \to n\pi}(z - n\pi)f(z) = \lim_{z \to n\pi} \frac{\cos z - (z - n\pi)\sin z}{\cos z} = \frac{\cos n\pi}{\cos n\pi} = 1,$$

so f has a pole of order 1 at $n\pi$ for all $n \in \mathbb{Z}$.

Next, define

$$g(z) = f(1/z) = \frac{\cos(1/z)}{\sin(1/z)}.$$

It can be seen that g is not analytic at $1/n\pi$ for all $n \in \mathbb{Z}$, and so g has a nonisolated singularity at 0. Therefore f has a nonisolated singularity at ∞. ∎

Exercise 7.19 (AN4.1.3). Find Laurent expansions of

$$f(z) = \frac{7z - 2}{z(z+1)(z-2)}$$

in the regions $\Omega_1 = A_{0,1}(-1)$, $\Omega_2 = A_{1,3}(-1)$, and $\Omega_3 = A_{3,\infty}(-1)$.

Solution. Partial fraction decomposition gives

$$f(z) = -\frac{3}{z+1} + \frac{1}{z} + \frac{2}{z-2}$$

for all $z \in \mathbb{C} \setminus \{-1, 0, 2\}$.

Let $f_1(z) = 1/z$, which is analytic on Ω_1. It is good practice to use the results of Chapter 6 and §7.1 to determine the Laurent expansion of f_1 on Ω_1. By Theorem 7.3,

$$f_1(z) = \sum_{n \in \mathbb{Z}} a_n (z+1)^n, \quad a_n = \frac{1}{2\pi i} \oint_{C_{1/2}(-1)} \frac{f_1(w)}{(w+1)^{n+1}}\, dw$$

for all $z \in \Omega_1$. In fact, since f_1 is analytic on $B_1(-1) \supseteq \Omega_1$, and $\gamma = C_{1/2}(-1)$ is a closed path in $B_1(-1)$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $n \notin B_1(-1)$, by Exercise 6.34 (setting $z = -1$) we have

$$a_n = \frac{\operatorname{wn}(\gamma, -1) f_1^{(n)}(-1)}{n!} = \frac{(1)(-n!)}{n!} = -1$$

for $n \geq 0$. As for $n = -1$,

$$a_{-1} = \frac{1}{2\pi i} \oint_\gamma \frac{1}{w}\, dw = \operatorname{wn}(\gamma, 0) = 0$$

by Theorem 6.22. Finally for $n \le -2$, by Cauchy's Integral Formula and the observation that $z \mapsto (z+1)^{-n-1}$ is entire,

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{1/w}{(w+1)^{n+1}}\,dw = \frac{1}{2\pi i}\oint_\gamma \frac{(w+1)^{-n-1}}{w}\,dw = (0+1)^{-n-1}\,\mathrm{wn}(\gamma,0) = 0.$$

Therefore

$$\frac{1}{z} = \sum_{n\in\mathbb{Z}} a_n(z+1)^n = -\sum_{n=0}^{\infty}(z+1)^n. \tag{7.15}$$

Next, let $f_2(z) = 2/(z-2)$, which is also analytic on $B_1(-1)$. We will determine the Laurent expansion of f_2 on Ω_1 more quickly using geometric series:

$$f(z) = \frac{2}{z-2} = \frac{-2}{3-(z+1)} = -\frac{2}{3}\cdot\frac{1}{1-\frac{z+1}{3}} = -\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n. \tag{7.16}$$

We now have

$$f(z) = -\frac{3}{z+1} - \sum_{n=0}^{\infty}(z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n = -\frac{3}{z+1} - \sum_{n=0}^{\infty}\left(\frac{2}{3^{n+1}}+1\right)(z+1)^n$$

as the Laurent expansion of f in Ω_1.

In the region Ω_2 where $1 < |z+1| < 3$, the series in (7.15) no longer converges, and so a different series representation for $1/z$ is necessary:

$$\frac{1}{z} = \frac{1}{(z+1)-1} = \frac{\frac{1}{z+2}}{1-\frac{1}{z+1}} = \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n,$$

and thus

$$f(z) = -\frac{3}{z+1} + \sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

is the Laurent expansion of f in Ω_2.

Finally, in the region Ω_3 where $|z+1| > 3$, the series in (7.16) is divergent, and so a different series representation for $2/(z-2)$ must be found:

$$\frac{2}{z-2} = \frac{2}{(z+1)-3} = \frac{\frac{2}{z+1}}{1-\frac{3}{z+1}} = \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n,$$

and thus

$$f(z) = -\frac{3}{z+1} + \sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} + \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

is the Laurent expansion of f in Ω_3. ∎

Exercise 7.20 (AN4.1.5). Find the first few terms of the Laurent expansion of

$$f(z) = \frac{1}{z^2(e^z - e^{-z})}$$

on $B'_\pi(0)$.

Solution. Clearly f is analytic on $B'_\pi(0)$, and since (using L'Hôpital's Rule)

$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{z}{e^z - e^{-z}} = \lim_{z\to 0} \frac{1}{e^z + e^{-z}} = \frac{1}{2},$$

f has a pole of order 3 at 0. Define $\varphi : B_\pi(0) \to \mathbb{C}$ by

$$\varphi(z) = \begin{cases} z^3 f(z), & z \in B'_\pi(0) \\ 1/2, & z = 0 \end{cases}$$

so φ is analytic on $B_\pi(0)$ by Corollary 4.22. By Theorem 4.29

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in B_\pi(0)$, where $a_n = \varphi^{(n)}(0)/n!$ for all $n \geq 0$. Since determining $\varphi^{(n)}(0)$ is increasingly labor-intensive for higher values of n, we proceed as follows: for $0 < |z| < \pi$,

$$\frac{z}{e^z - e^{-z}} = \sum_{n=0}^{\infty} a_n z^n,$$

and hence

$$(e^z - e^{-z}) \sum_{n=0}^{\infty} a_n z^n = z.$$

Recalling that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we obtain

$$(a_0 + a_1 z + a_2 z^2 + \cdots)\left(2z + \frac{2z^3}{3!} + \frac{2z^5}{5!} + \cdots\right) = z,$$

and finally

$$(a_0 + a_1 z + a_2 z^2 + \cdots)\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) = \frac{1}{2}.$$

The first few coefficient values are

$$a_0 = \frac{1}{2}, \quad a_1 = 0, \quad a_2 + \frac{c_0}{6} = 0, \quad a_3 = 0, \quad \frac{c_0}{120} + \frac{c_2}{6} + a_4 = 0.$$

Solving, we have $a_0 = 1/2$, $a_1 = 0$, $a_2 = -1/12$, $a_3 = 0$, $a_4 = 7/720$, so that

$$\varphi(z) = \frac{1}{2} - \frac{1}{12} z^2 + \frac{7}{720} z^4 + \cdots$$

for all $z \in B_\pi(0)$. Since $f(z) = \varphi(z)/z^3$ on $B'_\pi(0)$, it follows that

$$f(z) = \frac{1}{2z^3} - \frac{1}{12z} + \frac{7}{720} z + \cdots$$

for all $z \in B'_\pi(0)$. ∎

Exercise 7.21 (AN4.1.9(a)). Show that if f is analytic on $\overline{\mathbb{C}}$, then f is constant.

Solution. Suppose that f is analytic on $\overline{\mathbb{C}}$, so by Definition 7.13 f is analytic on $\mathbb{C}$ and has a removable singularity at ∞. Define the function $g : \mathbb{C}_* \to \mathbb{C}$ by $g(z) = f(1/z)$, which is analytic on $\mathbb{C}_*$ by the Chain Rule. By Proposition 7.6 g has a removable singularity at 0, and so there exists some $a \in \mathbb{C}$ such that

$$\hat{g}(z) = \begin{cases} g(z), & z \in \mathbb{C}_* \\ a, & z = 0 \end{cases}$$

is analytic on $\mathbb{C}$ by Corollary 4.22. It follows that $\hat{g}$ is bounded on $\overline{B}_{1/r}(0)$ for any $r > 0$, which is to say there exists some $M > 0$ such that $|\hat{g}(z)| \leq M$ for all $z \in \overline{B}_{1/r}(0)$. Now,

$$z \in A_{r,\infty}(0) \;\Rightarrow\; |z| > r \;\Rightarrow\; 0 < \left|\frac{1}{z}\right| < \frac{1}{r} \;\Rightarrow\; \frac{1}{z} \in \overline{B}'_{1/r}(0),$$

so

$$z \in A_{r,\infty}(0) \;\Rightarrow\; |f(z)| = |g(1/z)| \leq M$$

and we find that f is bounded on $A_{r,\infty}(0)$. Therefore f is a bounded entire function, and Liouville's Theorem implies there exists some constant z_0 such that $f \equiv z_0$ on $\mathbb{C}$. Since

$$f(\infty) = \lim_{z\to 0} f(1/z) = \lim_{z\to 0} z_0 = z_0$$

as well, we conclude that f is constant on $\overline{\mathbb{C}}$. (Indeed, since $g(z) \to a$ as $z \to 0$, it can be seen that $f \equiv a$ on $\overline{\mathbb{C}}$.) ∎

Exercise 7.22 (AN4.1.9(b)). Suppose f is entire and there exist some $k, M, R > 0$ such that $|f(z)| \leq M|z|^k$ for all $z \in A_{R,\infty}(0)$. Show that f is a polynomial function of degree at most k.

Solution. By Theorem 4.29,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. Let $r > R$ be arbitrary. According to Cauchy's Estimate,

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \max_{z\in C_r(0)} |f(z)| \leq \frac{n!}{r^n} \max_{z\in C_r(0)} M|z|^k = \frac{n!}{r^n} \cdot Mr^k = \frac{n!M}{r^{n-k}}.$$

If $n > k$, then $n!M/r^{n-k} \to 0$ as $r \to \infty$, and then the Squeeze Theorem implies that $|f^{(n)}(0)| = 0$. Hence

$$f(z) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} z^n$$

and we conclude that f is a polynomial function such that $\deg(f) \leq k$. ∎

Exercise 7.23 (AN4.1.9(c)). Prove that if f is entire and has a nonessential singularity at ∞, then f is a polynomial function.

Solution. Suppose that f is entire and has a nonessential singularity at ∞. It follows immediately that f cannot have a nonisolated singularity at ∞, and so either f has a removable singularity or a pole at ∞.

If f has a removable singularity at ∞, then by definition f is analytic on $\overline{\mathbb{C}}$ and so must be constant on $\mathbb{C}$ by Exercise 7.21. That is, f is a polynomial function of degree at most 0.

Suppose f has a pole of order k at ∞, where of course $k \geq 1$. Then the function $g(z) = f(1/z)$ is analytic on $\mathbb{C}_*$ and has a pole of order k at 0, so that

$$\lim_{z\to 0} z^k f(1/z) = \lim_{z\to 0} z^k g(z) = a$$

for some $a \in \mathbb{C}_*$. Thus there exists some $r > 0$ such that

$$z \in B'_{1/r}(0) \;\;\Rightarrow\;\; |z^k f(1/z) - a| < 1.$$

Since $z \in A_{r,\infty}(0)$ implies that $1/z \in B'_{1/r}(0)$, we find

$$z \in A_{r,\infty}(0) \;\;\Rightarrow\;\; \left|\left(\frac{1}{z}\right)^k f(z) - a\right| < 1 \;\;\Rightarrow\;\; |z|^{-k}|f(z)| < |a| + 1 \;\;\Rightarrow\;\; |f(z)| < (|a|+1)|z|^k,$$

where $|a| + 1 > 0$. That is, there exist some $k, M, r > 0$ such that $|f(z)| \leq M|z|^k$ for all $z \in A_{r,\infty}(0)$, and therefore f is a polynomial function of degree at most k by Exercise 7.22. ■

Exercise 7.24 (AN4.1.12). If f is entire and $f(\mathbb{C})$ is not dense in $\mathbb{C}$, then f is constant.

Solution. Suppose f has an essential singularity at ∞, so that $g(z) = f(1/z)$ has an essential singularity at 0. By the Casorati-Weierstrass Theorem $g(B'_1(0))$ is dense in $\mathbb{C}$. But $z \in B'_1(0)$ if and only if $1/z \in A_{1,\infty}(0)$, so that

$$\begin{aligned} w \in g(B'_1(0)) \;\;&\Leftrightarrow\;\; \exists z \in B'_1(0)\big(g(z) = w\big) \\ &\Leftrightarrow\;\; \exists z^{-1} \in A_{1,\infty}(0)\big(f(z^{-1}) = w\big) \\ &\Leftrightarrow\;\; w \in f(A_{1,\infty}(0)), \end{aligned}$$

and hence $f(A_{1,\infty}(0)) = g(B'_1(0))$. It follows that $f(A_{1,\infty}(0))$ is dense in $\mathbb{C}$, and since $f(A_{1,\infty}(0)) \subseteq f(\mathbb{C})$, we are forced to conclude that $f(\mathbb{C})$ is dense in $\mathbb{C}$ as well, which is a contradiction. Therefore f cannot have an essential singularity at ∞, and since f is entire, by Exercise 7.23 we conclude that f is a polynomial function.

Suppose that f is not a constant function. Fix $w \in \mathbb{C}$, and define $\varphi : \mathbb{C} \to \mathbb{C}$ by $\varphi(z) = f(z) - w$. Then φ is a non constant polynomial function, and by the Fundamental Theorem of Algebra there exists some $z_0 \in \mathbb{C}$ such that $\varphi(z_0) = 0$, and thus $f(z_0) = w$. Since w is arbitrary, it follows that $f(\mathbb{C}) = \mathbb{C}$, which certainly implies that $f(\mathbb{C})$ is dense in $\mathbb{C}$, a contradiction. Therefore f must be constant.[6] ■

[6] The solution given in [AN] employs Liouville's Theorem and turns out to be somewhat slicker.

7.3 – Meromorphic Functions

Given an analytic function $f : \Omega \to \mathbb{C}$, we let $P(f) \subseteq \Omega$ denote the set of complex-valued poles of f, and $\overline{P}(f) \subseteq \Omega \cup \{\infty\}$ denote the set of poles of f on the extended complex plane $\overline{\mathbb{C}}$. Given an arbitrary set S, we also define $P(f, S) = P(f) \cap S$ and $\overline{P}(f, S) = \overline{P}(f) \cap S$. Henceforth we let $\operatorname{ord}(f, z)$ denote the order of either a zero or pole of f at z.

Definition 7.25. *Let Ω be open in $\mathbb{C}$. A function f is **meromorphic on** Ω if f is analytic on $\Omega \setminus P(f, \Omega)$, where $P(f, \Omega)$ has no limit points in Ω.*

*Let Ω be open in $\overline{\mathbb{C}}$. A function f is **meromorphic on** Ω if f is analytic on $\Omega \setminus \overline{P}(f, \Omega)$, where $\overline{P}(f, \Omega)$ has no limit points in Ω.*

In particular a function f is meromorphic on $\overline{\mathbb{C}}$ if it is meromorphic on $\mathbb{C}$ and has either a pole at ∞ or is analytic at ∞ (i.e. has a removable singularity at ∞).

Proposition 7.26. *Let $\Omega \subseteq \mathbb{C}$ be a region. If f and g are meromorphic and not identically zero on Ω, then $f + g$, $f - g$, fg, and f/g have no essential singularities in Ω.*

Proof. The analysis of $f + g$, $f - g$, and fg can be done using Taylor and Laurent series representations and Definition 7.7. Only f/g seems better handled by a different approach involving cases.

Suppose f and g are meromorphic and not identically zero on Ω. Fix $z_0 \in \Omega$. Then

$$L_f = \lim_{z \to z_0} f(z) \quad \text{and} \quad L_g = \lim_{z \to z_0} g(z)$$

exist in $\overline{\mathbb{C}}$ by Theorem 7.10(2). Let $L = \lim_{z \to z_0} f(z)/g(z)$.

If $L_f = L_g = 0$, then f and g are analytic at z_0, and since f and g are not identically zero on any neighborhood of z_0 (otherwise one or the other would be identically zero on Ω by the Identity Theorem), by L'Hôpital's Rule L exists in $\overline{\mathbb{C}}$.

If $L_f = L_g = \infty$, then f and g each have a pole at z_0, hence are each analytic on a deleted neighborhood of z_0, and therefore L exists in $\overline{\mathbb{C}}$ by Exercise 6.44.

If $L_f = \infty$ and $L_g \in \mathbb{C}$, then $L = \infty$; and if $L_f \in \mathbb{C}$ and $L_g = \infty$, then $L = 0$.

Finally, suppose that $L_f, L_g \in \mathbb{C}$. If $L_g \neq 0$, then $L \in \mathbb{C}$; and if $L_g = 0$ and $L_f \neq 0$, then $L = \infty$.

We see that $L \in \overline{\mathbb{C}}$ in all possible cases. Therefore f/g has no essential singularity at z_0 by Theorem 7.10(3). ∎

Proposition 7.27. *If f is meromorphic on $\overline{\mathbb{C}}$, then f has a finite number of poles.*

Proof. Suppose f is meromorphic on $\overline{\mathbb{C}}$, so that f has a pole at ∞ or is analytic at ∞. In either case f has an isolated singularity at ∞, and so there exists some $r > 0$ such that f is analytic on $A_{r,\infty}(0)$. Hence $P(f) \subseteq \overline{B}_r(0)$, and so if $P(f)$ is an infinite set, then since $\overline{B}_r(0)$ is compact it follows that $P(f)$ has a limit point $w \in \overline{B}_r(0)$. But then f has a nonisolated singularity at w since f is not analytic on any deleted neighborhood of w (every deleted neighborhood contains a pole), which contradicts the hypothesis that f is meromorphic on $\overline{\mathbb{C}}$. Therefore $P(f)$ must be a finite set. ∎

Exercise 7.28 (AN4.1.9(d)). Prove that if f is meromorphic on $\overline{\mathbb{C}}$, then f is a rational function.

Solution. By Proposition 7.27 the set $P(f)$ is finite. If $P(f) = \varnothing$, then f is entire with nonessential singularity at ∞, and so we conclude by Exercise 7.23 that f is a polynomial function, and hence also a rational function.

Suppose that $P(f) \neq \varnothing$, so that $P(f) = \{z_1, \dots, z_m\}$ for some $m \in \mathbb{N}$. For each $1 \leq k \leq m$ let n_k be the order of the pole at z_k, so that

$$\lim_{z \to z_k} (z - z_k)^{n_k} f(z) = a_k \in \mathbb{C}_*.$$

Let $\Omega = \mathbb{C} \setminus P(f)$, and define $\varphi : \Omega \to \mathbb{C}$ by

$$\varphi(z) = \prod_{k=1}^{m} (z - z_k)^{n_k} f(z),$$

which is analytic on Ω with nonessential singularity at ∞, seeing as $\lim_{z\to 0} f(1/z)$ exists in $\overline{\mathbb{C}}$, as does

$$\lim_{z \to 0} \left(\frac{1}{z} - z_k \right)$$

for each $1 \leq k \leq m$. Now, φ has a removable singularity at each z_k since

$$b_k := \lim_{z \to z_k} \varphi(z) = \lim_{z \to z_k} \left[\prod_{\ell \neq k} (z - z_\ell)^{n_\ell} \cdot (z - z_k)^{n_k} f(z) \right] = \prod_{\ell \neq k} (z_k - z_\ell)^{n_\ell} \cdot a_k \in \mathbb{C},$$

so if we define $p : \mathbb{C} \to \mathbb{C}$ by

$$p(z) = \begin{cases} \varphi(z), & z \in \Omega \\ b_k, & z = z_k \end{cases}$$

then p is continuous on $\mathbb{C}$ and analytic on Ω, and therefore p is entire by Corollary 4.22. Moreover p has a nonessential singularity at ∞ since

$$\lim_{z \to 0} p(1/z) = \lim_{z \to 0} \varphi(1/z) \in \overline{\mathbb{C}}.$$

(Note: there exists some $r > 0$ such that $A_{r,\infty}(0) \subseteq \Omega$, so $p(z) = \varphi(z)$ for $z \in A_{r,\infty}(0)$, and hence $p(1/z) = \varphi(1/z)$ for $z \in B'_{1/r}(0)$. Now recall that $\varphi(z)$ has a nonessential singularity at ∞ iff $\varphi(1/z)$ has a nonessential singularity at 0 iff $\lim_{z\to 0} \varphi(1/z)$ exists in $\overline{\mathbb{C}}$.) By Exercise 7.23 it follows that p is a polynomial function on $\mathbb{C}$. Now, for $z \in \Omega$,

$$p(z) = \varphi(z) = \prod_{k=1}^{m} (z - z_k)^{n_k} f(z),$$

and since

$$\prod_{k=1}^{m} (z - z_k)^{n_k} \neq 0$$

we obtain

$$f(z) = \frac{p(z)}{\prod_{k=1}^{m} (z - z_k)^{n_k}},$$

which is a rational function on Ω. ■

Theorem 7.29 (**Cauchy's Theorem on Partial Fraction Expansions**)**.** *Let f be meromorphic on $\mathbb{C}$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of simple closed paths, and for each n let Ω_n be the homologically simply connected region for which $\partial\Omega_n = \gamma_n$. Suppose that:*

1. $\operatorname{wn}(\gamma_n, 0) = 1$*,* $\gamma_n^* \cap P(f) = \varnothing$*, and* $\Omega_n \subseteq \Omega_{n+1}$ *for each* n*.*
2. $\lim_{n\to\infty} \operatorname{dist}(0, \gamma_n^*) = \infty$*.*

Let $P(f) = \{b_k : k \in \mathbb{W}\}$ be ordered so $b_0, b_1, \ldots, b_{m_n} \in \Omega_n$ for each $n \in \mathbb{N}$, with $m_n \leq m_{n+1}$. Denote the principal part of the Laurent representation of f at b_k by G_k for each $k \in \mathbb{W}$, and suppose

$$\limsup_{n\to\infty} \oint_{\gamma_n} \frac{|f(z)|}{|z|^{p+1}} |dz| < \infty$$

for some integer $p \geq -1$. If f is analytic at z, then there exists a sequence of polynomials $(P_k)_{k\in\mathbb{W}}$ with $\deg(P_k) \leq p$ such that

$$\left(\sum_{k=0}^{m_n} (G_k + P_k) \right)_{n\in\mathbb{N}}$$

converges uniformly to f on compact subsets of $\mathbb{C} \setminus P(f)$. If $p = -1$, then $P_k = 0$ for all $k \in \mathbb{W}$.

Thus we have

$$f(z) = \lim_{n\to\infty} \sum_{k=0}^{m_n} \big[G_k(z) + P_k(z) \big]$$

pointwise on $\mathbb{C} \setminus P(f)$, so that

$$f(z) = \sum_{k=0}^{\infty} \big[G_k(z) + P_k(z) \big]$$

in the case when the set $P(f)$ is unbounded. In particular

$$f(z) = \sum_{k=0}^{\infty} \frac{\operatorname{res}(f, b_k)}{z - b_k}$$

when

$$\limsup_{n\to\infty} \oint_{\gamma_n} |f(z)||dz| = \limsup_{n\to\infty} \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt < \infty$$

and f has only simple poles, assuming $\gamma_n : [a, b] \to \mathbb{C}$ for each $n \in N$.

7.4 – The Residue Theorem

Definition 7.30. *Let f have an isolated singularity at z_0, so that $f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ in some deleted neighborhood of z_0. The **residue** of f at z_0 is* $\operatorname{res}(f, z_0) = a_{-1}$.

The uniqueness provision in Theorem 7.3 guarantees that the residue of f at an isolated singularity z_0 is well-defined.

Proposition 7.31. *Let f have an isolated singularity at z_0, so that f is analytic on $B'_\rho(z_0)$ for some $\rho > 0$. If γ is a closed path or cycle in $B'_\rho(z_0)$ such that* $\operatorname{wn}(\gamma, z_0) = 1$, *then*

$$\oint_\gamma f = 2\pi i \operatorname{res}(f, z_0).$$

Proof. Suppose that γ is a closed path or cycle in $B'_\rho(z_0)$ such that $\operatorname{wn}(\gamma, z_0) = 1$. Fix $r \in (0, \rho)$, and let $C_r(z_0) \subseteq B'_\rho(z_0)$ be parameterized by $t \mapsto re^{it}$ for $t \in [0, 2\pi]$. Since $\operatorname{wn}(C_r(z_0), z_0) = \operatorname{wn}(\gamma, z_0) = 1$, and $\operatorname{wn}(C_r(z_0), z) = \operatorname{wn}(\gamma, z) = 0$ for all $z \notin B_\rho(z_0)$ by Proposition 6.20(4), it follows that

$$\oint_{C_r(z_0)} f = \oint_\gamma f$$

by Corollary 6.32. On the other hand

$$\oint_{C_r(z_0)} f = 2\pi i a_{-1}$$

obtains by setting $n = -1$ in the equation (7.3) in Theorem 7.3, and therefore

$$\oint_\gamma f = 2\pi i a_{-1} = 2\pi i \operatorname{res}(f, z_0)$$

by Definition 7.30. ∎

Proposition 7.32. *Let f have an isolated singularity at z_0, so that f is analytic on $B'_\rho(z_0)$ for some $\rho > 0$. For fixed $k \in \mathbb{C}$, define $\varphi_k : B'_\rho(z_0) \to \mathbb{C}$ by*

$$\varphi_k(z) = f(z) - \frac{k}{z - z_0}.$$

Then φ_k has a primitive on $B'_\rho(z_0)$ if and only if $k = \operatorname{res}(f, z_0)$.

Proof. Suppose that φ_k has a primitive on $B'_\rho(z_0)$. For $0 < r < \rho$ we have, by Proposition 7.31 and Theorem 6.22,

$$\begin{aligned}
\oint_{C_r(z_0)} \varphi_k &= \oint_{C_r(z_0)} f(z)dz - k\oint_{C_r(z_0)} \frac{1}{z - z_0}dz \\
&= 2\pi i \operatorname{res}(f, z_0) - 2\pi i k \operatorname{wn}(C_r(z_0), z_0) \\
&= 2\pi i \operatorname{res}(f, z_0) - 2\pi i k.
\end{aligned}$$

On the other hand, since φ_k is continuous and has a primitive on $B'_\rho(z_0)$, the Fundamental Theorem of Path Integrals implies that

$$\oint_{C_r(z_0)} \varphi_k = 0.$$

Therefore $2\pi i \operatorname{res}(f, z_0) - 2\pi i k = 0$, giving $k = \operatorname{res}(f, z_0)$.

For the converse, suppose that $k = \operatorname{res}(f, z_0)$. Let

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

for all $z \in B'_\rho(z_0)$. Then, since $k = a_{-1}$,

$$\varphi_k(z) = f(z) - \frac{k}{z - z_0} = \sum_{n \in \mathbb{Z}\setminus\{-1\}} a_n (z - z_0)^n$$

for all $z \in B'_\rho(z_0)$, where the Laurent series is absolutely convergent on $B'_\rho(z_0)$ by Theorem 7.3. It follows that the series

$$\Phi_k(z) = \sum_{n \in \mathbb{Z}\setminus\{-1\}} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

is likewise absolutely convergent on $B'_\rho(z_0)$, and so

$$\Phi'_k(z) = \sum_{n \in \mathbb{Z}\setminus\{-1\}} \frac{d}{dz}\left[\frac{a_n}{n+1}(z - z_0)^{n+1}\right] = \sum_{n \in \mathbb{Z}\setminus\{-1\}} a_n (z - z_0)^n = \varphi_k(z)$$

by Theorem 7.4. That is, Φ_k is a primitive for φ_k on $B'_\rho(z_0)$. ■

Proposition 7.33. *Suppose f has a pole of order m at z_0. Then*

$$\operatorname{res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right),$$

and $\operatorname{res}(f'/f, z_0) = -m$.

Proof. Suppose that f has a pole of order m at z_0. Thus there exists some $r > 0$ such that f is analytic on $B'_r(z_0)$ and

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \alpha \in \mathbb{C}_*.$$

Define $\varphi : B_r(z_0) \to \mathbb{C}$ by

$$\varphi(z) = \begin{cases} (z - z_0)^m f(z), & z \in B'_r(z_0) \\ \alpha, & z = z_0 \end{cases}$$

Since φ is analytic on $B_r(z_0)$ by Corollary 4.22, it follows by Theorem 4.29 that φ has a Taylor series representation

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!} (z - z_0)^n$$

on $B_r(z_0)$. On the other hand, by Theorem 7.3 and Definition 7.7(2), f has a Laurent series representation

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^{n-m} \tag{7.17}$$

on $B'_r(z_0)$, where $a_{-m} \neq 0$. Now,

$$(z-z_0)^m f(z) = \varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n$$

for all $z \in B'_r(z_0)$, and so

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}. \tag{7.18}$$

for all $z \in B'_r(z_0)$. Comparing (7.17) and (7.18) and recalling that the Laurent series of an analytic function on a punctured disc is unique, we conclude that

$$\operatorname{res}(f, z_0) = a_{-1} = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!},$$

and therefore

$$\operatorname{res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \varphi^{(m-1)}(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[(z-z_0)^m f(z)\right]^{(m-1)}$$

since the analyticity of φ on $B_r(z_0)$ implies the continuity of $\varphi^{(m-1)}$ on $B_r(z_0)$.

Next, for $z \in B'_r(z_0)$ we have $f(z) = (z-z_0)^{-m}\varphi(z)$, and so

$$(f'/f)(z) = \frac{(z-z_0)^{-m}\varphi'(z) - m(z-z_0)^{-m-1}\varphi(z)}{(z-z_0)^{-m}\varphi(z)} = \frac{\varphi'(z)}{\varphi(z)} - \frac{m}{z-z_0}.$$

Since $\varphi(z_0) \neq 0$ and φ is analytic on $B_r(z_0)$, there exists some $0 < \rho \leq r$ such that $\varphi(z) \neq 0$ for all $z \in B_\rho(z_0)$, so that φ'/φ is analytic on $B_\rho(z_0)$ and by Theorem 4.29

$$(\varphi'/\varphi)(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

for all $z \in B_\rho(z_0)$. In particular $\operatorname{res}(\varphi'/\varphi, z_0) = b_{-1} = 0$, and since

$$(f'/f)(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n - m(z-z_0)^{-1}$$

for all $z \in B'_\rho(z_0)$, we conclude that $\operatorname{res}(f'/f, z_0) = -m$. ∎

If f has a simple pole (i.e. a pole of order 1) at z_0, then

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} (z-z_0)f(z)$$

in particular.

Proposition 7.34. *If f has a zero of order m at z_0, then f'/f has a simple pole at z_0 and $\operatorname{res}(f'/f, z_0) = m$.*

Proof. Suppose that f has a zero of order m at z_0. Then $f(z_0) = 0$, there exists some $r > 0$ such that f is analytic on $B_r(z_0)$, and there exists some analytic function $g : B_r(z_0) \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in B_r(z_0)$. By the continuity of g there is some $0 < \rho \leq r$ such that $g(z) \neq 0$ for all $z \in B_\rho(z_0)$, and hence $f(z) \neq 0$ for all $z \in B'_\rho(z_0)$. This implies that f'/f is analytic on $B'_\rho(z_0)$ and has an isolated singularity at z_0. Indeed, since

$$\begin{aligned}\lim_{z\to z_0}(z - z_0)(f'/f)(z) &= \lim_{z\to z_0}\left[(z - z_0) \cdot \frac{(z - z_0)^m g'(z) + m(z - z_0)^{m-1} g(z)}{(z - z_0)^m g(z)}\right] \\ &= \lim_{z\to z_0} \frac{(z - z_0)g'(z) + mg(z)}{g(z)} = m \cdot \frac{g(z_0)}{g(z_0)} = m,\end{aligned}$$

where $m \neq 0$ since $m \in \mathbb{N}$, it follows that f'/f has a pole of order 1 at z_0. By Proposition 7.33

$$\operatorname{res}(f'/f, z_0) = \frac{1}{0!} \lim_{z\to z_0} \left[(z - z_0)(f'/f)(z)\right]^{(0)} = \lim_{z\to z_0}(z - z_0)(f'/f)(z),$$

and therefore $\operatorname{res}(f'/f, z_0) = m$. ■

Theorem 7.35 (**Residue Theorem**). *Let f be analytic on $\Omega \setminus S(f)$, where Ω is some open set and $S(f) \subseteq \Omega$ consists of isolated singularities of f. If γ is a closed cycle in $\Omega \setminus S(f)$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$, then*

$$\oint_\gamma f = 2\pi i \sum_{z\in S(f)} \operatorname{res}(f, z) \operatorname{wn}(\gamma, z).$$

Theorem 7.36 (**Argument Principle**). *Let f be analytic on Ω, with f not identically zero on any component of Ω. If γ is a closed path in $\Omega \setminus Z(f)$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$, then*

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z\in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z).$$

Proof. Suppose γ is a closed path in $\Omega \setminus Z(f)$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$. Since f is analytic on $\Omega \setminus Z(f)$, $\gamma^* \subseteq \Omega \setminus Z(f)$, and $0 \notin (f \circ \gamma)^*$, Corollary 6.23 implies that

$$\operatorname{wn}(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_\gamma f'/f. \tag{7.19}$$

Now, f'/f is analytic on $\Omega \setminus Z(f)$. The set $Z(f)$ has no limit point in Ω, for if it did, then by the Identity Theorem f must be identically zero on the component of Ω containing the limit point. Thus, for each $z \in Z(f)$, there exists some $\epsilon > 0$ such that $B'_\epsilon(z) \subseteq \Omega$ and $B'_\epsilon(z) \cap Z(f) = \varnothing$, and we see that $Z(f)$ consists of isolated singularities of f'/f. Therefore, by the Residue Theorem,

$$\oint_\gamma f'/f = 2\pi i \sum_{z\in Z(f)} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z), \tag{7.20}$$

and combining (7.19) and (7.20) yields

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z\in Z(f)} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z).$$

Finally, fix $z \in Z(f)$, and let Ω' be the component of Ω that contains z. Since f is analytic and not identically zero on the open connected set Ω', $z \in \Omega'$, and $f(z) = 0$, it follows by Proposition 5.15 that $\operatorname{ord}(f, z) = m$ for some $m \in \mathbb{N}$, and hence $\operatorname{res}(f'/f, z) = m = \operatorname{ord}(f, z)$ by Proposition 7.34. Therefore

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z \in Z(f)} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z)$$

as was to be shown. ∎

Theorem 7.37. *Let f be meromorphic on Ω, with f not identically zero on any component of Ω. If γ is a closed path in $\Omega \setminus (Z(f) \cup P(f))$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$, then*

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z) - \sum_{z \in P(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z).$$

Proof. Let $S = Z(f) \cup P(f)$ and suppose γ is a closed path in $\Omega \setminus S$ such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$. Since f is analytic on $\Omega \setminus S$, $\gamma^* \subseteq \Omega \setminus S$, and $0 \notin (f \circ \gamma)^*$, Corollary 6.23 implies that

$$\operatorname{wn}(f \circ \gamma, 0) = \frac{1}{2\pi i} \oint_\gamma f'/f. \tag{7.21}$$

Now, f'/f is analytic on $\Omega \setminus S$. As shown in the proof of the Argument Principle, $Z(f)$ has no limit point in Ω, and also $P(f)$ has no limit point in Ω since poles are a type of isolated singularity. Hence S has no limit point in Ω and so is precisely the set of isolated singularities of f'/f. By the Residue Theorem,

$$\oint_\gamma f'/f = 2\pi i \sum_{z \in S} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z), \tag{7.22}$$

and combining (7.21) and (7.22) yields

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z \in S} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z).$$

If $z \in Z(f)$, then $\operatorname{ord}(f, z) \in \mathbb{N}$ by Proposition 5.15, and $\operatorname{res}(f'/f, z) = \operatorname{ord}(f, z)$ by Proposition 7.34. If $z \in P(f)$, then $\operatorname{res}(f'/f, z) = -\operatorname{ord}(f, z)$ by Proposition 7.33. From

$$\operatorname{wn}(f \circ \gamma, 0) = \sum_{z \in Z(f)} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z) + \sum_{z \in P(f)} \operatorname{res}(f'/f, z) \operatorname{wn}(\gamma, z)$$

the desired result now readily obtains. ∎

Theorem 7.38 (**Rouché's Theorem**)**.** *Suppose f and g are analytic on Ω, with neither f nor g identically zero on any component of Ω. Let γ be a closed path in Ω such that $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \Omega$. If $|f + g| < |f| + |g|$ on γ^*, then*

$$\sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(g)} \operatorname{ord}(g, z) \operatorname{wn}(\gamma, z).$$

Remark. Note that if $|f+g| < |f| + |g|$ on γ^*, then neither f nor g can have any zeros on γ^*. If the closed path $\gamma : [a,b] \to \mathbb{C}$ in Rouché's Theorem is a simple curve—meaning $\gamma(t_1) \neq \gamma(t_2)$ for all $t_1, t_2 \in (a,b)$ such that $t_1 \neq t_2$, and hence the curve does not cross itself—then there is a clearly defined region $S \subseteq \Omega$ such that $\partial S = \gamma^*$, and moreover $\operatorname{wn}(\gamma, z) \in \{0,1\}$ for all $z \in Z(f) \cup Z(g)$. Thus, in the case when γ is a simple closed path, Rouché's Theorem may be accurately interpreted as saying that f and g have the same number of zeros in the region bounded by γ, counting multiplicities. Many authors insert the requirement that γ be simple in the statement of Rouché's Theorem. Also there is a version of the theorem adapted for meromorphic functions.[7]

Exercise 7.39 (AN4.2.1). Let

$$f(z) = \frac{(z-1)(z-3+4i)}{(z+2i)^2},$$

and let $\gamma : [a,b] \to \mathbb{C}$ be the path shown in Figure 17. Find $\operatorname{wn}(f \circ \gamma, 0)$.

Solution. We see that f is meromorphic on $\mathbb{C}$, with a single pole of order 2 at $-2i$, and zeros of order 1 at 1 and $3-4i$. Since $\gamma^* \subseteq \mathbb{C} \setminus [Z(f) \cup P(f)]$, Theorem 7.37 yields

$$\begin{aligned} \operatorname{wn}(f \circ \gamma, 0) &= \operatorname{ord}(f,1)\operatorname{wn}(\gamma,1) + \operatorname{ord}(f,3-4i)\operatorname{wn}(\gamma,3-4i) - \operatorname{ord}(f,-2i)\operatorname{wn}(\gamma,-2i) \\ &= (1)(1) + (1)(0) - (2)(1) = -1. \end{aligned}$$

There is an alternate method for computing $\operatorname{wn}(f \circ \gamma, 0)$. Define $\gamma_1 = \gamma - 1$, $\gamma_2 = \gamma - 3 + 4i$, and $\gamma_3 = \gamma + 2i$. Since $0 \notin \gamma_1^* \cup \gamma_2^* \cup \gamma_3^*$, we have $0 \notin (\gamma_1\gamma_2)^* \cup (\gamma_3^2)^*$, where $\gamma_1\gamma_2$ and γ_3^2 are closed curves. For any $t \in [a,b]$,

$$(f \circ \gamma)(t) = \frac{[\gamma(t)-1][\gamma(t)-3+4i]}{[\gamma(t)+2i]^2} = \frac{\gamma_1(t)\gamma_2(t)}{\gamma_3^2(t)} = (\gamma_1\gamma_2/\gamma_3^2)(t),$$

so that $f \circ \gamma = \gamma_1\gamma_2/\gamma_3^2$. By Proposition 6.20(3),

$$\operatorname{wn}(f \circ \gamma, 0) = \operatorname{wn}(\gamma_1\gamma_2/\gamma_3^2, 0) = \operatorname{wn}(\gamma_1\gamma_2, 0) - \operatorname{wn}(\gamma_3^2, 0) = \operatorname{wn}(\gamma_1, 0) + \operatorname{wn}(\gamma_2, 0) - 2\operatorname{wn}(\gamma_3, 0),$$

and since $-2i, 1, 3-4i \notin \gamma^*$, Proposition 6.20(2) implies that

$$\operatorname{wn}(\gamma_1, 0) = \operatorname{wn}(\gamma - 1, 0) = \operatorname{wn}(\gamma, 1), \quad \operatorname{wn}(\gamma_2, 0) = \operatorname{wn}(\gamma - 3 + 4i, 0) = \operatorname{wn}(\gamma, 3-4i),$$

and

$$\operatorname{wn}(\gamma_3, 0) = \operatorname{wn}(\gamma + 2i, 0) = \operatorname{wn}(\gamma, -2i),$$

and so

$$\operatorname{wn}(f \circ \gamma, 0) = \operatorname{wn}(\gamma, 1) + \operatorname{wn}(\gamma, 3-4i) - 2\operatorname{wn}(\gamma, -2i) = 1 + 0 - 2(1) = -1$$

obtains once more. ■

Exercise 7.40 (AN4.2.3). Prove that any polynomial function of degree $n \geq 1$ has exactly n zeros, counting multiplicities.

[7]See Conway's *Functions of One Complex Variable I.*

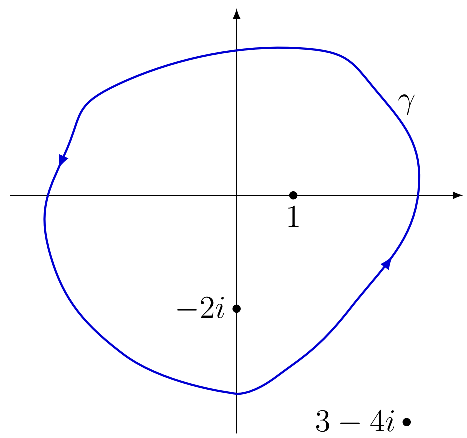

FIGURE 17.

Solution. Let f be a polynomial function of degree $n \geq 1$, so f is entire with

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

for all $z \in \mathbb{C}$, where $a_n \neq 0$. Define g by $g(z) = -a_n z^n$ for all $z \in \mathbb{C}$, also an entire function. Now, since

$$\frac{1}{|a_n|} \cdot \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \leq \frac{1}{|a_n|} \cdot \left(\frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right),$$

for $z \neq 0$, it is straightforward to show that

$$\begin{aligned} \lim_{z \to \infty} \frac{|f(z) + g(z)|}{|g(z)|} &= \lim_{z \to \infty} \frac{|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|}{|a_n z^n|} \\ &= \lim_{z \to \infty} \left(\frac{1}{|a_n|} \cdot \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \right) = 0, \end{aligned}$$

and so there exists some $r > 0$ such that

$$\frac{|f(z) + g(z)|}{|g(z)|} < 1,$$

and hence $|f(z) + g(z)| < |g(z)|$, for all $z \in A_{r,\infty}(0)$. If γ is any closed path in $A_{r,\infty}(0)$ such that $\operatorname{wn}(\gamma, 0) = 1$. Then

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

for all $z \in \gamma^*$. By Rouché's Theorem, since g has just a single zero of order n at 0,

$$\sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(g)} \operatorname{ord}(g, z) \operatorname{wn}(\gamma, z) = \operatorname{ord}(g, 0) \operatorname{wn}(\gamma, 0) = (n)(1) = n,$$

and therefore f has n zeros, counting multiplicities. ■

Exercise 7.41 (AN4.2.6a). Find the residue at $z = 0$ of $\csc^2 z$.

Solution. Applying L'Hôpital's Rule twice,

$$\lim_{z\to 0} z^2 \csc^2 z = \lim_{z\to 0} \frac{z^2}{\sin^2 z} = \lim_{z\to 0} \frac{2z}{2\sin z \cos z} = \lim_{z\to 0} \frac{1}{\cos^2 z - \sin^2 z} = 1,$$

and so $\csc^2 z$ has a pole of order 2 at $z = 0$ by Theorem 7.10(1). Thus, by Proposition 7.33,

$$\operatorname{res}(\csc^2, 0) = \lim_{z\to 0} \frac{d}{dz}(z^2 \csc^2 z) = \lim_{z\to 0} \frac{2z - 2z^2 \cot z}{\sin^2 z} = 0,$$

which may be obtained by applying L'Hôpital's Rule thrice. ■

Exercise 7.42 (AN4.2.6c). Find the residue at $z = 0$ of $z\cos(1/z)$.

Solution. In Exercise 7.16 it was found that $z\cos(1/z)$ has an essential singularity at $z = 0$, and thus Proposition 7.33 may not be used to find the residue as in previous exercise. Instead we take a direct approach:

$$z\cos(1/z) = z\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!z^{2n-1}} = z - \frac{1}{2}z^{-1} + \frac{1}{24}z^{-3} - \cdots .$$

Therefore $\operatorname{res}(z\cos(1/z), 0) = -1/2$. ■

Exercise 7.43 (AN4.2.10). Prove that all zeros of $f(z) = z^4 + 6z + 3$ are in $B_2(0)$, and three of them are in $A_{1,2}(0)$.

Solution. Let $g(z) = -z^4$, and define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = 2e^{it}$. For $z \in \gamma^*$ we have $|z| = 2$, so that

$$|f(z) + g(z)| = |6z + 3| \leq |6z| + 3 = 15 < 16 = 2^4 = |z^4| = |g(z)|,$$

and hence $|f + g| < |f| + |g|$ on γ^*. Observing that f and g are analytic and not identically zero on $\mathbb{C}$, and γ is a closed path in $\mathbb{C}$, by Rouché's Theorem

$$\sum_{z\in Z(f)} \operatorname{ord}(f, z)\operatorname{wn}(\gamma, z) = \sum_{z\in Z(g)} \operatorname{ord}(g, z)\operatorname{wn}(\gamma, z) = \operatorname{ord}(g, 0)\operatorname{wn}(\gamma, 0) = (4)(1) = 4,$$

and therefore f has four zeros in $B_2(0)$, counting multiplicities. Since f has exactly four zeros in all by Exercise 7.40, we conclude that all zeros of f are in $B_2(0)$.

Next, let $h(z) = -6z - 3$, and define $\xi : [0, 2\pi] \to \mathbb{C}$ by $\xi(t) = e^{it}$. For $z \in \xi^*$ we have $|z| = 1$, so that

$$|f(z) + h(z)| = |z^4| = |z|^4 = 1 < 3 = \big||6z| - 3\big| \leq |6z + 3| = |h(z)|,$$

and hence $|f + h| < |f| + |h|$ on ξ^*. By Rouché's Theorem

$$\sum_{z\in Z(f)} \operatorname{ord}(f, z)\operatorname{wn}(\gamma, z) = \sum_{z\in Z(h)} \operatorname{ord}(h, z)\operatorname{wn}(\gamma, z) = \operatorname{ord}(h, -1/2)\operatorname{wn}(\gamma, -1/2) = (1)(1) = 1,$$

and therefore f has one zero in $\mathbb{B}$.

It remains to verify that f cannot have a zero on $\partial\mathbb{B}$. For any $z \in \partial\mathbb{B}$ we have $z = e^{it}$ for some $t \in [0, 2\pi]$, so that

$$|f(z)| = |e^{i4t} + 6e^{it} + 3| \geq \big||6e^{it} + 3| - |e^{i4t}|\big| = \big|3|2e^{it} + 1| - 1\big|$$

$$\geq 3|2e^{it}+1|-1 \geq 3\big||2e^{it}|-1\big|-1 = 3(2-1)-1 = 2,$$

which shows that $f(z) \neq 0$. Hence f has exactly one zero in $\overline{\mathbb{B}}$ and four zeros in $B_2(0)$, which implies that f has three zeros in $B_2(0) \setminus \overline{\mathbb{B}} = A_{1,2}(0)$. ■

Exercise 7.44 (AN4.2.11). Suppose f is analytic on an open set $\Omega \supset \overline{\mathbb{B}}$, and $|f(z)| < 1$ for all $z \in \partial\mathbb{B}$. Show that for each $n \in \mathbb{N}$, the function $\varphi(z) = f(z) - z^n$ has exactly n zeros in $\mathbb{B}$, counting multiplicities. In particular, f has exactly one fixed point in $\mathbb{B}$.

Solution. Fix $n \in \mathbb{N}$, let $\psi(z) = z^n$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$. For each $z \in \gamma^*$,

$$|\varphi(z) + \psi(z)| = |f(z)| < 1 = |z|^n = |z^n| = |\psi(z)|,$$

and so $|\varphi + \psi| < |\varphi| + |\psi|$ on γ^*. Discounting any components of Ω that are disjoint from $\overline{\mathbb{B}}$, we can assume that Ω is connected. It is clear that φ and ψ are analytic on Ω, and ψ is not identically zero on Ω. Moreover, $|f| < 1$ on $\partial\mathbb{B}$ implies that $\varphi \neq 0$ on $\partial\mathbb{B}$, and thus φ is also not identically zero on Ω. Since γ is a closed path in Ω and $\mathrm{wn}(\gamma, z) = 0$ for any $z \notin \Omega$ (since $\overline{\mathbb{B}} \subseteq \Omega$), by Rouché's Theorem

$$\sum_{z \in Z(\varphi)} \mathrm{ord}(\varphi, z)\,\mathrm{wn}(\gamma, z) = \sum_{z \in Z(\psi)} \mathrm{ord}(\psi, z)\,\mathrm{wn}(\gamma, z) = \mathrm{ord}(\psi, 0)\,\mathrm{wn}(\gamma, 0) = (n)(1) = n.$$

Thus φ has exactly n zeros in $\mathbb{B}$, counting multiplicities. In the case when $n = 1$, we conclude that $\varphi(z) = f(z) - z$ has exactly one zero in $\mathbb{B}$; that is, there exists exactly one $z_0 \in \mathbb{B}$ such that $\varphi(z_0) = 0$, which is to say there exists exactly one $z_0 \in \mathbb{B}$ such that $f(z_0) = z_0$. ■

Exercise 7.45 (AN4.2.16). Show that the equation $3z = e^{-z}$ has exactly one root in $\mathbb{B}$.

Solution. Let $f(z) = 3z - e^{-z}$, and let $g(z) = -3z$. If $z \in \partial\mathbb{B}$, then $|z| = 1$ and $\mathrm{Re}(z) \in [-1, 1]$, and so

$$|f(z) + g(z)| = |-e^{-z}| = \frac{1}{|e^z|} = \frac{1}{e^{\mathrm{Re}(z)}} \leq e < 3 = |-3z| = |g(z)|.$$

Thus, if $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = e^{it}$, then $|f + g| < |f| + |g|$ on γ^*. Since f and g are analytic on $\mathbb{C}$, by Rouché's Theorem

$$\sum_{z \in Z(f)} \mathrm{ord}(f, z)\,\mathrm{wn}(\gamma, z) = \sum_{z \in Z(g)} \mathrm{ord}(g, z)\,\mathrm{wn}(\gamma, z) = \mathrm{ord}(g, 0)\,\mathrm{wn}(\gamma, 0) = (1)(1) = 1.$$

Therefore f has exactly one zero in $\mathbb{B}$, which implies that $3z = e^{-z}$ has exactly one root in $\mathbb{B}$. ■

Exercise 7.46 (AN4.2.17). Let f be analytic on $\mathbb{B}$ with $f(0) = 0$. Show that if

$$\min_{z \in C_r(0)} |f(z)| \geq \epsilon$$

for some $\epsilon > 0$ and $0 < r < 1$, then $B_\epsilon(0) \subseteq f(B_r(0))$.

Solution. Since $|f(z)| \geq \epsilon$ for all $z \in C_r(0)$, it is clear that f is not identically zero on $\mathbb{B}$. Let $z_0 \in B_\epsilon(0)$, and define the analytic function $g : \mathbb{B} \to \mathbb{C}$ by $g(z) = z_0 - f(z)$, which is also not identically zero on $\mathbb{B}$. Also define the path $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = re^{it}$, so that $\gamma^* = C_r(0)$ and $\mathrm{wn}(\gamma, 0) = 1$. Since

$$|f(z) + g(z)| = |z_0| < \epsilon \leq \min_{w \in C_r(0)} |f(w)| \leq |f(z)| \leq |f(z)| + |g(z)|$$

for every $z \in \gamma^*$, by Rouché's Theorem

$$\sum_{z \in Z(g)} \mathrm{ord}(g, z) \,\mathrm{wn}(\gamma, z) = \sum_{z \in Z(f)} \mathrm{ord}(f, z) \,\mathrm{wn}(\gamma, z) \geq \mathrm{ord}(f, 0) \,\mathrm{wn}(\gamma, 0) \geq 1.$$

Thus there exists some $w \in Z(g)$ such that $\mathrm{ord}(g, w) \,\mathrm{wn}(\gamma, w) \geq 1$, which implies $\mathrm{ord}(g, w) \geq 1$ and $\mathrm{wn}(\gamma, w) = 1$. Thus $w \in B_r(0)$ is such that $g(w) = 0$, giving $f(w) = z_0$ and hence $z_0 \in f(B_r(0))$. Since $z_0 \in B_\epsilon(0)$ is arbitrary, it follows that $B_\epsilon(0) \subseteq f(B_r(0))$. ■

Exercise 7.47 (AN4.2.20). Prove the equation $e^z - 3z^7 = 0$ has seven roots in $\mathbb{B}$. More generally, show that if $|a| > e$ and $n \in \mathbb{N}$, then $e^z - az^n = 0$ has exactly n roots in $\mathbb{B}$.

Solution. Define $f(z) = e^z - 3z^7$ and $g(z) = 3z^7$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$. Since $\mathrm{Re}(z) \in [-1, 1]$ and $|z| = 1$ for any $z \in \gamma^*$,

$$|f(z) + g(z)| = |e^z| = e^{\mathrm{Re}(z)} \leq e < 3 = 3|z|^7 = |3z^7| = |g(z)| \leq |g(z)| + |f(z)|,$$

and so $|f + g| < |f| + |g|$ on γ^*. Observing that f and g are analytic on $\mathbb{C}$, and neither f nor g is identically zero on $\mathbb{C}$, by Rouché's Theorem

$$\sum_{z \in Z(f)} \mathrm{ord}(f, z) \,\mathrm{wn}(\gamma, z) = \sum_{z \in Z(g)} \mathrm{ord}(g, z) \,\mathrm{wn}(\gamma, z) = \mathrm{ord}(g, 0) \,\mathrm{wn}(\gamma, 0) = (7)(1) = 7,$$

which implies that f has exactly seven zeros in $\mathbb{B}$, and therefore $e^z - 3z^7 = 0$ has exactly seven roots in $\mathbb{B}$.

Now suppose that $|a| > e$, $n \in \mathbb{N}$, $f(z) = e^z - az^n$, and $g(z) = az^n$. Define the path γ as before. For any $z \in \gamma^* = \partial\mathbb{B}$,

$$|f(z) + g(z)| = |e^z| = e^{\mathrm{Re}(z)} \leq e < |a| = |a||z|^n = |az^n| = |g(z)| \leq |g(z)| + |f(z)|,$$

and so $|f + g| < |f| + |g|$ on γ^*. by Rouché's Theorem

$$\sum_{z \in Z(f)} \mathrm{ord}(f, z) \,\mathrm{wn}(\gamma, z) = \sum_{z \in Z(g)} \mathrm{ord}(g, z) \,\mathrm{wn}(\gamma, z) = \mathrm{ord}(g, 0) \,\mathrm{wn}(\gamma, 0) = (n)(1) = n,$$

which implies that f has exactly n zeros in $\mathbb{B}$, and therefore $e^z - az^n = 0$ has exactly n roots in $\mathbb{B}$. ■

Exercise 7.48 (AN4.2.22). Show that $f(z) = z^7 - 5z^4 + z^2 - 2$ has exactly 4 zeros in $\mathbb{B}$.

Solution. Let $g(z) = -z^7 + 5z^4$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$. For any $z \in \gamma^* = \partial\mathbb{B}$ we have $|z| = 1$, so

$$|f(z) + g(z)| = |z^2 - 2| \leq |z|^2 + 2 = 3 < 4 = \big||z|^3 - 5\big|$$

$$\le |z^3 - 5| = |z|^4|z^3 - 5| = |z^7 - 5z^4| = |g(z)|,$$

and hence $|f + g| < |f| + |g|$ on γ^*. Since f and g are analytic on $\mathbb{C}$, by Rouché's Theorem

$$\sum_{z\in Z(f)} \operatorname{ord}(f, z)\operatorname{wn}(\gamma, z) = \sum_{z\in Z(g)} \operatorname{ord}(g, z)\operatorname{wn}(\gamma, z) = \operatorname{ord}(g, 0)\operatorname{wn}(\gamma, 0) = (4)(1) = 4.$$

(Since $g(z) = z^4(5 - z^3)$, any zero z_0 of g besides 0 will be such that $|z_0| = \sqrt[3]{5} > 1$, and hence $\operatorname{wn}(\gamma, z_0) = 0$.) Therefore f has exactly 4 zeros in $\mathbb{B}$, counting multiplicities. ■

Exercise 7.49 (AN4.2.25). Let Ω be an open connected set, and let $(f_n : \Omega \to \mathbb{C})_{n\in\mathbb{N}}$ be a sequence of analytic functions that converges uniformly to $f : \Omega \to \mathbb{C}$ on compact subsets of Ω. Assume that f is not identically zero, and let $z_0 \in \Omega$. Prove that $f(z_0) = 0$ if and only if there exists a sequence $(z_n)_{n\in\mathbb{N}}$ and subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $z_n \to z_0$ and $f_{n_k}(z_k) = 0$ for all k.

Solution. Suppose that $f(z_0) = 0$. Let $r > 0$ be such that $\overline{B}_r(z_0) \subseteq \Omega$. Since $Z(f)$ has no limit points in Ω by the Identity Theorem, there exists some $0 < \epsilon < r/2$ such that $B'_{2\epsilon}(z_0)\cap Z(f) = \varnothing$. For each $k \in \mathbb{N}$ let $\gamma_k : [0, 2\pi] \to \Omega$ be given by $\gamma_k(t) = \epsilon k^{-1}e^{it}$, so that

$$\gamma_k^* = C_{\epsilon/k}(z_0) \subseteq B'_{2\epsilon}(z_0) \subseteq \overline{B}_r(z_0) \subseteq \Omega$$

and $\operatorname{wn}(\gamma_k, z_0) = 1$. Since $f(z) \neq 0$ for all $z \in \gamma_k^*$ and f is continuous on the compact set γ_k^*,

$$m_k = \min_{z\in\gamma_k^*} |f(z)| > 0.$$

Now, $f_n \xrightarrow{\text{u}} f$ on γ_k^*, and so there exists some $n_k \in \mathbb{N}$ such that $|f_n(z) - f(z)| < m_k$ for all $n \ge n_k$ and $z \in \gamma_k^*$. In particular

$$\big|f(z) + [-f_{n_k}(z)]\big| = |f(z) - f_{n_k}(z)| < m_k \le |f(z)| \le |f(z)| + |-f_{n_k}(z)|$$

for all $z \in \gamma_k^*$. Since $f_n \to f$ pointwise on Ω and f is not identically zero, we can assume n_k is sufficiently large that f_{n_k} is also not identically zero. Therefore by Rouché's Theorem,

$$\sum_{z\in Z(-f_{n_k})} \operatorname{ord}(f_{n_k}, z)\operatorname{wn}(\gamma_k, z) = \sum_{z\in Z(f)} \operatorname{ord}(f, z)\operatorname{wn}(\gamma_k, z) \ge \operatorname{ord}(f, z_0)\operatorname{wn}(\gamma_k, z_0) = \operatorname{ord}(f, z_0) \ge 1,$$

which implies that $-f_{n_k}$, and hence f_{n_k}, has at least one zero $z_k \in B_{\epsilon/k}(z_0)$. In this way we construct a sequence $(z_n)_{n\in\mathbb{N}}$ and subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $z_n \to z_0$ and $f_{n_k}(z_k) = 0$ for all $k \in \mathbb{N}$.

Conversely, suppose there's a sequence $(z_n)_{n\in\mathbb{N}}$ and subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $z_n \to z_0$ and $f_{n_k}(z_k) = 0$ for all k. Let $\epsilon > 0$. The function f is continuous at z_0 by Theorem 2.54, so there exists some $\delta > 0$ such that $|z - z_0| \le \delta$ implies $|f(z) - f(z_0)| < \epsilon/2$. Let $S = \overline{B}_\delta(z_0)$, which is a compact subset of Ω. Since $f_{n_k} \xrightarrow{\text{u}} f$ on S, there exists some $k_0 \in \mathbb{N}$ such that

$$|f_{n_k}(z) - f(z)| < \epsilon/2$$

for all $k \ge k_0$ and $z \in S$. Since $z_k \to z_0$, there exists some $k_1 \in \mathbb{N}$ such that $k \ge k_1$ implies $|z_k - z_0| < \delta$. Suppose that $k \ge \max\{k_0, k_1\}$. Then $|z_k - z_0| < \delta$, so that $z_k \in S$, and we obtain

$$|f(z_k) - f(z_0)| < \epsilon/2 \quad \text{and} \quad |f_{n_k}(z_k) - f(z_k)| < \epsilon/2,$$

and thus by the Triangle Inequality,

$$|f_{n_k}(z_k) - f(z_0)| < \epsilon.$$

However, $f_{n_k}(z_k) = 0$, which implies that $|f(z_0)| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $f(z_0) = 0$. ■

7.5 – Improper Integrals

In this section we employ residue theory to compute various kinds of proper and improper Riemann integrals of both real- and complex-valued functions. If $\varphi : [a, \infty) \to \mathbb{C}$, we define

$$\int_a^\infty \varphi = \int_a^\infty \operatorname{Re}\varphi + i\int_a^\infty \operatorname{Im}\varphi, \tag{7.23}$$

provided the integrals at right exist (i.e. converge) in $\mathbb{R}$. Similarly, if $\varphi : (-\infty, b] \to \mathbb{C}$, we define

$$\int_{-\infty}^b \varphi = \int_{-\infty}^b \operatorname{Re}\varphi + i\int_{-\infty}^b \operatorname{Im}\varphi, \tag{7.24}$$

provided the integrals at right exist in $\mathbb{R}$. Finally, if $\varphi : \mathbb{R} \to \mathbb{C}$, we define

$$\int_{-\infty}^\infty \varphi = \int_{-\infty}^\infty \operatorname{Re}\varphi + i\int_{-\infty}^\infty \operatorname{Im}\varphi. \tag{7.25}$$

The improper Riemann integrals on the right-hand sides of equations (7.23), (7.24), and (7.25) are taken to be either of the "first kind" or "mixed," as defined in §8.7 of the *Calculus Notes*. Similar definitions are made for integrals of complex-valued functions that are improper integrals of the "second kind."

As the next two propositions show, many of the properties of real-valued improper integrals known from calculus and elementary analysis carry over to complex-valued improper integrals.

Proposition 7.50. *Let $s, t \in \mathbb{R}$.*

1. *Suppose $\varphi : [a, \infty) \to \mathbb{C}$. Then*

$$\lim_{r\to\infty}\int_a^r \varphi = s + it \quad \text{iff} \quad \int_a^\infty \operatorname{Re}\varphi = s \text{ and } \int_a^\infty \operatorname{Im}\varphi = t.$$

2. *Suppose $\varphi : (-\infty, b] \to \mathbb{C}$. Then*

$$\lim_{r\to-\infty}\int_r^b \varphi = s + it \quad \text{iff} \quad \int_{-\infty}^b \operatorname{Re}\varphi = s \text{ and } \int_{-\infty}^b \operatorname{Im}\varphi = t.$$

Proof. We prove only the first part, for the proof of the second part is similar. Suppose $\lim_{r\to\infty}\int_a^r \varphi = s + it$, so

$$\lim_{r\to\infty}\left(\int_a^r \operatorname{Re}\varphi + i\int_a^r \operatorname{Im}\varphi\right) = s + it$$

by the definition of $\int_a^r \varphi$. Then

$$\lim_{r\to\infty}\int_a^r \operatorname{Re}\varphi = s \quad \text{and} \quad \lim_{r\to\infty}\int_a^r \operatorname{Im}\varphi = t \tag{7.26}$$

by Proposition 2.16, and so $\int_a^\infty \operatorname{Re}\varphi = s$ and $\int_a^\infty \operatorname{Im}\varphi = t$ by definition.

Now suppose $\int_a^\infty \operatorname{Re}\varphi = s$ and $\int_a^\infty \operatorname{Im}\varphi = t$. By definition this means (7.26) holds, and so

$$\lim_{r\to\infty}\int_a^r \varphi = \lim_{r\to\infty}\left(\int_a^r \operatorname{Re}\varphi + i\int_a^r \operatorname{Im}\varphi\right) = \int_a^\infty \operatorname{Re}\varphi + i\int_a^\infty \operatorname{Im}\varphi = s + it$$

by Theorem 2.15. ■

From (7.23) and Proposition 7.50 we see that

$$\lim_{r\to\infty}\int_a^r \varphi = \int_a^\infty \varphi$$

whenever either side of the equation is known to exist in $\mathbb{C}$, and similarly

$$\lim_{r\to-\infty}\int_r^b \varphi = \int_{-\infty}^b \varphi.$$

Proposition 7.51. *Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is such that $\int_{-\infty}^{\infty}\varphi$ converges.*

1. *If $c \in \mathbb{R}$, then*

$$\int_{-\infty}^{\infty}\varphi = \int_{-\infty}^{c}\varphi + \int_{c}^{\infty}\varphi.$$

2. *Then*

$$\int_{-\infty}^{\infty}\varphi = \lim_{r\to\infty}\int_{-r}^{r}\varphi.$$

Proof.
Proof of Part (1). Fix $c \in \mathbb{R}$. The convergence of $\int_{-\infty}^{\infty}\varphi$ implies the convergence of $\int_{-\infty}^{\infty}\operatorname{Re}\varphi$ and $\int_{-\infty}^{\infty}\operatorname{Im}\varphi$, with

$$\int_{-\infty}^{\infty}\varphi = \int_{-\infty}^{c}\operatorname{Re}\varphi + i\int_{-\infty}^{c}\operatorname{Re}\varphi \quad\text{and}\quad \int_{-\infty}^{\infty}\varphi = \int_{-\infty}^{c}\operatorname{Im}\varphi + i\int_{-\infty}^{c}\operatorname{Im}\varphi$$

from elementary analysis. Now,

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi &= \int_{-\infty}^{\infty}\operatorname{Re}\varphi + i\int_{-\infty}^{\infty}\operatorname{Im}\varphi\\ &= \left(\int_{-\infty}^{c}\operatorname{Re}\varphi + \int_{c}^{\infty}\operatorname{Re}\varphi\right) + i\left(\int_{-\infty}^{c}\operatorname{Im}\varphi + \int_{c}^{\infty}\operatorname{Im}\varphi\right)\\ &= \left(\int_{-\infty}^{c}\operatorname{Re}\varphi + i\int_{-\infty}^{c}\operatorname{Im}\varphi\right) + \left(\int_{c}^{\infty}\operatorname{Re}\varphi + i\int_{c}^{\infty}\operatorname{Im}\varphi\right) = \int_{-\infty}^{c}\varphi + \int_{c}^{\infty}\varphi\end{aligned}$$

by equation (7.25).

Proof of Part (2). The convergence of $\int_{-\infty}^{\infty}\operatorname{Re}\varphi$ and $\int_{-\infty}^{\infty}\operatorname{Im}\varphi$ implies that

$$\int_{-\infty}^{\infty}\operatorname{Re}\varphi = \lim_{r\to\infty}\int_{-r}^{r}\operatorname{Re}\varphi \in \mathbb{R} \quad\text{and}\quad \int_{-\infty}^{\infty}\operatorname{Im}\varphi = \lim_{r\to\infty}\int_{-r}^{r}\operatorname{Im}\varphi \in \mathbb{R}$$

from elementary analysis. Now,

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi = \int_{-\infty}^{\infty}\operatorname{Re}\varphi + i\int_{-\infty}^{\infty}\operatorname{Im}\varphi &= \lim_{r\to\infty}\int_{-r}^{r}\operatorname{Re}\varphi + i\lim_{r\to\infty}\int_{-r}^{r}\operatorname{Re}\varphi\\ &= \lim_{r\to\infty}\left(\int_{-r}^{r}\operatorname{Re}\varphi + i\int_{-r}^{r}\operatorname{Re}\varphi\right) = \lim_{r\to\infty}\int_{-r}^{r}\varphi,\end{aligned}$$

as desired. $\blacksquare$

Example 7.52. Suppose $a, b \in \mathbb{R}$ with $a > 0$. Show that

$$\int_0^\infty e^{-(a+ib)t}dt = \frac{1}{a+ib}.$$

Solution. By Example 3.25 we have

$$\int_0^r e^{-(a+ib)t}dt = \frac{1-e^{-(a+ib)r}}{a+ib}.$$

Now, since $a > 0$,

$$|e^{-(a+ib)r}| = |e^{-ar}e^{-ibr}| = e^{-ar} \to 0$$

as $r \to \infty$, and so

$$\int_0^\infty e^{-(a+ib)t}dt = \lim_{r\to\infty}\int_0^r e^{-(a+ib)t}dt = \lim_{r\to\infty}\frac{1-e^{-(a+ib)r}}{a+ib} = \frac{1}{a+ib}$$

by Proposition 7.50 and limit laws. ■

Theorem 7.53. *If $\varphi : \mathbb{R} \to \mathbb{C}$ is such that $\int_{-\infty}^\infty |\varphi|$ converges, then $\int_{-\infty}^\infty \varphi$ also converges.*

Proof. Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is such that $\int_{-\infty}^\infty |\varphi|$ converges. Since $|\operatorname{Re}\varphi| \le |\varphi|$ and $|\operatorname{Im}\varphi| \le |\varphi|$, the Comparison Test for Integrals known from calculus implies that $\int_{-\infty}^\infty |\operatorname{Re}\varphi|$ and $\int_{-\infty}^\infty |\operatorname{Im}\varphi|$ converge, and hence $\int_{-\infty}^\infty \operatorname{Re}\varphi$ and $\int_{-\infty}^\infty \operatorname{Im}\varphi$ converge by a result analogous to Theorem 7.53 established in calculus for integrals of real-valued functions. Now it is clear from (7.25) that $\int_{-\infty}^\infty \varphi$ converges in $\mathbb{C}$. ■

Similar theorems could be stated for integrals of the form $\int_a^\infty \varphi$ and $\int_{-\infty}^b \varphi$, and they would be proven just as easily using analogous theorems established in elementary analysis for real-valued functions. Alternatively one could employ Theorem 7.53 in every instance by noting that

$$\int_a^\infty \varphi = \int_{-\infty}^\infty \varphi\chi_{(a,\infty)} \quad \text{and} \quad \int_{-\infty}^b \varphi = \int_{-\infty}^\infty \varphi\chi_{(-\infty,b)},$$

recalling the indicator function

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

from measure theory.

7.6 – The Calculus of Residues

The **calculus of residues** consists in large part of the evaluation of Riemann integrals of real- or complex-valued functions using the Residue Theorem and related tools.

Exercise 7.54 (AN4.2.4a). Evaluate
$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4} dx.$$

Solution. Let f be given by
$$f(z) = \frac{ze^{iaz}}{z^4+4}$$
for all $z \in \mathbb{C}$ such that $z^4 \neq -4$, and let
$$S = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}.$$

Since
$$f(z) = \frac{ze^{iaz}}{\big(z-\sqrt{2}e^{i\pi/4}\big)\big(z-\sqrt{2}e^{i3\pi/4}\big)\big(z-\sqrt{2}e^{i5\pi/4}\big)\big(z-\sqrt{2}e^{i7\pi/4}\big)},$$
it is seen that f is meromorphic on $\mathbb{C}$, with simple poles at $\sqrt{2}e^{i\theta}$ for each $\theta \in S$. For convenience define
$$z_k = \sqrt{2}e^{i(2k-1)\pi/4}$$
for $k = 1, 2, 3, 4$.

For $r > \sqrt{2}$, define $\gamma_1 : [-r, r] \to \mathbb{C}$ by $\gamma_1(t) = t$, and define $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let $\gamma : [0, 1] \to \mathbb{C}$ be the concatenation $\gamma = \gamma_1 * \gamma_2$. As shown in Figure 18, the path γ is closed, and by Proposition 3.28 we have
$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

By Proposition 7.33 and L'Hôpital's Rule,
$$\begin{aligned}\operatorname{res}(f, z_k) &= \lim_{z \to z_k}(z - z_k)f(z) = \lim_{z \to z_k} \frac{(z-z_k)ze^{iaz}}{z^4+4}\\ &= \lim_{z \to z_k} \frac{(2z-z_k)e^{iaz} + ia(z-z_k)ze^{iaz}}{4z^3} = \frac{e^{iaz_k}}{4z_k^2}.\end{aligned}$$

Since $\operatorname{wn}(\gamma, z_k) = 1$ for $k \in \{1, 2\}$, and $\operatorname{wn}(\gamma, z_k) = 0$ for $k \in \{3, 4\}$, we apply the Residue Theorem with $\Omega = \mathbb{C}$ and $S(f) = \{z_1, z_2, z_3, z_4\}$ to obtain
$$\begin{aligned}\oint_\gamma f &= 2\pi i \sum_{k=1}^{4} \operatorname{res}(f, z_k)\operatorname{wn}(\gamma, z_k) = 2\pi i\big[\operatorname{res}(f, z_1) + \operatorname{res}(f, z_2)\big]\\ &= \frac{\pi i}{4}\left[\frac{\exp\big(ia\sqrt{2}e^{i\pi/4}\big)}{e^{i\pi/2}} + \frac{\exp\big(ia\sqrt{2}e^{i3\pi/4}\big)}{e^{i3\pi/2}}\right] = \frac{\pi}{2}ie^{-a}\sin a.\end{aligned}$$

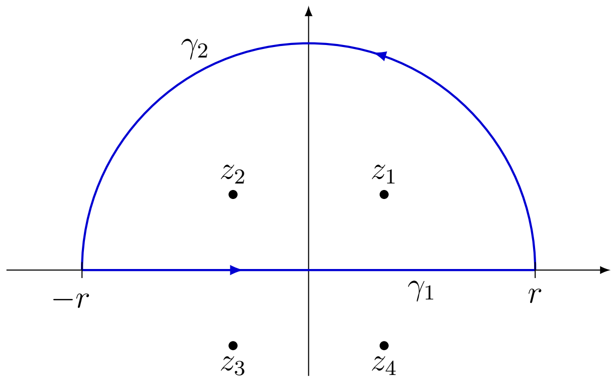

Figure 18.

By Theorem 3.23, noting that $r^4 > 4$ and $iare^{it} = -ar\sin t + iar\cos t$,

$$\begin{aligned}\left|\int_{\gamma_2} f\right| &\le \mathcal{L}(\gamma_2)\sup_{z\in\gamma_2^*}|f(z)| = \pi r \sup_{t\in[0,\pi]}|f(re^{it})| = \pi r \sup_{t\in[0,\pi]}\frac{|re^{it}||e^{iare^{it}}|}{|r^4e^{i4t}+4|}\\ &= \pi r \sup_{t\in[0,\pi]}\frac{re^{\mathrm{Re}(iare^{it})}}{|r^4e^{i4t}+4|} = \sup_{t\in[0,\pi]}\frac{\pi r^2e^{-ar\sin t}}{|r^4e^{i4t}+4|} \le \sup_{t\in[0,\pi]}\frac{\pi r^2}{|r^4e^{i4t}+4|}\\ &\le \sup_{t\in[0,\pi]}\frac{\pi r^2}{\left||r^4e^{i4t}|-4\right|} = \sup_{t\in[0,\pi]}\frac{\pi r^2}{r^4-4} = \frac{\pi r^2}{r^4-4},\end{aligned}$$

and hence

$$\lim_{r\to\infty}\int_{\gamma_2} f = 0.$$

Next,

$$\int_{\gamma_1} f = \int_{-r}^{r} f(t)dt = \int_{-r}^{r}\frac{te^{iat}}{t^4+4}dt = \int_{-r}^{r}\frac{t\cos at}{t^4+4}dt + i\int_{-r}^{r}\frac{t\sin at}{t^4+4}dt.$$

Performing the substitution $u = t^2$, we have

$$\int_0^r \frac{t}{t^4+4}dt = \frac{1}{2}\int_0^{r^2}\frac{1}{u^2+4}du = \left[\frac{1}{4}\tan^{-1}\left(\frac{u}{2}\right)\right]_0^{r^2} = \frac{1}{4}\tan^{-1}\left(\frac{r^2}{2}\right)$$

for all $r > 0$, and so

$$\int_0^\infty \frac{t}{t^4+4}dt = \lim_{r\to\infty}\frac{1}{4}\tan^{-1}\left(\frac{r^2}{2}\right) = \frac{\pi}{8}.$$

Since

$$0 \le \left|\frac{t\cos at}{t^4+4}\right| \le \frac{t}{t^4+4}$$

for all $t \in [0,\infty)$, it follows by the Comparison Test for Integrals in §8.8 of the *Calculus Notes* that

$$\int_0^\infty \left|\frac{t\cos at}{t^4+4}\right| dt$$

converges, and hence

$$\int_0^\infty \frac{t\cos at}{t^4+4}dt$$

converges by Proposition 8.36 in the *Calculus Notes*. Similar arguments will show that

$$\int_{-\infty}^{0} \frac{t\cos at}{t^4+4}dt, \quad \int_{0}^{\infty} \frac{t\sin at}{t^4+4}dt, \quad \text{and} \quad \int_{-\infty}^{0} \frac{t\sin at}{t^4+4}dt$$

also converge, so that

$$\int_{-\infty}^{\infty} \frac{t\cos at}{t^4+4}dt = \int_{-\infty}^{0} \frac{t\cos at}{t^4+4}dt + \int_{0}^{\infty} \frac{t\cos at}{t^4+4}dt$$

and

$$\int_{-\infty}^{\infty} \frac{t\sin at}{t^4+4}dt = \int_{-\infty}^{0} \frac{t\sin at}{t^4+4}dt + \int_{0}^{\infty} \frac{t\sin at}{t^4+4}dt$$

converge, and therefore

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} \operatorname{Re} f(t)dt + i\int_{-\infty}^{\infty} \operatorname{Im} f(t)dt$$

converges since

$$\operatorname{Re} f(t) = \frac{t\cos at}{t^4+4} \quad \text{and} \quad \operatorname{Im} f(t) = \frac{t\sin at}{t^4+4}.$$

By Proposition 7.51,

$$\lim_{r\to\infty} \int_{\gamma_1} f = \lim_{r\to\infty} \int_{-r}^{r} f(t)dt = \int_{-\infty}^{\infty} f(t)dt \in \mathbb{C}.$$

Now,

$$\begin{aligned}\frac{\pi}{2}ie^{-a}\sin a = \lim_{r\to\infty} \frac{\pi}{2}ie^{-a}\sin a = \lim_{r\to\infty} \oint_{\gamma} f = \lim_{r\to\infty}\left(\int_{\gamma_1} f + \int_{\gamma_2} f\right) \\ = \lim_{r\to\infty}\int_{\gamma_1} f + \lim_{r\to\infty}\int_{\gamma_2} f = \int_{-\infty}^{\infty} f(t)dt,\end{aligned}$$

and thus

$$\int_{-\infty}^{\infty} \operatorname{Re} f(t)dt + i\int_{-\infty}^{\infty} \operatorname{Im} f(t)dt = \frac{\pi}{2}ie^{-a}\sin a.$$

Equating imaginary parts finally yields

$$\int_{-\infty}^{\infty} \frac{t\sin at}{t^4+4}dt = \int_{-\infty}^{\infty} \operatorname{Im} f(t)dt = \frac{\pi}{2}e^{-a}\sin a$$

at last. ∎

Exercise 7.55 (AN4.2.4b). Evaluate

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}dx.$$

Solution. Let f be given by

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)},$$

a function that is meromorphic on $\mathbb{C}$ with simple poles in the set $S = \{\pm i, -1 \pm i\}$. For $r > \sqrt{2}$, define $\gamma_1 : [-r, r] \to \mathbb{C}$ by $\gamma_1(t) = t$, and define $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let

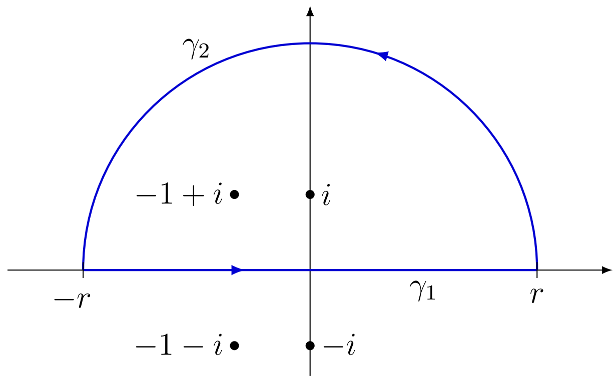

FIGURE 19.

$\gamma : [0,1] \to \mathbb{C}$ be the concatenation $\gamma = \gamma_1 * \gamma_2$. As shown in Figure 19, the path γ is closed, with $\mathrm{wn}(\gamma, -1+i) = 1$, $\mathrm{wn}(\gamma, -1-i) = 0$, $\mathrm{wn}(\gamma, i) = 1$, and $\mathrm{wn}(\gamma, -i) = 0$. By Proposition 7.33 and L'Hôpital's Rule,

$$\begin{aligned}\operatorname{res}(f, i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\left(\frac{z^2 - iz}{z^4 + 2z^3 + 3z^2 + 2z + 2}\right)\\ &= \lim_{z\to i}\left(\frac{2z - i}{4z^3 + 6z^2 + 6z + 2}\right) = \frac{i}{4i^3 + 6i^2 + 6i + 2} = \frac{1}{10} - \frac{1}{5}i,\end{aligned}$$

and

$$\begin{aligned}\operatorname{res}(f, -1+i) &= \lim_{z\to -1+i}\left(\frac{z^2 + z - iz}{z^4 + 2z^3 + 3z^2 + 2z + 2}\right)\\ &= \lim_{z\to -1+i}\left(\frac{2z + 1 - i}{4z^3 + 6z^2 + 6z + 2}\right) = -\frac{1}{10} + \frac{3}{10}i.\end{aligned}$$

Now, by the Residue Theorem,

$$\oint_\gamma f = 2\pi i\big[\operatorname{res}(f, i) + \operatorname{res}(f, -1+i)\big] = 2\pi i \cdot \frac{1}{10}i = -\frac{\pi}{5}$$

for all $r > \sqrt{2}$. On the other hand,

$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

for all $r > \sqrt{2}$, where by Theorem 3.23

$$\begin{aligned}\left|\int_{\gamma_2} f\right| &\le \mathcal{L}(\gamma_2)\sup_{z\in\gamma_2^*}|f(z)| = \pi r \sup_{t\in[0,\pi]}\frac{|re^{it}|}{|r^4e^{4it} + 2r^3e^{3it} + 3r^2e^{2it} + 2re^{it} + 2|}\\ &= \sup_{t\in[0,\pi]}\frac{\pi r^2}{|r^4e^{4it} + 2r^3e^{3it} + 3r^2e^{2it} + 2re^{it} + 2|},\end{aligned}$$

and hence

$$\lim_{r\to\infty}\int_{\gamma_2} f = 0.$$

We also have

$$\lim_{r\to\infty}\int_{\gamma_1} f = \lim_{r\to\infty}\int_{-r}^{r} \frac{t}{t^4+2t^3+3t^2+2t+2}dt = \int_{-\infty}^{\infty} \frac{t}{t^4+2t^3+3t^2+2t+2}dt,$$

where the last equality is justified since both

$$\int_{-\infty}^{0} \frac{t}{t^4+2t^3+3t^2+2t+2}dt \quad \text{and} \quad \int_{0}^{\infty} \frac{t}{t^4+2t^3+3t^2+2t+2}dt$$

can be shown to converge by the Comparison Test for Integrals. (For instance,

$$\frac{t}{t^4+2t^3+3t^2+2t+2} \le \frac{t}{t^4+2}$$

on $[0,\infty)$, and

$$\int_0^\infty \frac{t}{t^4+2}dt$$

can be shown to converge by direct evaluation—see the previous exercise.) Finally,

$$-\frac{\pi}{5} = \lim_{r\to\infty} -\frac{\pi}{5} = \lim_{r\to\infty}\oint_\gamma f = \lim_{r\to\infty}\int_{\gamma_1} f + \lim_{r\to\infty}\int_{\gamma_2} f = \int_{-\infty}^{\infty} \frac{t}{t^4+2t^3+3t^2+2t+2}dt,$$

which shows that

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}dx = -\frac{\pi}{5}$$

and we're done. ∎

Exercise 7.56 (AN4.2.4e). Evaluate

$$\int_0^\infty \frac{1}{x^4+a^4}dx,$$

where $a > 0$.

Solution. Define f by

$$f(z) = \frac{1}{z^4+a^4},$$

which is meromorphic on $\mathbb{C}$ with simple poles at

$$z_k = ae^{i(2k-1)\pi/4}, \quad k = 1,2,3,4.$$

Thus, if we let $S = \{z_1,z_2,z_3,z_4\}$, then f is analytic on $\mathbb{C}\setminus S$. Note $f(-x) = f(x)$ for any $x \in \mathbb{R}$, so that $\int_0^\infty f = \int_{-\infty}^0 f$, and therefore

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{0} f + \int_0^\infty f = 2\int_0^\infty f.$$

For this reason we may achieve our objective by evaluating $\int_{-\infty}^\infty f$.

For $r > a$ define $\gamma_1 : [-r,r] \to \mathbb{C}$ by $\gamma_1(t) = t$, and define $\gamma_2 : [0,\pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let $\gamma = \gamma_1 * \gamma_2$. The path γ is closed (see Figure 18), with $\mathrm{wn}(\gamma, z_k) = 1$ for $k = 1,2$, and

$\text{wn}(\gamma, z_k) = 0$ for $k = 3, 4$. We apply Proposition 7.33 and L'Hôpital's Rule to obtain necessary residues:

$$\text{res}(f, z_k) = \lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} \frac{z - z_k}{z^4 + a^4} = \lim_{z \to z_k} \frac{1}{4z^3} = \frac{1}{4z_k^3},$$

and so by the Residue Theorem

$$\oint_\gamma f = 2\pi i \big[\text{res}(f, z_1) + \text{res}(f, z_2) \big] = \frac{2\pi i}{4a^3} \big(e^{-i3\pi/4} + e^{-i9\pi/4} \big)$$
$$= \frac{\pi i}{2a^3} \left[\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right) \right] = \frac{\pi}{a^3\sqrt{2}}.$$

On the other hand,

$$\left| \int_{\gamma_2} f \right| \le \mathcal{L}(\gamma_2) \sup_{t \in [0,\pi]} |f(\gamma_2(t))| = \pi r \sup_{t \in [0,\pi]} \frac{1}{|r^4 e^{i4t} + a^4|}$$

for all large r, implying that $\int_{\gamma_2} f \to 0$ as $r \to \infty$, and so

$$\frac{\pi}{a^3\sqrt{2}} = \lim_{r \to \infty} \oint_\gamma f = \lim_{r \to \infty} \int_{\gamma_1} f + \lim_{r \to \infty} \int_{\gamma_2} f = \lim_{r \to \infty} \int_{-r}^{r} \frac{1}{t^4 + a^4} dt = \int_{-\infty}^{\infty} \frac{1}{t^4 + a^4} dt,$$

where the last equality holds since the improper integral is known to converge by the Comparison Test for Integrals. Therefore

$$\int_0^\infty \frac{1}{x^4 + a^4} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4 + a^4} dx = \frac{\pi\sqrt{2}}{4a^3}$$

for all $a > 0$. ■

Exercise 7.57 (AN4.2.4f). Evaluate

$$\int_0^\infty \frac{\cos x}{x^2 + 1} dx.$$

Solution. Define f by

$$f(z) = \frac{e^{iz}}{z^2 + 1},$$

which is meromorphic on $\mathbb{C}$ with simple poles at $\pm i$. Note $f(-x) = f(x)$ for any $x \in \mathbb{R}$, so that $\int_0^\infty f = \int_{-\infty}^0 f$, and therefore

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{0} f + \int_0^\infty f = 2 \int_0^\infty f.$$

The exercise can be done by evaluating $\int_{-\infty}^{\infty} f$.

For $r > 1$ define $\gamma_1 : [-r, r] \to \mathbb{C}$ by $\gamma_1(t) = t$, and define $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let $\gamma = \gamma_1 * \gamma_2$. The path γ is closed, with $\text{wn}(\gamma, i) = 1$ and $\text{wn}(\gamma, -i) = 0$. By Proposition 7.33 and L'Hôpital's Rule,

$$\text{res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{(z - i)e^{iz}}{z^2 + 1} = \lim_{z \to i} \frac{(z - i)ie^{iz} + e^{iz}}{2z} = \frac{e^{i^2}}{2i} = -\frac{1}{2e} i,$$

and so by the Residue Theorem

$$\oint_\gamma f = 2\pi i \operatorname{res}(f,i) = \frac{\pi}{e}.$$

On the other hand, since

$$\left|\int_{\gamma_2} f\right| \leq \pi r \sup_{t\in[0,\pi]} \left|\frac{e^{ire^{it}}}{r^2e^{i2t}+1}\right| \leq \sup_{t\in[0,\pi]} \frac{\pi r}{|r^2e^{i2t}+1|} \leq \sup_{t\in[0,\pi]} \frac{\pi r}{r^2-1},$$

it is clear that $\int_{\gamma_2} f \to 0$ as $r\to\infty$, and so

$$\begin{aligned}\frac{\pi}{e} &= \lim_{r\to\infty}\oint_\gamma f = \lim_{r\to\infty}\int_{\gamma_1} f + \lim_{r\to\infty}\int_{\gamma_2} f = \lim_{r\to\infty}\int_{-r}^{r}\frac{e^{it}}{t^2+1}dt = \int_{-\infty}^{\infty}\frac{e^{it}}{t^2+1}dt \\ &= \int_{-\infty}^{\infty}\frac{\cos t}{t^2+1}dt + i\int_{-\infty}^{\infty}\frac{\sin t}{t^2+1}dt.\end{aligned}$$

Comparing real parts yields

$$\int_{-\infty}^{\infty}\frac{\cos t}{t^2+1}dt = \frac{\pi}{e},$$

and therefore

$$\int_0^\infty \frac{\cos x}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}dx = \frac{\pi}{2e}.$$

■

Exercise 7.58 (AN4.2.8a). Show that for any $r>0$,

$$\int_0^{\pi/2} e^{-r\sin\theta}d\theta \leq \frac{\pi}{2r}(1-e^{-r}) \quad \text{and} \quad \int_{\pi/2}^{\pi} e^{-r\sin\theta}d\theta \leq \frac{\pi}{2r}(1-e^{-r}).$$

Solution. Let $r>0$. For all $\theta\in[0,\pi/2]$ we have $\sin\theta \geq 2\theta/\pi$, where

$$\sin\theta \geq \frac{2\theta}{\pi} \quad\Rightarrow\quad -r\sin\theta \leq -\frac{2\theta}{\pi}r \quad\Rightarrow\quad e^{-r\sin\theta} \leq e^{-\frac{2\theta}{\pi}r}.$$

Thus

$$\int_0^{\pi/2} e^{-r\sin\theta}d\theta \leq \int_0^{\pi/2} e^{-2r\theta/\pi}d\theta = \left[-\frac{\pi}{2r}e^{-2r\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{2r}(1-e^{-r}).$$

Next, make the substitution $u=\pi-\theta$ and use the fact that $\sin(\pi-u)=\sin u$ to obtain

$$\int_{\pi/2}^{\pi} e^{-r\sin\theta}d\theta = -\int_{\pi/2}^{0} e^{-r\sin(\pi-u)}du = \int_0^{\pi/2} e^{-r\sin u}du \leq \frac{\pi}{2r}(1-e^{-r}),$$

as desired. ■

The two results of Exercise 7.58 taken together yield

$$\int_0^{\pi} e^{-r\sin\theta}d\theta = \int_0^{\pi/2} e^{-r\sin\theta}d\theta + \int_{\pi/2}^{\pi} e^{-r\sin\theta}d\theta \leq \frac{\pi}{r}(1-e^{-r}), \tag{7.27}$$

itself a potentially useful result.

Exercise 7.59 (AN4.2.8b). Suppose f has a simple pole at z_0, and for $\epsilon > 0$ and $\alpha \in (0, 2\pi]$ let $\gamma_\epsilon : [\alpha_0, \alpha_0 + \alpha] \to \mathbb{C}$ be given by $\gamma_\epsilon(t) = z_0 + \epsilon e^{it}$. Prove that

$$\lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} f = \alpha i \operatorname{res}(f, z_0).$$

Solution. The function f is analytic on $B_r'(z_0)$ for some $r > 0$. Let $k = \operatorname{res}(f, z_0)$. By Proposition 7.33,

$$\lim_{z \to z_0} (z - z_0) f(z) = k.$$

Let $\mu > 0$ be arbitrary. There exists some $\delta > 0$ sufficiently small that $0 < |z - z_0| < \delta$ implies

$$\big|(z - z_0) f(z) - k\big| < \frac{\mu}{\alpha}.$$

Thus, if $0 < \epsilon < \delta$, then

$$\big|\epsilon e^{it} f(z_0 + \epsilon e^{it}) - k\big| < \frac{\mu}{\alpha}$$

holds for all $t \in [\alpha_0, \alpha_0 + \alpha]$. Now,

$$\begin{aligned}
\left|\int_{\gamma_\epsilon} f - \alpha i k\right| &= \left|\int_{\alpha_0}^{\alpha_0+\alpha} i\epsilon e^{it} f(z_0 + \epsilon e^{it}) dt - \int_{\alpha_0}^{\alpha_0+\alpha} ik\, dt\right| \\
&= \left|\int_{\alpha_0}^{\alpha_0+\alpha} \Big(\epsilon e^{it} f(z_0 + \epsilon e^{it}) - k\Big) dt\right| \\
&\leq \int_{\alpha_0}^{\alpha_0+\alpha} \big|\epsilon e^{it} f(z_0 + \epsilon e^{it}) - k\big| dt \leq \int_{\alpha_0}^{\alpha_0+\alpha} \frac{\mu}{\alpha} dt = \mu,
\end{aligned}$$

which shows that

$$\lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} f = \alpha i k = \alpha i \operatorname{res}(f, z_0),$$

as desired.

If we let $\alpha = \pi$ (so that γ_ϵ is a semicircular arc of radius ϵ), then

$$\lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} f = \alpha i k = \pi i \operatorname{res}(f, z_0)$$

in particular. ■

Exercise 7.60 (AN4.2.9a). Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

Solution. In Example 8.39 of the *Calculus Notes* it was shown that the mixed improper integral

$$\int_0^\infty \frac{\sin x}{x} dx$$

is convergent, which implies that the limits

$$\int_0^1 \frac{\sin x}{x} dx = \lim_{r \to \infty} \int_{1/r}^1 \frac{\sin x}{x} dx \quad \text{and} \quad \int_1^\infty \frac{\sin x}{x} dx = \lim_{r \to \infty} \int_1^r \frac{\sin x}{x} dx$$

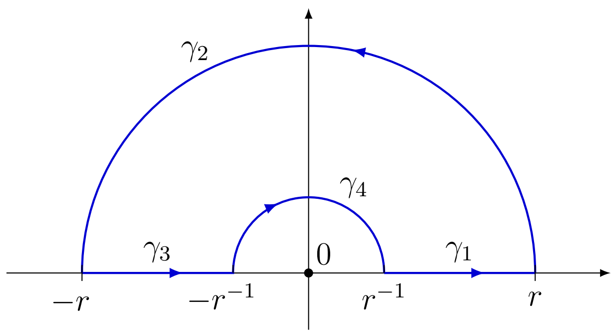

FIGURE 20.

exist in $\mathbb{R}$, and

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx$$

Hence

$$\begin{aligned}\lim_{r\to\infty}\int_{1/r}^r \frac{\sin x}{x}dx &= \lim_{r\to\infty}\left(\int_{1/r}^1 \frac{\sin x}{x}dx + \int_1^r \frac{\sin x}{x}dx\right)\\ &= \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{\sin x}{x}dx,\end{aligned}$$

a fact we will need before the end.

Let $f(z) = e^{iz}/z$, which has a simple pole at 0 and is analytic on $\mathbb{C}_*$. For $r > 1$ define the closed path $\gamma_r = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$, where

$$\begin{aligned}&\gamma_1(t) = t, \quad r^{-1} \le t \le r\\ &\gamma_2(t) = re^{it}, \quad 0 \le t \le \pi\\ &\gamma_3(t) = t, \quad -r \le t \le -r^{-1}\\ &\gamma_4(t) = r^{-1}e^{i(\pi-t)}, \quad 0 \le t \le \pi.\end{aligned}$$

This is an indented semicircle, as shown in Figure 20. By the Residue Theorem, since $\operatorname{wn}(\gamma_r, 0) = 0$,

$$\oint_{\gamma_r} f = 2\pi i \operatorname{res}(f, 0)\operatorname{wn}(\gamma_r, 0) = 0. \tag{7.28}$$

On the other hand, for any $r > 1$,

$$\oint_{\gamma_r} f = \int_{1/r}^r \frac{e^{it}}{t}dt + \int_0^\pi ie^{ire^{it}}dt + \int_{-r}^{-1/r}\frac{e^{it}}{t}dt - \int_0^\pi ie^{-ir^{-1}e^{-it}}dt. \tag{7.29}$$

Now,

$$\left|\int_{\gamma_2} f\right| = \left|\int_0^\pi ie^{ire^{it}}dt\right| \le \int_0^\pi \left|e^{ire^{it}}\right|dt = \int_0^\pi e^{\operatorname{Re}(ire^{it})}dt = \int_0^\pi e^{-r\sin t}dt \le \frac{\pi}{r}(1-e^{-r})$$

by (7.27), which shows that

$$\lim_{r\to\infty}\int_{\gamma_2} f = 0.$$

Also, for $\overline{\gamma}_4(t) = \gamma_4(\pi - t)$ for $t \in [0, \pi]$, we have

$$\lim_{r\to\infty} \int_{\overline{\gamma}_4} f = \pi i \operatorname{res}(f, 0) = \pi i \lim_{z\to 0} z f(z) = \pi i \lim_{z\to 0} e^{iz} = \pi i,$$

by Exercise 7.59, and thus by Proposition 3.27 (since $r \to \infty$ iff $r^{-1} \to 0^+$),

$$\lim_{r\to\infty} \int_{\gamma_4} f = -\pi i.$$

Finally,

$$\int_{\gamma_1} f + \int_{\gamma_3} f = \int_{1/r}^{r} \frac{e^{it}}{t} dt + \int_{-r}^{-1/r} \frac{e^{it}}{t} dt = \int_{1/r}^{r} \frac{e^{it}}{t} dt + \int_{1/r}^{r} \frac{e^{-it}}{t} dt = 2i \int_{1/r}^{r} \frac{\sin t}{t} dt,$$

so that

$$\lim_{r\to\infty} \left(\int_{\gamma_1} f + \int_{\gamma_3} f \right) = 2i \int_0^\infty \frac{\sin t}{t} dt.$$

Recalling (7.28) and (7.29), we have

$$0 = \lim_{r\to\infty} \int_{\gamma_r} f = \lim_{r\to 0} \int_{\gamma_2} f + \lim_{r\to 0} \int_{\gamma_4} f + \lim_{r\to\infty} \left(\int_{\gamma_1} f + \int_{\gamma_3} f \right) = -\pi i + 2i \int_0^\infty \frac{\sin t}{t} dt,$$

giving

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

By Example 8.40 of the *Calculus Notes*, then,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_0^\infty \frac{\sin x}{x} dx = \pi.$$

■

Exercise 7.61 (AN4.2.9b). Show that

$$\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Solution. Let $f(z) = e^{iz^2}$, and take

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

as given. Define the closed path $\gamma = \gamma_1 * \gamma_2 * \gamma_3$, where

$$\gamma_1(t) = t, \quad 0 \le t \le r$$

$$\gamma_2(t) = re^{it}, \quad 0 \le t \le \pi/4$$

$$\gamma_3(t) = (r - t)e^{i\pi/4}, \quad 0 \le t \le r$$

Since f is analytic on $\mathbb{C}$, by Cauchy's Theorem (Theorem 6.29)

$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f = 0$$

for all $r > 0$.

Now, making the substitution $\theta = 2t$ and applying

$$\left|\int_{\gamma_2} f\right| \leq \int_0^{\pi/4} \left|e^{ir^2 e^{i2t}} \cdot ire^{it}\right| dt = \int_0^{\pi/4} re^{\mathrm{Re}(ir^2 e^{i2t})} dt = r\int_0^{\pi/4} e^{-r^2 \sin 2t} dt$$
$$= \frac{r}{2}\int_0^{\pi/2} e^{-r^2 \sin\theta} d\theta \leq \frac{r}{2} \cdot \frac{\pi}{2r^2}(1 - e^{-r^2}) = \frac{\pi}{4r}(1 - e^{-r^2}),$$

it is clear that

$$\lim_{r\to\infty} \int_{\gamma_2} f = 0.$$

Next, making the substitution $\tau = r - t$,

$$\int_{\gamma_3} f = -\int_0^r e^{i(r-t)^2 e^{i\pi/2}} e^{i\pi/4} dt = -e^{i\pi/4}\int_0^r e^{-(r-t)^2} dt = -e^{i\pi/4}\int_0^r e^{-\tau^2} d\tau$$

and so

$$\lim_{r\to\infty} \int_{\gamma_3} f = -e^{i\pi/4}\int_0^\infty e^{-\tau^2} d\tau = -\frac{\sqrt{\pi}}{2} e^{i\pi/4}.$$

Thus,

$$\lim_{r\to\infty} \int_{\gamma_1} f = \lim_{r\to\infty} \int_{\gamma} f - \lim_{r\to\infty} \int_{\gamma_2} f - \lim_{r\to\infty} \int_{\gamma_3} f = \frac{\sqrt{\pi}}{2} e^{i\pi/4} = \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right),$$

whence

$$\lim_{r\to\infty}\left(\int_0^r \cos(t^2) dt + \int_0^r \sin(t^2) dt\right) = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}.$$

It follows by (a minor variant of) Proposition 2.16 that

$$\lim_{r\to\infty}\int_0^r \cos(t^2) dt = \frac{\sqrt{2\pi}}{4} \quad \text{and} \quad \lim_{r\to\infty}\int_0^r \sin(t^2) dt = \frac{\sqrt{2\pi}}{4}.$$

Therefore

$$\int_0^\infty \cos(t^2) dt = \int_0^\infty \sin(t^2) dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

and the fun is done. ■

For the next exercise, recall from §6.1 that $\mathrm{Log} = \log_{-\pi} : \mathbb{C}_* \to H_{-\pi}$ is analytic on $\mathbb{C}\setminus(-\infty, 0]$, and for all $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$\mathrm{Log}'(z) = \log'_{-\pi}(z) = \frac{1}{z}.$$

Letting $\varphi(z) = z + i$, then $\mathrm{Log} \circ \varphi : \mathbb{C} \setminus \{-i\} \to H_{-\pi}$ is given by $(\mathrm{Log} \circ \varphi)(z) = \mathrm{Log}(z + i)$, which is analytic on $\Omega = \mathbb{C} \setminus (-\infty, 0] - i$ shown in Figure 21. By the Chain Rule, $\mathrm{Log}'(z + i) = (z + i)^{-1}$ for all $z \in \Omega$.

Exercise 7.62 (AN4.2.9c). Evaluate

$$\int_0^\infty \frac{\ln(x^2 + 1)}{x^2 + 1} dx.$$

Solution. Let
$$f(z) = \frac{\operatorname{Log}(z+i)}{z^2+1},$$
which has simple poles at i and $-i$. For $r > 2$, define $\gamma_1 : [-r, r] \to \mathbb{C}$ by $\gamma_1(t) = t$, and define $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let $\gamma : [0, 1] \to \mathbb{C}$ be the concatenation $\gamma = \gamma_1 * \gamma_2$. As shown in Figure 22, the path γ is closed, with $\operatorname{wn}(\gamma, i) = 1$ and $\operatorname{wn}(\gamma, -i) = 0$. By Proposition 7.33 and L'Hôpital's Rule, making use of the fact that $\operatorname{Log}(z + i)$ is continuous on the region Ω shown in Figure 21, we have
$$\begin{aligned}\operatorname{res}(f, i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-i)\operatorname{Log}(z+i)}{z^2+1} = \lim_{z\to i}\frac{\dfrac{z-i}{z+i} + \operatorname{Log}(z+i)}{2z}\\ &= \frac{\operatorname{Log}(2i)}{2i} = \frac{\exp_{-\pi}^{-1}(2i)}{2i} = \frac{\ln 2 + \pi i/2}{2i} = \frac{\pi}{4} - \frac{\ln 2}{2}i,\end{aligned}$$
and thus by the Residue Theorem
$$\oint_\gamma f = 2\pi i \operatorname{res}(f, i)\operatorname{wn}(\gamma, i) = 2\pi i\left(\frac{\pi}{4} - \frac{\ln 2}{2}i\right) = \pi \ln 2 + \frac{\pi^2}{2}i. \tag{7.30}$$
On the other hand,
$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_{-r}^{r}\frac{\operatorname{Log}(t+i)}{t^2+1}dt + \int_0^\pi \frac{\operatorname{Log}(i+re^{it})}{r^2e^{i2t}+1}ire^{it}dt \tag{7.31}$$
for all $r > 2$.

By Proposition 6.2,
$$\begin{aligned}\operatorname{Log}(i+re^{it}) &= \ln|i+re^{it}| + i\operatorname{Arg}(i+re^{it})\\ &= \frac{1}{2}\ln(r^2+2r\sin t+1) + i\operatorname{Arg}(i+re^{it}),\end{aligned}$$
and thus for $t \in [0, \pi]$, since $\operatorname{Arg} : \mathbb{C}_* \to [-\pi, \pi)$,
$$|\operatorname{Log}(i+re^{it})| \le \frac{1}{2}\ln(r^2+2r\sin t+1) + \pi \le \frac{1}{2}\ln(r^2+2r+1) + \pi = \ln(r+1) + \pi.$$
From this we then obtain, for $t \in [0, \pi]$ and $r > 2$,
$$\left|\frac{ire^{it}\operatorname{Log}(i+re^{it})}{r^2e^{i2t}+1}\right| \le \frac{r\ln(r+1)+\pi r}{r^2-1},$$

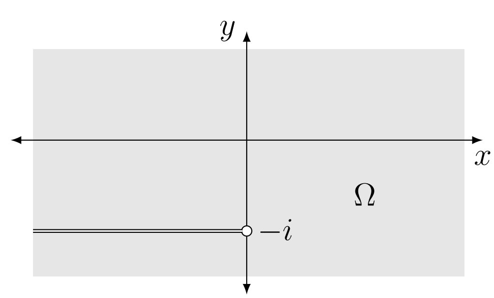

FIGURE 21.

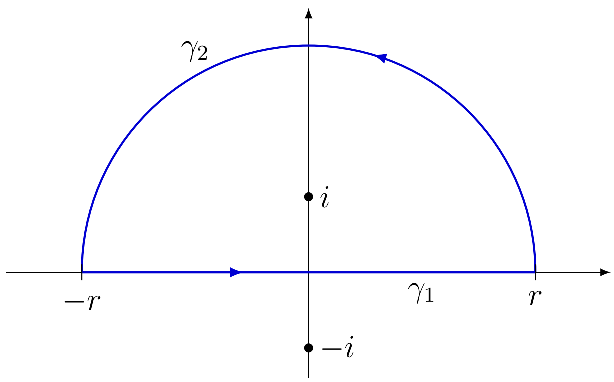

FIGURE 22.

and therefore

$$\left|\int_{\gamma_2} f\right| \le \int_0^\pi \left|\frac{\operatorname{Log}(i+re^{it})}{r^2e^{i2t}+1}ire^{it}\right| dt \le \int_0^\pi \frac{r\ln(r+1)+\pi r}{r^2-1}dt = \frac{\pi r\ln(r+1)+\pi^2 r}{r^2-1}$$

for all $r > 2$. This shows that

$$\lim_{r\to\infty}\int_{\gamma_2} f = \lim_{r\to\infty}\int_0^\pi \frac{\operatorname{Log}(i+re^{it})}{r^2e^{i2t}+1}ire^{it}dt = 0. \tag{7.32}$$

Next, by Proposition 6.2,

$$\int_{-r}^r \frac{\operatorname{Log}(t+i)}{t^2+1}dt = \int_{-r}^r \frac{\ln\sqrt{t^2+1}}{t^2+1}dt + i\int_{-r}^r \frac{\operatorname{Arg}(t+i)}{t^2+1}dt,$$

and since the Comparison Test for Integrals can be applied to show that

$$\int_0^\infty \frac{\ln\sqrt{t^2+1}}{t^2+1}dt \quad \text{and} \quad \int_0^\infty \frac{\operatorname{Arg}(t+i)}{t^2+1}dt$$

converge (as do the corresponding $\int_{-\infty}^0$ integrals), it follows that

$$\lim_{r\to\infty}\int_{-r}^r \frac{\ln\sqrt{t^2+1}}{t^2+1}dt = \int_{-\infty}^\infty \frac{\ln\sqrt{t^2+1}}{t^2+1}dt$$

and

$$\lim_{r\to\infty}\int_0^\infty \frac{\operatorname{Arg}(t+i)}{t^2+1}dt = \int_{-\infty}^\infty \frac{\operatorname{Arg}(t+i)}{t^2+1}dt.$$

With these considerations, together with (7.30), (7.31), and (7.32), we obtain

$$\begin{aligned}\pi\ln 2 + \frac{\pi^2}{2}i = \lim_{r\to\infty}\oint_\gamma f = \lim_{r\to\infty}\int_{-r}^r \frac{\operatorname{Log}(t+i)}{t^2+1}dt \\ = \int_{-\infty}^\infty \frac{\ln\sqrt{t^2+1}}{t^2+1}dt + i\int_{-\infty}^\infty \frac{\operatorname{Arg}(t+i)}{t^2+1}dt,\end{aligned}$$

which shows that

$$\int_{-\infty}^\infty \frac{\ln\sqrt{t^2+1}}{t^2+1}dt = \pi\ln 2.$$

Since the integrand is an even function,

$$\int_0^\infty \frac{\ln\sqrt{t^2+1}}{t^2+1}dt = \frac{\pi \ln 2}{2},$$

and therefore

$$\int_0^\infty \frac{\ln(t^2+1)}{t^2+1}dt = \pi \ln 2. \tag{7.33}$$

Note that we also obtain

$$\int_{-\infty}^\infty \frac{\mathrm{Arg}(t+i)}{t^2+1}dt = \frac{\pi^2}{2}$$

as an added bonus. ■

Exercise 7.63 (AN4.2.9d). Derive formulas for

$$\int_0^{\pi/2} \ln(\cos\theta)d\theta \quad \text{and} \quad \int_0^{\pi/2} \ln(\sin\theta)d\theta.$$

Solution. Let $t = \tan\theta$ in (7.33) to obtain

$$\begin{aligned}\pi \ln 2 &= \int_0^\infty \frac{\ln(t^2+1)}{t^2+1}dt = \lim_{r\to\infty}\int_0^r \frac{\ln(t^2+1)}{t^2+1}dt = \lim_{r\to\infty}\int_0^{\tan^{-1} r} \ln(\sec^2\theta)d\theta \\ &= \int_0^{\pi/2} \ln(\cos^{-2}\theta)d\theta = -2\int_0^{\pi/2}\ln(\cos\theta)d\theta,\end{aligned}$$

and hence

$$\int_0^{\pi/2} \ln(\cos\theta)d\theta = -\frac{\pi}{2}\ln 2.$$

Next, with the substitution $u = \pi/2 - \theta$, we have

$$\begin{aligned}\int_0^{\pi/2} \ln(\sin\theta)d\theta &= \int_0^{\pi/2} \ln\big(\cos(\pi/2-\theta)\big)d\theta = \lim_{\epsilon\to 0^+}\int_\epsilon^{\pi/2} \ln\big(\cos(\pi/2-\theta)\big)d\theta \\ &= \lim_{\epsilon\to 0^+}\int_{\pi/2-\epsilon}^0 -\ln(\cos u)du = \lim_{\epsilon\to 0^+}\int_0^{\pi/2-\epsilon}\ln(\cos u)du \\ &= \int_0^{\pi/2}\ln(\cos u)du,\end{aligned}$$

and therefore

$$\int_0^{\pi/2} \ln(\sin\theta)d\theta = -\frac{\pi}{2}\ln 2$$

also. ■

8

Conformal Mappings

8.1 – Open Mappings

As in the past, we define a **region** to be a nonempty open connected subset of $\mathbb{C}$. It is a fact that any open set $\Omega \subseteq \mathbb{C}$ that is not connected is a disjoint union of regions.

Lemma 8.1. *Let f be a nonconstant analytic function on a region Ω. Let $z_0 \in \Omega$, $w_0 = f(z_0)$, and $k = \operatorname{ord}(f - w_0, z_0)$.*

1. *There exists $\epsilon > 0$ such that $\overline{B}'_\epsilon(z_0) \subseteq \Omega$, and $f(z) \neq w_0$ and $f'(z) \neq 0$ for all $z \in \overline{B}'_\epsilon(z_0)$.*
2. *Let γ be a path such that $\gamma^* = C_\epsilon(z_0)$ and $\operatorname{wn}(\gamma, z_0) = 1$, let W_0 be the component of $\mathbb{C} \setminus (f \circ \gamma)^*$ containing w_0, and let $\Omega' = B'_\epsilon(z_0) \cap f^{-1}(W_0)$. Then $f : \Omega' \to W_0 \setminus \{w_0\}$ is a surjective k-to-one map.*
3. *The map $f : \Omega' \cup \{z_0\} \to W_0$ is a bijection if and only if $f'(z_0) \neq 0$.*

Proof.
Proof of Part (1). Let $\epsilon_0 > 0$ such that $B_{2\epsilon_0}(z_0) \subseteq \Omega$. The analytic function $f - w_0$ is nonconstant, and hence not identically zero, on the region Ω; and so by the Identity Theorem $Z(f - w_0)$ has no limit point in Ω and there exists some $\epsilon_1 > 0$ such that $Z(f - w_0) \cap B'_{\epsilon_1}(z_0) = \varnothing$. Similarly, f' is not identically zero on Ω by Theorem 3.26, and so there exists some $\epsilon_2 > 0$ such that $Z(f') \cap B'_{\epsilon_2}(z_0) = \varnothing$. Setting $\epsilon = \min\{\epsilon_0, \epsilon_1, \epsilon_2\}$, we conclude that $\overline{B}'_\epsilon(z_0) \subseteq \Omega$, and $f(z) \neq w_0$ and $f'(z) \neq 0$ for all $z \in \overline{B}'_\epsilon(z_0)$.

Proof of Part (2). Since $f - w_0$ is not identically zero on Ω, we have $k \in \mathbb{N}$ by Proposition 5.15. Take $\epsilon > 0$ to be as defined in Part (1). Fix $w \in W_0 \setminus \{w_0\}$. Since $f \circ \gamma$ is a closed curve, by Proposition 6.21 the function $z \mapsto \operatorname{wn}(f \circ \gamma, z)$ constant on W_0, and so

$$\operatorname{wn}(f \circ \gamma, w) = \operatorname{wn}(f \circ \gamma, w_0). \tag{8.1}$$

Since $(f - w_0)' = f' \neq 0$ on $B'_\epsilon(z_0)$, by Proposition 5.11 any zero for $f - w_0$ in $B'_\epsilon(z_0)$ must be of order 1, and by hypothesis $\operatorname{ord}(f - w_0, z_0) = k$. Now, $w_0 \in W_0 \subseteq \mathbb{C} \setminus (f \circ \gamma)^*$ makes clear that $w_0 \notin (f \circ \gamma)^*$, so

$$\operatorname{wn}(f \circ \gamma, w_0) = \operatorname{wn}(f \circ \gamma - w_0, 0) = \operatorname{wn}((f - w_0) \circ \gamma, 0) \tag{8.2}$$

by Proposition 6.20(2), and then by the Argument Principle, recalling that $f - w_0 \neq 0$ on the set

$$\overline{B}'_\epsilon(z_0) = B'_\epsilon(z_0) \cup C_\epsilon(z_0),$$

we obtain

$$\operatorname{wn}((f - w_0) \circ \gamma, 0) = \sum_{z \in Z(f-w_0)} \operatorname{ord}(f - w_0, z) \operatorname{wn}(\gamma, z) = \operatorname{ord}(f - w_0, z_0) \operatorname{wn}(\gamma, z_0) = (k)(1) = k.$$

Hence by (8.1) and (8.2) we obtain $\operatorname{wn}(f \circ \gamma, w) = k$.

Since $(f - w)' = f' \neq 0$ on $B'_\epsilon(z_0)$, by Proposition 5.11 any zero for $f - w$ in $B'_\epsilon(z_0)$ must be of order 1. Moreover $w \neq w_0$ implies that $f(z_0) - w \neq 0$, which is to say z_0 is not a zero for $f - w$. Thus, noting that $w \notin (f \circ \gamma)^*$, we have

$$\begin{aligned} k = \operatorname{wn}(f \circ \gamma, w) &= \operatorname{wn}(f \circ \gamma - w, 0) = \operatorname{wn}((f - w) \circ \gamma, 0) \\ &= \sum_{z \in Z(f-w)} \operatorname{ord}(f - w, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(f-w)} \operatorname{wn}(\gamma, z) \end{aligned}$$

by the Argument Principle. Since $z_0 \notin Z(f-w)$ and $\operatorname{wn}(\gamma, z) \in \{0, 1\}$ for all $z \in Z(f-w)$, there exist distinct points $z_1, \dots, z_k \in B'_\epsilon(z_0)$ such that $(f - w)(z_j) = 0$, and hence $f(z_j) = w \in W_0$, for $1 \leq j \leq k$.

We conclude that, for each $w \in W_0 \setminus \{w_0\}$, there exist precisely k distinct elements of $B'_\epsilon(z_0)$ which f maps to w. Therefore $f : \Omega' \to W_0 \setminus \{w_0\}$ is a surjective k-to-one map.

Proof of Part (3). Suppose that $f : \Omega' \cup \{z_0\} \to W_0$ is a bijection. By Proposition 5.15 $\operatorname{ord}(f - w_0, z_0) = k$ for some $k \in \mathbb{N}$, but because $f : \Omega' \to W_0 \setminus \{w_0\}$ is a surjective one-to-one map, by Part (2) it must be that $k = 1$. Hence $(f - w_0)'(z_0) \neq 0$ by Proposition 5.11, and since $(f - w_0)' = f'$, we obtain $f'(z_0) \neq 0$.

For the converse, suppose that $f'(z_0) \neq 0$. Then $(f - w_0)'(z_0) \neq 0$, so that $\operatorname{ord}(f - w_0, z_0) = 1$ by Proposition 5.11, and we conclude that $f : \Omega' \to W_0 \setminus \{w_0\}$ is a bijection by Part (2). By Part (1), $f(z) \neq w_0$ for all $z \in \overline{B}'_\epsilon(z_0)$, and since

$$\Omega' = B'_\epsilon(z_0) \cap f^{-1}(W_0) \subseteq \overline{B}'_\epsilon(z_0),$$

it is clear that $f(z) \neq w_0$ for all $z \in \Omega'$, and therefore $f : \Omega' \cup \{z_0\} \to W_0$ is a bijection. ∎

Given topological spaces X and Y, a map $f : X \to Y$ is said to be an **open map** if, for any set U that is open in X, the set $f(U)$ is open in V.

Theorem 8.2 (**Open Mapping Theorem**). *If Ω is a region and $f : \Omega \to \mathbb{C}$ is a nonconstant analytic function, then f is an open map.*

Proof. Suppose Ω is a region and $f : \Omega \to \mathbb{C}$ is a nonconstant analytic function. We will first show that $f(\Omega)$ is open. Let $w_0 \in f(\Omega)$, so there exists some $z_0 \in \Omega$ such that $f(z_0) = w_0$. Define $\epsilon > 0$ as in Lemma 8.1(1), and define the path γ and sets Ω' and W_0 as in Lemma 8.1(2). Then $f : \Omega' \to W_0 \setminus \{w_0\}$ is surjective by Lemma 8.1(2), and hence $f : \Omega' \cup \{z_0\} \to W_0$ is also surjective. That is, $f(\Omega' \cup \{z_0\}) = W_0$, so $W_0 \subseteq f(\Omega)$. Since $w_0 \in W_0$ and W_0 is an open set (W_0 is a component of the open set $\mathbb{C} \setminus (f \circ \gamma)^*$, and components of an open set in the complex plane are always open), it follows that w_0 is an interior point of $f(\Omega)$. Thus $f(\Omega)$ consists entirely of interior points and is therefore open.

Next, let $U \subseteq \Omega$ be a region. Suppose that $f \equiv w$ on U for some $w \in \mathbb{C}$. Then the analytic function $f - w : \Omega \to \mathbb{C}$ is identically zero on U, implying that $Z(f - w)$ has a limit point in U, and hence in Ω. The Identity Theorem now leads to the conclusion that $f - w \equiv 0$ (i.e. $f \equiv w$) on Ω, in contradiction to the hypothesis that f is nonconstant on Ω. Therefore f must be nonconstant on U, and it follows that $f(U)$ is open by the argument of the previous paragraph.

Finally, let $U \subseteq \Omega$ be an open set, not necessarily connected. Then U is expressible as a disjoint union of regions,

$$U = \bigsqcup_{\alpha \in I} U_\alpha,$$

where I is an index set, so that

$$f(U) = \bigcup_{\alpha \in I} f(U_\alpha).$$

Since $f(U_\alpha)$ is an open set for each $\alpha \in I$, we conclude that $f(U)$ is open also. ■

Corollary 8.3. *Let Ω be open. If $f : \Omega \to \mathbb{C}$ is analytic and nonconstant on each component of Ω, then f is an open map.*

Theorem 8.4 (Inverse Function Theorem). *Let Ω and S be nonempty open sets in $\mathbb{C}$. If $f : \Omega \to S$ is an analytic bijection, then $f^{-1} : S \to \Omega$ is also analytic, with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

for all $w \in S$.

Proof. Suppose $f : \Omega \to S$ is an analytic bijection. Assume that Ω is a region. Since $f(\Omega) = S$ and S is open, it is clear that f is nonconstant. For any open set $U \subseteq \Omega$ we have

$$(f^{-1})^{-1}(U) = \{w \in S : f^{-1}(w) \in U\} = \{w \in S : w \in f(U)\} = f(U)$$

is open by the Open Mapping Theorem, implying that the nonconstant bijection $f^{-1} : S \to \Omega$ is continuous.

Let $z_0 \in \Omega$ and $w_0 = f(z_0)$, and define the path γ and sets Ω' and W_0 as in Lemma 8.1(2). Then $f : \Omega' \to W_0 \setminus \{w_0\}$ is surjective by Lemma 8.1(2), so that $f : \Omega' \cup \{z_0\} \to W_0$ is also surjective and hence a bijection. Thus $f'(z_0) \neq 0$ by Lemma 8.1(3), and since $z_0 \in \Omega$ is arbitrary it follows that $f' \neq 0$ on Ω. This result, coupled with the knowledge that $f : \Omega \to S$ is analytic, $f^{-1} : S \to \Omega$ is continuous, and $f(f^{-1}(z)) = z$ for all $z \in S$, leads via Theorem 3.6 to the conclusion that f^{-1} is analytic on S, and

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

on S.

Now assume that Ω is not connected. Then there exists a family of disjoint regions $\{\Omega_\alpha : \alpha \in I\}$ in Ω such that $\Omega = \bigsqcup_{\alpha \in I} \Omega_\alpha$. For each $\alpha \in I$ let $S_\alpha = f(\Omega_\alpha)$. Then $f : \Omega_\alpha \to S_\alpha$ is an analytic bijection, and since Ω_α is a region and f is nonconstant on Ω_α (owing to it being injective), the Open Mapping Theorem implies that $f(\Omega_\alpha) = S_\alpha$ is open. Now the argument above (when Ω was assumed to be a region) may be applied here to conclude that f^{-1} is analytic on S_α, and

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \tag{8.3}$$

for all $w \in S_\alpha$. This holds for all $\alpha \in I$, and since

$$S = \bigsqcup_{\alpha \in I} S_\alpha,$$

we conclude that f^{-1} is analytic on S and (8.3) holds for all $w \in S$. ∎

Proposition 8.5. *Suppose f and g are analytic on Ω and f is injective. Then for each $z_0 \in \Omega$ and $r > 0$ such that $\overline{B}_r(z_0) \subseteq \Omega$,*

$$g(f^{-1}(w)) = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{g(z)f'(z)}{f(z) - w} dw$$

for all $w \in B_r(z_0)$.

Exercise 8.6 (AN4.3.1). Suppose that g is analytic at z_0 and f has a simple pole at z_0. Show that $\operatorname{res}(fg, z_0) = g(z_0)\operatorname{res}(f, z_0)$. Also show that the result is false if the word "simple" is omitted.

Solution. By Theorem 7.9(1) there exists some $a \in \mathbb{C}_*$ such that

$$\lim_{z \to z_0} (z - z_0)f(z) = a,$$

and in fact $a = \operatorname{res}(f, z_0)$ by Proposition 7.31. Of course, the analyticity of g at z_0 implies that $g(z) \to g(z_0)$ as $z \to z_0$, and hence

$$\lim_{z \to z_0} (z - z_0)(fg)(z) = \lim_{z \to z_0} \big[(z - z_0)f(z) \cdot g(z)\big] = \lim_{z \to z_0} (z - z_0)f(z) \cdot \lim_{z \to z_0} g(z) = ag(z_0)$$

by a law of limits. If $g(z_0) \neq 0$ then $ag(z_0) \in \mathbb{C}_*$, so that Theorem 7.9(1) implies fg has a simple pole at z_0, and then by Proposition 7.31 we conclude that

$$\operatorname{res}(fg, z_0) = ag(z_0) = g(z_0)\operatorname{res}(f, z_0)$$

as desired.

Suppose that $g(z_0) = 0$. There exists some $r > 0$ such that g is analytic on $B_r(z_0)$, and so by Proposition 5.15 either g is identically zero on $B_r(z_0)$, or there exists some $m \in \mathbb{N}$ such that $\operatorname{ord}(g, z_0) = m$. In the former case we easily find that $fg \to 0$ as $z \to z_0$; and in the latter case there exists some analytic function $h : B_r(z_0) \to \mathbb{C}$ such that $h(z_0) \neq 0$ and $g(z) = (z - z_0)^m h(z)$ for all $z \in B_r(z_0)$, and thus

$$\lim_{z \to z_0} (fg)(z) = \lim_{z \to z_0} (z - z_0)^m f(z)h(z) = \begin{cases} ah(z_0), & m = 1 \\ 0, & m > 1 \end{cases}$$

In either case the limit exists in $\mathbb{C}$, so that fg has a removable singularity at z_0 by Theorem 7.9, and therefore $\operatorname{res}(fg, z_0) = 0$ by Definition 7.7(1). We have

$$g(z_0)\operatorname{res}(f, z_0) = 0 \cdot \operatorname{res}(f, z_0) = 0 = \operatorname{res}(fg, z_0)$$

once again.

Suppose $g(z) = z$ and $f(z) = 1/z^2$, so g is analytic at 0 while f has a pole of order 2 at 0. Clearly $\operatorname{res}(f, 0) = 0$, and since $(fg)(z) = 1/z$ we have $\operatorname{res}(fg, 0) = 1$. Now,

$$\operatorname{res}(fg, 0) = 1 \neq 0 = g(0)\operatorname{res}(f, 0),$$

which shows that "simple" may not be omitted in the statement of the result. ■

Exercise 8.7 (AN4.3.3). Let f be meromorphic on an open set $\Omega \subseteq \mathbb{C}$, and let $P(f) \subseteq \Omega$ be the set of poles for f. Define $f(z) = \infty$ for each $z \in P(f)$, so as to define a continuous map $f : (\Omega, d) \to (\overline{\mathbb{C}}, \bar{d})$ that is analytic on $\Omega' = \Omega \setminus P(f)$. Prove that if f is nonconstant on each component of Ω, then $f(\Omega)$ is open in $(\overline{\mathbb{C}}, \bar{d})$.

Solution. First assume that Ω is connected (hence a region), and $P(f) = \{z_0\}$ for some some $z_0 \in \Omega$. Choose $r > 0$ such that $|f(z)| > 1$ for all $z \in B'_r(z_0)$, which implies that $B'_r(z_0) \subseteq \Omega'$, and define $g : B_r(z_0) \to \mathbb{C}$ by

$$g(z) = \begin{cases} 1/f(z), & z \in B'_r(z_0) \\ 0, & z = z_0 \end{cases}$$

Since g is analytic on $B'_r(z_0)$ and continuous on $B_r(z_0)$, Corollary 4.22 implies that g is analytic on $B_r(z_0)$. By Lemma 8.1(1) there's some $\epsilon > 0$ such that $\overline{B}_\epsilon(z_0) \subseteq B_r(z_0)$, and $g(z), g'(z) \neq 0$ for all $z \in \overline{B}_\epsilon(z_0) \setminus \{z_0\}$. Let γ be a path such that $\gamma^* = C_\epsilon(z_0)$ and $\operatorname{wn}(\gamma, z_0) = 1$, let W be the component of $\mathbb{C} \setminus (g \circ \gamma)^*$ containing 0, and let $\Omega_0 = B_\epsilon(z_0) \cap g^{-1}(W)$. By Lemma 8.1(2) the map $g : \Omega_0 \to W$ is surjective.

Let $\rho > 0$ such that $B_\rho(0) \subseteq W$, so for any $w \in B_\rho(0)$ there exists some $z \in \Omega_0$ with $g(z) = w$. Let $\alpha = 1/\rho$, and suppose $w \in A_{\alpha,\infty}(0)$. Since

$$w \in A_{\alpha,\infty}(0) \;\Leftrightarrow\; |w| > \alpha > 0 \;\Leftrightarrow\; 0 < |1/w| < \rho \;\Leftrightarrow\; 1/w \in B'_\rho(0),$$

there exists some $z \in \Omega_0 \setminus \{z_0\} \subseteq \Omega'$ such that $g(z) = 1/w$, and thus $f(z) = w$. Hence $w \in f(\Omega')$ and we obtain $A_{\alpha,\infty}(0) \subseteq f(\Omega')$.

Let $\delta = (\alpha^2 + 1)^{-1/2} > 0$. Suppose

$$w \in B_{\#,\delta}(\infty) = \{\zeta \in \overline{\mathbb{C}} : \bar{d}(\zeta, \infty) < \delta\}.$$

It is clear that $\infty \in f(\Omega)$, so assume $w \neq \infty$. Now,

$$\bar{d}(w, \infty) < \delta \;\Rightarrow\; \frac{1}{\sqrt{|w|^2 + 1}} < \frac{1}{\sqrt{\alpha^2 + 1}} \;\Rightarrow\; |w| > \alpha \;\Rightarrow\; w \in A_{\alpha,\infty}(0),$$

so there exists some $z \in \Omega'$ such that $f(z) = w$, which shows $w \in f(\Omega') \subseteq f(\Omega)$, and thus $B_{\#,\delta}(\infty) \subseteq f(\Omega)$. Hence ∞ is an interior point of $f(\Omega)$.

Since Ω' is open in $(\mathbb{C}, d)$, and $f : \Omega' \to \mathbb{C}$ is analytic and nonconstant on Ω', we find by Corollary 8.3 that $f(\Omega')$ is open in $(\mathbb{C}, d)$, and hence open in $(\overline{\mathbb{C}}, \bar{d})$ by Proposition 6.39. Let $w \in f(\Omega) = f(\Omega') \cup \{\infty\}$. If $w = \infty$, then w is an interior point of $f(\Omega)$ with respect to $\bar{d}$ as shown above. If $w \in f(\Omega')$, then w is an interior point of $f(\Omega')$ since $f(\Omega')$ is open in $(\overline{\mathbb{C}}, \bar{d})$, and thus w is again an interior point of $f(\Omega)$ with respect to $\bar{d}$. Therefore $f(\Omega)$ is comprised entirely of interior points with respect to $\bar{d}$, which leads us to conclude that $f(\Omega)$ is open in $(\overline{\mathbb{C}}, \bar{d}\,)$.

As the next case, suppose that $P(f)$ contains more than one point in Ω. For each $z_p \in P(f)$ there exists some $\epsilon > 0$ such that $B'_\epsilon(z_p) \subseteq \Omega'$. We may carry out the same analysis with z_p as was done for z_0 above to determine that $A_{\alpha,\infty}(0) \subseteq f(\Omega')$ for some $\alpha > 0$. Once again this will imply that ∞ is an interior point of $f(\Omega)$, and since $f(\Omega')$ is again open in $(\mathbb{C}, d)$ by Corollary 8.3 and $f(\Omega) = f(\Omega') \cup \{\infty\}$, we are led once more to conclude that $f(\Omega)$ is open in $(\overline{\mathbb{C}}, \bar{d}\,)$.

The general case in which Ω is not connected, f is nonconstant on each component of Ω, and $|P(f)| \geq 1$ is now easily treated. Let Ω_c be a component of Ω, and let $P_c(f) = P(f) \cap \Omega_c$. If $P_c(f) = \varnothing$ then $f(\Omega_c)$ is open in $(\mathbb{C}, d)$ by the Open Mapping Theorem, hence open in $(\overline{\mathbb{C}}, \bar{d}\,)$; and if $|P_c(f)| \geq 1$, then again $f(\Omega_c)$ is open in $(\overline{\mathbb{C}}, \bar{d}\,)$ by the arguments above. Since $f(\Omega)$ is the union of all $f(\Omega_c)$, it follows that $f(\Omega)$ is also open in $(\overline{\mathbb{C}}, \bar{d}\,)$. ■

Proposition 8.8. *Let f have a pole at z_0, and let $r > 0$ be such that f is analytic and $|f| > 1$ on $B'_r(z_0)$.*

1. *There exists $\epsilon > 0$ such that $\overline{B}_\epsilon(z_0) \subseteq B_r(z_0)$ and $(1/f)'(z) \neq 0$ for all $z \in \overline{B}'_\epsilon(z_0)$.*
2. *For any $0 < \delta < \epsilon$ there exists $\alpha > 0$ such that $A_{\alpha,\infty}(0) \subseteq f(B'_\delta(z_0))$.*

Proof. Suppose f has a pole at z_0, so there exists some $s > 0$ such that f is meromorphic on $\Omega = B_s(z_0)$ and analytic on $\Omega' = B'_s(z_0)$. Since $|f(z)| \to +\infty$ as $z \to z_0$ by Theorem 7.9(2), it is clear that f is nonconstant on Ω. Taking $0 < r < s$, we may now apply verbatim the arguments made in the first two paragraphs of the solution to Exercise 8.7 above. The proof of Part (1) derives directly from the first paragraph. The proof of Part (2) follows from the last two sentences of the first paragraph and all of the second paragraph, only we replace ϵ with an arbitrary $\delta \in (0, \epsilon)$. Noting that $\Omega_0 \setminus \{z_0\} \subseteq B'_\delta(z_0)$, we obtain the chain of implications

$$w \in A_{\alpha,\infty}(0) \;\;\Rightarrow\;\; \exists z \in \Omega_0 \setminus \{z_0\} \left(g(z) = \frac{1}{w} \right) \;\;\Rightarrow\;\; \exists z \in B'_\delta(z_0)\big(f(z) = w\big) \;\;\Rightarrow\;\; w \in B'_\delta(z_0),$$

and hence $A_{\alpha,\infty}(0) \subseteq f(B'_\delta(z_0))$ as desired. ■

Exercise 8.9 (AN4.3.5). Let f be a nonconstant analytic function on a region Ω. Show that the functions $\operatorname{Re} f$ and $\operatorname{Im} f$ can have no local maximum in Ω.

Solution. Fix $z_0 \in \Omega$, and let $\epsilon > 0$ be such that $B_\epsilon(z_0) \subseteq \Omega$. By the Open Mapping Theorem $S = f(B_\epsilon(z_0))$ is open, and since $f(z_0) = u_0 + iv_0 \in S$, there exists some $\delta > 0$ such that $B_{2\delta}(u_0 + iv_0) \subseteq S$. Then $(u_0 + \delta) + iv_0 \in S$, which implies there exists some $z \in B_\epsilon(z_0)$ such that $f(z) = (u_0 + \delta) + iv_0$, and moreover

$$(\operatorname{Re} f)(z) = u_0 + \delta > u_0 = (\operatorname{Re} f)(z_0).$$

Since $\epsilon > 0$ is arbitrary it follows that $\operatorname{Re} f$ does not have a local maximum at z_0, and therefore $\operatorname{Re} f$ has no local maximum in Ω since $z_0 \in \Omega$ is arbitrary. The argument that $\operatorname{Im} f$ has no local maximum in Ω is much that same. ■

8.2 – Möbius Transformations

Definition 8.10. *Let $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The* ***Möbius transformation*** *associated with a, b, c, d is the mapping $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by*

$$T(z) = \begin{cases} \frac{az+b}{cz+d}, & z \neq -\frac{d}{c}, \infty \\ \frac{a}{c}, & z = \infty, c \neq 0 \\ \infty, & z = \infty, c = 0 \\ \infty, & z = -\frac{d}{c}, c \neq 0 \end{cases}$$

The condition $ad - bc \neq 0$ prevents two situations from occurring: first, there is no $z \in \overline{\mathbb{C}}$ such that $T(z)$ becomes a $0/0$ indeterminate form; and second, T cannot be a constant function. The ∞/∞ form can arise if $a, c \neq 0$ and $z = \infty$, but Definition 8.10 makes clear that in such an instance we take $T(\infty) = a/c$. If we subscribe to the conventions

$$\frac{a \in \mathbb{C}}{\infty} = 0, \quad \frac{\infty}{a \in \mathbb{C}} = \infty, \quad \frac{a \in \mathbb{C}_*}{0} = \infty, \quad (a \in \mathbb{C}_*)(\infty) = \infty, \quad (a \in \mathbb{C}) + \infty = \infty,$$

then we may write simply

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d}, & z \neq \infty \\ \dfrac{a}{c}, & z = \infty \end{cases}$$

without ambiguity.

To see that the $0/0$ form is unattainable, note that if $z = \infty$, then $T(\infty) = a/c$ only if $c \neq 0$, otherwise $T(\infty) = \infty$. If $z = 0$, then $0/0$ can only arise if $b = d = 0$, which is not possible given $ad - bc \neq 0$. Suppose that $z \in \mathbb{C}_*$ is such that $az + b = 0$ and $cz + d = 0$. Then $a \neq 0$ since otherwise $a = b = 0$ in violation of $ad - bc \neq 0$, and $c \neq 0$ since otherwise $c = d = 0$ in violation of $ad - bc \neq 0$. Thus $z = -b/a$ and $z = -d/c$, which yields $b/a = d/c$ and finally $ad = bc$, again contradicting the condition $ad - bc \neq 0$!

It is clear that no permissible combination of complex numbers a, b, c, d will result in $T \equiv \infty$. On the other hand we cannot have $T \equiv z_0$ for any $z_0 \in \mathbb{C}$, since either $T(\infty) = \infty \neq z_0$ if $c = 0$, or $T(-d/c) = \infty \neq z_0$ if $c \neq 0$.

Proposition 8.11. *A Möbius transformation*

$$T(z) = \frac{az+b}{cz+d}$$

is a continuous bijection of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ that is analytic on $\overline{\mathbb{C}} \setminus \{-d/c\}$, with a simple pole at $-d/c$ and a zero of order 1 at $-b/a$.

Special kinds of Möbius transformations are **translations** $z \mapsto z + w$ for fixed $w \in \mathbb{C}$, **rotations** $z \mapsto \lambda z$ for $\lambda \in \mathbb{S}$, **dilations** $z \mapsto \rho z$ for $\rho > 1$, **contractions** $z \mapsto \rho z$ for $0 < \rho < 1$, and the **inversion** operation $z \mapsto 1/z$. Compositions of these basic transformations can construct any given Möbius transformation. For instance, for the transformation

$$T(z) = \frac{z-2}{3z+1}$$

let

$$f_1(z) = 3z, \quad f_2(z) = z + 1, \quad f_3(z) = \frac{1}{z}, \quad f_4(z) = -\frac{7}{3}z, \quad f_5(z) = z + \frac{1}{3};$$

then

$$z \xrightarrow{f_1} 3z \xrightarrow{f_2} 3z + 1 \xrightarrow{f_3} \frac{1}{3z+1} \xrightarrow{f_4} \frac{-\frac{7}{3}}{3z+1} \xrightarrow{f_5} \frac{-\frac{7}{3}}{3z+1} + \frac{1}{3} = \frac{z-2}{3z+1},$$

and thus $T = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

Recall that a function f is defined to be analytic at ∞ if it has a removable singularity there, in which case we may set

$$f(\infty) = \lim_{z \to 0} f(1/z).$$

Let f be a function with domain $\overline{\mathbb{C}}$, and let $\overline{P}(f)$ denote the set of poles of f in $\overline{\mathbb{C}}$. Recalling Definition 7.25, f is meromorphic on $\overline{\mathbb{C}}$ if it is meromorphic on $\mathbb{C}$, and has either a removable singularity or a pole at ∞. It's immediate that all Möbius transformations are meromorphic on $\overline{\mathbb{C}}$, and some are entire with pole at ∞. No Möbius transformation can be analytic on $\overline{\mathbb{C}}$, since Exercise 7.21 would then imply the transformation is a constant function.

Proposition 8.12. *Let*

$$S = \{z : az\overline{z} + \overline{\beta}z + \beta\overline{z} + c = 0\},$$

where $a, c \in \mathbb{R}$, $\beta \in \mathbb{C}$, and $\beta\overline{\beta} - ac > 0$.

1. *If $a \neq 0$, then S is a circle; and if $a = 0$, then S is a line.*
2. *Given any line L or circle C, there exist $a, c \in \mathbb{R}$ and $\beta \in \mathbb{C}$ such that $S = L$ or $S = C$.*

Proposition 8.13. *Let T be a Möbius transformation. If L is a line, then $T(L)$ is a line; and if C is a circle, then $T(C)$ is a circle.*

Exercise 8.14 (AN4.5.1). Show that if T_1 and T_2 are Möbius transformations, then $T_1 \circ T_2$ is also; and if T is a Möbius transformations, then T^{-1} is also.

Solution. Let T_1 and T_2 be Möbius transformations, so that for $i \in \{1, 2\}$ we have $T_i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$T_i(z) = \begin{cases} \frac{a_i z + b_i}{c_i z + d_i}, & z \neq -\frac{d_i}{c_i}, \infty \\ \frac{a_i}{c_i}, & z = \infty, c_i \neq 0 \\ \infty, & z = \infty, c_i = 0 \\ \infty, & z = -\frac{d_i}{c_i}, c_i \neq 0 \end{cases}$$

for $a_i, b_i, c_i, d_i \in \mathbb{C}$ such that $a_i d_i - b_i c_i \neq 0$. If we define

$$\mathbf{A}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

then $\det(\mathbf{A}_i) = a_i d_i - b_i c_i \neq 0$ for each i, and so

$$\det(\mathbf{A}_1\mathbf{A}_2) = \det(\mathbf{A}_1)\det(\mathbf{A}_2) \neq 0.$$

Letting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{bmatrix} = \mathbf{A}_1\mathbf{A}_2,$$

it follows that $ad - bc \neq 0$. We will need this result soon enough.

Suppose that $z \in \overline{\mathbb{C}}$ is such that $z \neq -d/c, \infty$. A little algebra reveals that

$$z \neq -\frac{d}{c} \quad \Leftrightarrow \quad \frac{a_2 z + b_2}{c_2 z + d_2} \neq -\frac{d_1}{c_1} \quad \Leftrightarrow \quad T_2(z) \neq -\frac{d_1}{c_1}.$$

If we suppose that $z \neq -d_2/c_2$, then $T_2(z) \neq \infty$ also, and we obtain

$$(T_1 \circ T_2)(z) = \frac{a_1\left(\dfrac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\dfrac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} = \frac{az + b}{cz + d}.$$

Suppose that $z = -d_2/c_2$. Then $c_2 \neq 0$ since $z \neq \infty$, and so $T_2(z) = \infty$. We must have $c_1 \neq 0$, since otherwise $z \neq -d/c$ implies $z \neq -d_2/c_2$. Hence

$$(T_1 \circ T_2)(z) = T_1(\infty) = \frac{a_1}{c_1} = \frac{a_1(a_2 d_2 - b_2 c_2)}{c_1(a_2 d_2 - b_2 c_2)} = \frac{a(-d_2/c_2) + b}{c(-d_2/c_2) + d} = \frac{az + b}{cz + d}$$

once again. That is,

$$T(z) = \frac{az + b}{cz + d}$$

whenever $z \neq -d/c, \infty$.

Next, suppose that $z = \infty$ and $c \neq 0$. Then either $c_1 \neq 0$ or $c_2 \neq 0$. If $c_2 \neq 0$, then

$$c \neq 0 \quad \Leftrightarrow \quad a_2 c_1 + c_2 d_1 \neq 0 \quad \Leftrightarrow \quad \frac{a_2}{c_2} \neq -\frac{d_1}{c_1},$$

and so since $a_2/c_2 \neq \infty$ as well,

$$(T_1 \circ T_2)(z) = T_1(T_2(\infty)) = T_1(a_2/c_2) = \frac{a_1(a_2/c_2) + b_1}{c_1(a_2/c_2) + d_1} = \frac{a_1 a_2 + b_1 c_2}{a_2 c_1 + c_2 d_1} = \frac{a}{c}.$$

If $c_2 = 0$, then $a = a_1 a_2$ and $c = a_2 c_1$ with $a_2 \neq 0$ (otherwise $ad - bc = 0$ would result), and so

$$(T_1 \circ T_2)(z) = T_1(T_2(\infty)) = T_1(\infty) = \frac{a_1}{c_1} = \frac{a_1 a_2}{a_2 c_1} = \frac{a}{c}$$

once again. That is, $T(z) = a/c$ if $z = \infty$ and $c \neq 0$.

Now suppose that $z = \infty$ and $c = 0$. If $c_1, c_2 \neq 0$, then $c = 0$ implies $a_2/c_2 = -d_1/c_1$, so that

$$(T_1 \circ T_2)(z) = T_1(T_2(\infty)) = T_1(a_2/c_2) = T_1(-d_1/c_1) = \infty.$$

If $c_1, c_2 = 0$, then

$$(T_1 \circ T_2)(z) = T_1(T_2(\infty)) = T_1(\infty) = \infty.$$

The case $c_1 = 0$, $c_2 \neq 0$ cannot occur: from $c = 0$ would come $d_1 = 0$, and then $a_1 d_1 - b_1 c_1 = 0$ results. Similarly, the case $c_1 \neq 0$, $c_2 = 0$ cannot occur: from $c = 0$ would come $a_2 = 0$, and then $a_2 d_2 - b_2 c_2 = 0$ results. Therefore $T(z) = \infty$ if $z = \infty$ and $c = 0$.

Finally, suppose that $z = -d/c$ and $c \neq 0$. If $d/c = d_2/c_2$ (so that $c_2 \neq 0$), then some algebra leads to $c_1(a_2 d_2 - b_2 c_2) = 0$, whence $c_1 = 0$ obtains and then

$$(T_1 \circ T_2)(z) = T_1(T_2(-d/c)) = T_1(T_2(-d_2/c_2)) = T_1(\infty) = \infty.$$

If $d/c \neq d_2/c_2$, then we must have $c_1 \neq 0$ since otherwise $d/c = d_1d_2/c_2d_1 = d_2/c_2$. Now,

$$(T_1 \circ T_2)(z) = T_1\left(\frac{a_2(-d/c) + b_2}{c_2(-d/c) + d_2}\right) = T_1(-d_1/c_1) = \infty.$$

Therefore $T(z) = \infty$ if $z = \infty$ and $c \neq 0$.

Gathering all our findings, we have:

$$(T_1 \circ T_2)(z) = \begin{cases} \frac{az+b}{cz+d}, & z \neq -\frac{d}{c}, \infty \\ \frac{a}{c}, & z = \infty, c \neq 0 \\ \infty, & z = \infty, c = 0 \\ \infty, & z = -\frac{d}{c}, c \neq 0 \end{cases}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Therefore $T_1 \circ T_2$ is a Möbius transformation.

To show that T^{-1} is also a Möbius transformation requires much the same drudgery. We'll go only as far as finding an expression for $T^{-1}(z)$. Let T be defined as in Definition 8.10. For all $z, w \in \overline{\mathbb{C}}$ we have

$$T(z) = w \quad \Leftrightarrow \quad T^{-1}(w) = z,$$

and so for $z \in \mathbb{C} \setminus \{-d/c\}$,

$$T(z) = w \quad \Rightarrow \quad \frac{az+b}{cz+d} = w \quad \Rightarrow \quad z = \frac{dw - b}{-cw + a} \quad \Rightarrow \quad T^{-1}(w) = \frac{dw - b}{-cw + a},$$

from which we may conclude that

$$T^{-1}(z) = \begin{cases} \frac{dz-b}{a-cz}, & z \neq \frac{a}{c}, \infty \\ -\frac{d}{c}, & z = \infty, c \neq 0 \\ \infty, & z = \infty, c = 0 \\ \infty, & z = \frac{a}{c}, c \neq 0 \end{cases}$$

for all $z \in \overline{\mathbb{C}}$. Note

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \quad \Rightarrow \quad \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} \neq 0,$$

as desired. ∎

Exercise 8.15 (AN4.5.2a). Find a formula for the inverse of the Möbius transformation

$$T(z) = \frac{1+z}{1-z}.$$

Solution. Using the general formula for T^{-1} near the end of the previous exercise,

$$T^{-1}(z) = \begin{cases} \frac{z-1}{z+1}, & z \neq -1, \infty \\ 1, & z = \infty \\ \infty, & z = -1 \end{cases}$$

For the following exercise we shall let $\mathbb{I} = \{z \in \mathbb{C} : \operatorname{Re}(z) = 0\}$ denote the set of imaginary numbers, and $\overline{\mathbb{I}} = \mathbb{I} \cup \{\infty\}$.

Exercise 8.16 (AN4.5.2b). Consider the Möbius transformation

$$T(z) = \frac{1+z}{1-z}.$$

Show that T maps $\mathbb{B}$ onto $\{z : \text{Re}(z) > 0\}$, $\partial\mathbb{B}$ onto $\overline{\mathbb{I}}$, and $\overline{\mathbb{C}} \setminus \overline{\mathbb{B}}$ onto $\{z : \text{Re}(z) < 0\}$.

Solution. Let $z \in \mathbb{B}$, so that $z = u + iv$ with $u^2 + v^2 < 1$. Now,

$$T(z) = \frac{(1+u)+iv}{(1-u)-iv} = \frac{1-(u^2+v^2)}{(1-u)^2+v^2} + \frac{2v}{(1-u)^2+v^2}i,$$

and since

$$\text{Re}(T(z)) = \frac{1-(u^2+v^2)}{(1-u)^2+v^2} > 0$$

it is clear that $T : \mathbb{B} \to \{z : \text{Re}(z) > 0\}$. To show the mapping is surjective, let $w = r + is$ such that $r > 0$, so $w \neq -1, \infty$. We wish to find z such that $T(z) = w$, and by the formula for T^{-1} found in the previous exercise we have

$$z = T^{-1}(w) = \frac{w-1}{w+1}.$$

Since $r > 0$,

$$0 < |w-1|^2 = r^2 + s^2 + 1 - 2r < r^2 + s^2 + 1 + 2r = |w+1|^2,$$

whence

$$|z| = \frac{|w-1|}{|w+1|} < 1$$

obtains and we conclude that $z \in \mathbb{B}$. We conclude that

$$T(\mathbb{B}) = \{z : \text{Re}(z) > 0\}.$$

Next, let $z \in \partial\mathbb{B}$, so that $z = u + iv$ with $u^2 + v^2 = 1$. If $z = 1$, then $T(1) = \infty$; and if $z \neq 1$, then

$$T(z) = \frac{(1+u)+iv}{(1-u)-iv} = \frac{1-(u^2+v^2)}{(1-u)^2+v^2} + \frac{2v}{(1-u)^2+v^2}i = \frac{2v}{(1-u)^2+v^2}i = \frac{v}{1-u}i,$$

which shows that $\text{Re}(z) = 0$ and thus $T : \partial\mathbb{B} \to \overline{\mathbb{I}}$. To show the mapping is surjective, let $w = is$ for some $s \in \mathbb{R}$. We must find $v \in [-1, 1]$ and $u \in [-1, 1)$ such that $v/(1-u) = s$. Since $u^2 + v^2 = 1$, this means finding $u \in [-1, 1)$ such that

$$\frac{\sqrt{1-u^2}}{1-u} = s.$$

With some algebra this becomes

$$(s^2+1)u^2 - 2s^2u + (s^2-1) = 0,$$

whence

$$u = \frac{2s^2 \pm \sqrt{4s^4 - 4(s^2+1)(s^2-1)}}{2(s^2+1)} = \frac{s^2-1}{s^2+1}$$

obtains (discounting the alternate solution $u = 1$). Now we have

$$z = u + iv = \frac{s^2 - 1}{s^2 + 1} + i\sqrt{1 - \left(\frac{s^2 - 1}{s^2 + 1}\right)^2},$$

with $|z| = 1$ and $T(z) = is$. We conclude that

$$T\big(\partial\mathbb{B}\big) = \bar{\mathbb{I}}.$$

The function $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a bijection by Proposition 8.11, and since T maps $\mathbb{B}$ onto $\{z : \mathrm{Re}(z) > 0\}$ and $\partial\mathbb{B}$ onto $\bar{\mathbb{I}}$, it must map $\overline{\mathbb{C}} \setminus \overline{\mathbb{B}}$ onto $\{z : \mathrm{Re}(z) < 0\}$ (in particular $T(\infty) = -1$). ∎

Exercise 8.17 (AN4.5.3). Find a Möbius transformation T such that $T(1) = 1$, $T(i) = 0$, and $T(-1) = -1$.

Solution. Let

$$T(z) = \frac{az + b}{cz + d}.$$

Values for a, b, c, and d must be found such that

$$\frac{a + b}{c + d} = 1, \quad \frac{ai + b}{ci + d} = 0, \quad \frac{-a + b}{-c + d} = -1,$$

whence

$$a + b = c + d, \quad ai + b = 0, \quad b - a = c - d.$$

The sum of the first and third equations yields $b = c$, which together with the first equation leads to $a = d$. Since $b = -ai$ by the second equation, the required condition $ad - bc \neq 0$ implies that $a \neq 0$.

Let $a = 1$, so that $b = c = -i$ and $d = 1$. Then

$$T(z) = \frac{z - i}{1 - iz}$$

is easily verified to be such that $T(1) = 1$, $T(i) = 0$, and $T(-1) = -1$. ∎

Exercise 8.18 (AN4.5.5). Let f be an injective meromorphic function on $\mathbb{C}$. Show that f is a Möbius transformation by establishing the following.

1. f has at most one pole in $\mathbb{C}$, and thus ∞ is an isolated singularity of f.
2. $f(\mathbb{B})$ and $f(A_{1,\infty}(0))$ are disjoint open sets in $(\overline{\mathbb{C}}, \hat{d})$.
3. f has a pole or removable singularity at ∞, and thus f is meromorphic on $\overline{\mathbb{C}}$.
4. f has exactly one pole in $\overline{\mathbb{C}}$.
5. If ∞ is the pole of f, then f is a degree 1 polynomial; and if $z_p \in \mathbb{C}$ is the pole of f, then the function $g = 1/f$, with $g(z_p) = 0$, is analytic at z_p with $g'(z_p) \neq 0$.
6. f has a simple pole at z_p.
7. There exists some $c \in \mathbb{C}$ such that

   $$f(z) - \frac{\mathrm{res}(f, z_p)}{z - z_p} \equiv c$$

   on $\mathbb{C} \setminus \{z_p\}$, and hence f is a Möbius transformation.

Solution.
Part (1). Suppose f has distinct poles $z_1, z_2 \in \mathbb{C}$, so there exists some $r > 0$ such that f is analytic and $|f| > 1$ on $B'_r(z_1)$ and $B'_r(z_2)$. By Proposition 8.8 we can find $0 < \delta < r$ sufficiently small so that $B'_\delta(z_1) \cap B'_\delta(z_2) = \varnothing$, and there exist $\alpha, \beta > 0$ such that $A_{\alpha,\infty}(0) \subseteq f(B'_\delta(z_1))$ and $A_{\beta,\infty}(0) \subseteq f(B'_\delta(z_2))$. Thus, if $w \in \mathbb{C}$ with $|w| > \max\{\alpha, \beta\}$, then there exist $\zeta_1 \in B'_\delta(z_1)$ and $\zeta_2 \in B'_\delta(z_2)$ for which $f(\zeta_1) = f(\zeta_2) = w$, violating the hypothesized injectivity of f. We conclude that f can have at most one pole z_p in $\mathbb{C}$, which implies that f (being meromorphic) is analytic on $A_{s,\infty}(0)$ for $s \geq |z_p|$, and therefore f has an isolated singularity at ∞.

Part (2). First, if f has a pole at $z_p \in \mathbb{C}$, then we define $f(z_p) = \infty$ and thereby regard f as a continuous injective map $(\mathbb{C}, d) \to (\overline{\mathbb{C}}, \bar{d})$ that is analytic on $\mathbb{C} \setminus \{z_p\}$. It is immediate that $f(\mathbb{B})$ and $f(A_{1,\infty}(0))$ are disjoint sets in $\overline{\mathbb{C}}$. It remains to show they are open sets in $\overline{\mathbb{C}}$ (with respect to the chordal metric $\bar{d}$ of course).

If $z_p \in \mathbb{B}$, then f is meromorphic on $\mathbb{B}$, and also nonconstant on $\mathbb{B}$ by virtue of being injective, and so by Exercise 8.7 (with $P(f) = \{z_p\}$, $\Omega = \mathbb{B}$, and $\Omega' = \mathbb{B} \setminus \{z_p\}$) we conclude that $f(\mathbb{B})$ is open in $(\overline{\mathbb{C}}, \bar{d})$. If $z_p \notin \mathbb{B}$, then f is nonconstant analytic on $\mathbb{B}$, so that $f(\mathbb{B})$ is open in $(\mathbb{C}, d)$ by the Open Mapping Theorem, hence open in $(\overline{\mathbb{C}}, \bar{d})$ by Proposition 6.39. The same arguments also show $f(A_{1,\infty}(0))$ to be open in $(\overline{\mathbb{C}}, \bar{d})$ whether or not z_p is in $A_{1,\infty}(0)$.

Part (3). Since f has an isolated singularity at ∞, there exists some $r > 1$ such that f is analytic on $A_{r,\infty}(0)$. Suppose f has an essential singularity at ∞. By definition $h(z) = f(1/z)$ has an essential singularity at 0 and is analytic on $B'_{1/r}(0)$, so by the Casorati-Weierstrass Theorem

$$S := f(A_{r,\infty}(0)) = h(B'_{1/r}(0))$$

is dense in $\mathbb{C}$.

On the other hand we may choose $z_0 \in \mathbb{B}$ and $\epsilon > 0$ such that $B_\epsilon(z_0) \subseteq \mathbb{B}$ and f is analytic on $B_\epsilon(z_0)$. Now, $B \subseteq f(\mathbb{B})$ and $S \subseteq f(A_{1,\infty}(0))$, so $B \cap S = \varnothing$ by Part (2), and since B is open in $\mathbb{C}$ by the Open Mapping Theorem, we are led to the contradictory conclusion that S is not dense in $\mathbb{C}$.

It must be that the isolated singularity possessed by f at ∞ cannot be an essential singularity, and therefore f has a pole or removable singularity there. That is, f is either analytic at ∞ or f has a pole at ∞, and since f is meromorphic on $\mathbb{C}$ by hypothesis, we conclude that f is meromorphic on $\overline{\mathbb{C}}$.

Part (4). Suppose that f has more than one pole in $\overline{\mathbb{C}}$. In light of Part (1) this can only occur if f has a pole at ∞ and as some $z_p \in \mathbb{C}$. By Proposition 8.8 there is some $\epsilon_1 > 0$ such that, for any $0 < \delta < \epsilon_1$, there exists some $\alpha_1 > 0$ such that $A_{\alpha_1,\infty}(0) \subseteq f(B'_\delta(z_p))$. Also, since $h(z) = f(1/z)$ has a pole at 0, there is some $\epsilon_2 > 0$ such that, for any $0 < \delta < \epsilon_2$, there exists some $\alpha_2 > 0$ such that

$$A_{\alpha_2,\infty}(0) \subseteq h(B'_\delta(0)) = f(A_{1/\delta,\infty}(0)).$$

Choose $\delta > 0$ sufficiently small so that $\delta < \min\{\epsilon_1, \epsilon_2\}$ and $1/\delta > |z_p| + 1$. Then, since $\delta < 1$,

$$z \in A_{1/\delta,\infty}(0) \;\Rightarrow\; |z| > \frac{1}{\delta} \;\Rightarrow\; |z| > |z_p| + \delta \;\Rightarrow\; z \notin B'_\delta(z_p),$$

and

$$z \in B'_\delta(z_p) \;\Rightarrow\; 0 < |z - z_p| < \delta \;\Rightarrow\; |z| < |z_p| + \delta < \frac{1}{\delta} \;\Rightarrow\; z \notin A_{1/\delta,\infty}(0),$$

and hence $B'_\delta(z_p) \cap A_{1/\delta,\infty}(0) = \varnothing$. Let $\alpha = \max\{\alpha_1, \alpha_2\}$, and choose $w \in \mathbb{C}$ such that $|w| > \alpha$. That is, $w \in A_{\alpha,\infty}(0)$, and since

$$A_{\alpha,\infty}(0) \subseteq f(B'_\delta(z_p)) \quad \text{and} \quad A_{\alpha,\infty}(0) \subseteq f(A_{1/\delta,\infty}(0)),$$

there exist $z_1 \in B'_\delta(z_p)$ and $z_2 \in A_{1/\delta,\infty}(0)$ such that $f(z_1) = f(z_2) = w$. This contradicts the injectivity of f, and therefore f can have at most one pole in $\overline{\mathbb{C}}$.

Next, suppose that f has no pole in $\overline{\mathbb{C}}$. Then f is entire with removable singularity at ∞, which is to say f is analytic on $\overline{\mathbb{C}}$, and then by Exercise 7.21 we find that f is constant—again violating injectivity. So f must have at least one pole in $\overline{\mathbb{C}}$, and therefore f must have exactly one pole in $\overline{\mathbb{C}}$.

Part (5). Suppose ∞ is the pole of f. Then f is entire and has nonessential singularity at ∞, and by Exercise 7.23 it follows that f is a polynomial function. Since f is injective, it is nonconstant, and so $\deg(f) \geq 1$. By the Fundamental Theorem of Algebra there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$, where $\operatorname{ord}(f, z_0) \in \mathbb{N}$ by Proposition 5.15. Indeed, the first two parts of Lemma 8.1 lead to the conclusion that $\operatorname{ord}(f, z_0) = 1$, and so there exists some analytic function $q : \mathbb{C} \to \mathbb{C}$ such that $q(z_0) \neq 0$ and $f(z) = (z - z_0)q(z)$ for all $z \in \mathbb{C}$.

It is necessary to confirm that q is a polynomial function. Since $\deg(f) \geq 1$, we have

$$q(z) = \frac{f(z)}{z - z_0} = \frac{a_n z^n + \cdots + a_1 z + a_0}{z - z_0}$$

for all $z \in \mathbb{C} \setminus \{z_0\}$, where $n \geq 1$ and $a_n \neq 0$. Clearly q has an isolated singularity at ∞, and so $h(z) = q(1/z)$ has an isolated singularity at 0. If $n = 1$, then

$$\lim_{z \to 0} h(z) = \lim_{z \to 0} \frac{a_1 z^{-1} - a_0}{z^{-1} - z_0} = \lim_{z \to 0} \frac{a_1 + a_0 z}{1 - z_0 z} = a_1,$$

so that h has a removable singularity at 0, and hence q has a removable singularity at ∞ by Definition 7.8. If $n > 1$, then

$$\lim_{z \to 0} |h(z)| = \lim_{z \to 0} \left| \frac{a_n z^{-n} + \cdots + a_1 z^{-1} + a_0}{z^{-1} - z_0} \right| = \lim_{z \to 0} \left| \frac{a_n + a_{n-1} z + \cdots + a_1 z^{n-1} + a_0 z^n}{z^{n-1}(1 - z_0 z)} \right| = +\infty$$

since $a_n \neq 0$, so that h has a pole at 0 by Theorem 7.10(2), and hence q has a pole at ∞. Therefore q has a nonessential singularity at ∞, and since q is entire, Exercise 7.23 implies that q is a polynomial function.

Now, if $q(z_1) = 0$ for some $z_1 \neq z_0$, then

$$f(z_1) = (z_1 - z_0)q(z_1) = 0 = f(z_0),$$

which is impossible since f is injective. Therefore q is a polynomial function with no zeros in $\mathbb{C}$, so by the Fundamental Theorem of Algebra it follows that q is a (nonzero) constant function. That is, $q \equiv c$ for some $c \in \mathbb{C}_*$, so that $f(z) = c(z - z_0)$ for all $z \in \mathbb{C}$ and we conclude that f is a degree 1 polynomial function.

Next, suppose $z_p \in \mathbb{C}$ is the pole of f. Since $|f(z)| \to +\infty$ as $z \to z_p$ by Theorem 7.10(1), there exists some $r > 0$ such that f is analytic and $|f| > 0$ on $B'_r(z_0)$. The function $g = 1/f$ is thus analytic on $B'_r(z_p)$, and since

$$\lim_{z \to z_p} g(z) = 0 = g(z_p),$$

it's seen that g is continuous on $B_r(z_p)$ and hence analytic there by Corollary 4.22. The injectivity of f on $B'_r(z_p)$ implies that g is injective there, and hence g is injective (in particular nonconstant) on $B_r(z_p)$. If $\Omega' \subseteq B_r(z_p)$ and W_0 are defined as in Lemma 8.1(2), then the lemma together with the injectivity of g imply that $g : \Omega' \to W_0 \setminus \{0\}$ is a bijection, whence it follows that $g : \Omega' \cup \{z_p\} \to W_0$ is a bijection and therefore $g'(z_p) \neq 0$ by Lemma 8.1(3).

Part (6). Suppose ∞ is the pole of f. Then f is a degree 1 polynomial, so that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. Let $h(z) = f(1/z)$. Then

$$\lim_{z \to 0} zh(z) = \lim_{z \to 0}(a + bz) = a \in \mathbb{C}_*,$$

which by Theorem 7.10(1) shows that h has a simple pole at 0, and hence f has a simple pole at ∞.

Now suppose $z_p \in \mathbb{C}$ is the pole of f, and define function g as in Part (5). Then g is nonconstant analytic on $B_r(z_p)$ for some $r > 0$, and since $g'(z_p) \neq 0$, Proposition 5.11 implies that $\operatorname{ord}(g, z_p) = 1$. Thus there exists some analytic $\psi : B_r(z_p) \to \mathbb{C}$ such that $\psi(z_p) \neq 0$ and $g(z) = (z - z_p)\psi(z)$ for all $z \in B_r(z_p)$. Since $g \neq 0$ on $B'_r(z_p)$ implies $\psi \neq 0$ on $B'_r(z_p)$, it follows that

$$(z - z_p)f(z) = \frac{1}{\psi(z)}$$

for $z \in B'_r(z_p)$, and so

$$\lim_{z \to z_p} (z - z_p)f(z) = \lim_{z \to z_p} \frac{1}{\psi(z)} = \frac{1}{\psi(z_p)} \in \mathbb{C}_*.$$

Therefore f has a simple pole at z_p.

Part (7). If ∞ is the pole of f, then f is a degree 1 polynomial and therefore a Möbius transformation. Suppose $z_p \in \mathbb{C}$ is the pole of f, so that f has a removable singularity at ∞ and therefore

$$\lim_{z \to 0} f(1/z) = a$$

for some $a \in \mathbb{C}$. By Theorem 7.3,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_p)^n$$

for all $z \in \mathbb{C} \setminus \{z_p\}$; but since z_p is a simple pole, by Definition 7.7(2) we obtain

$$f(z) = \frac{\operatorname{res}(f, z_p)}{z - z_p} + \sum_{n=0}^{\infty} a_n (z - z_p)^n$$

on $\mathbb{C} \setminus \{z_p\}$, and hence

$$\varphi(z) := f(z) - \frac{\operatorname{res}(f, z_p)}{z - z_p} = \sum_{n=0}^{\infty} a_n (z - z_p)^n$$

on $\mathbb{C} \setminus \{z_p\}$. Defining $\varphi(z_p) = a_0$, it follows by Corollary 4.22 that φ is an entire function. Moreover,

$$\lim_{z \to 0} \varphi(1/z) = \lim_{z \to 0} \left[f(1/z) - \frac{\operatorname{res}(f, z_p)}{z^{-1} - z_p} \right] = \lim_{z \to 0} \left[f(1/z) - \frac{z \operatorname{res}(f, z_p)}{1 - z_p z} \right] = a + 0 = a,$$

which shows that φ has a removable singularity at ∞; that is, φ is analytic on $\overline{\mathbb{C}}$, so φ is a constant function by Exercise 7.21. In fact it is easy to see that $\varphi \equiv a = a_0$. We now have

$$f(z) - \frac{\operatorname{res}(f, z_p)}{z - z_p} = a$$

for all $z \in \mathbb{C} \setminus \{z_p\}$, or equivalently

$$f(z) = \frac{az + [\operatorname{res}(f, z_p) - az_p]}{z - z_p}$$

on $z \in \mathbb{C} \setminus \{z_p\}$. Since

$$(a)(-z_p) - [\operatorname{res}(f, z_p) - az_p](1) = -\operatorname{res}(f, z_p) \neq 0$$

shows the condition $ad - bc \neq 0$ to be satisfied, we conclude that f is a Möbius transformation. ■

8.3 – Conformal Mappings

Definition 8.19. *An analytic function $f : \Omega \to \mathbb{C}$ with $f'(z) \neq 0$ for all $z \in \Omega$ is a* ***conformal mapping*** *(or* ***conformal map****).*

In some of the literature there is another definition for the term "conformal mapping" that is stricter: in addition to the properties of f given in Definition 8.19, there is the additional requirement that f be injective on Ω. The two definitions are not equivalent! However, the next proposition does at least guarantee that a conformal mapping as defined here is *locally* injective.

Proposition 8.20. *Let f be analytic at z_0. Then $f'(z_0) \neq 0$ if and only if f is injective in a neighborhood of z_0.*

Proof. Suppose $f'(z_0) \neq 0$. Then there exists some $r > 0$ such that f is analytic on $B = B_r(z_0)$, and f must be nonconstant on B (otherwise $f'(z_0) = 0$). Using Lemma 8.1 we can find a region $\Omega \subseteq B$ containing z_0 such that, if $S = f(\Omega)$, then $f : \Omega \to S$ is a bijection. Therefore f is injective on Ω.

Next, suppose f is injective in a neighborhood Ω of z_0. This immediately implies that f is nonconstant on Ω, which we may take to be a region by passing to an open disc $B_r(z_0) \subseteq \Omega$ if necessary. The set $\Omega \cup \{z_0\}$ in Lemma 8.1 is an open set in Ω containing z_0, with the second part of the lemma (along with our hypothesis) implying that $f : \Omega' \cup \{z_0\} \to W_0$ is a surjective one-to-one map, and the third part in turn implying that $f'(z_0) \neq 0$. ■

By the Open Mapping Theorem, local injectivity implies local bijectivity between open sets in $\mathbb{C}$, and so it follows by the Inverse Function Theorem that a conformal mapping f is locally a diffeomorphism.

Definition 8.21. *Suppose f is defined on $B_r(z_0)$ such that $f(z) \neq f(z_0)$ for all $z \in B'_r(z_0)$. If there exists some $\lambda \in \mathbb{S}$ such that*

$$\lim_{\epsilon \to 0^+} \frac{f(z_0 + \epsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \epsilon e^{i\theta}) - f(z_0)|} = \lambda e^{i\theta},$$

then f ***preserves angles*** *at z_0.*

Theorem 8.22. *Suppose f is analytic at z_0. Then f preserves angles at z_0 if and only if $f'(z_0) \neq 0$.*

Proof. Suppose that $f'(z_0) \neq 0$. By Proposition 8.20 there exists some $r > 0$ such that $f : B_r(z_0) \to \mathbb{C}$ is an injective analytic function, implying $f(z) \neq f(z_0)$ for all $z \in B'_r(z_0)$ as required by Definition 8.21. Fix $\theta \in \mathbb{R}$, and define the functions

$$\varphi(\epsilon) = \frac{f(z_0 + \epsilon e^{i\theta}) - f(z_0)}{|f(z_0 + \epsilon e^{i\theta}) - f(z_0)|}, \quad \psi(\epsilon) = \frac{|f(z_0 + \epsilon e^{i\theta}) - f(z_0)|}{|\epsilon e^{i\theta}|}, \quad \omega(\epsilon) = \frac{|\epsilon e^{i\theta}|}{\epsilon e^{i\theta}},$$

for $\epsilon > 0$. Observe that both $\psi(\epsilon)$ and $\omega(\epsilon)$ are nonzero for $0 < \epsilon < r$. Let $H = \varphi\psi\omega$. Now, from

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

it follows that

$$\lim_{\epsilon\to 0^+} H(\epsilon) = \lim_{\epsilon\to 0^+}[\varphi(\epsilon)\psi(\epsilon)\omega(\epsilon)] = \lim_{\epsilon\to 0^+} \frac{f(z_0+\epsilon e^{i\theta}) - f(z_0)}{\epsilon e^{i\theta}} = f'(z_0),$$

and also

$$\lim_{\epsilon\to 0^+} \frac{1}{\psi(\epsilon)} = \frac{1}{|f'(z_0)|}$$

since $f'(z_0) \neq 0$. In addition,

$$\lim_{\epsilon\to 0^+} \frac{1}{\omega(\epsilon)} = \lim_{\epsilon\to 0^+} \frac{\epsilon e^{i\theta}}{|\epsilon e^{i\theta}|} = \lim_{\epsilon\to 0^+} e^{i\theta} = e^{i\theta}.$$

Finally,

$$\lim_{\epsilon\to 0^+} \varphi(\epsilon) = \lim_{\epsilon\to 0^+} \left[H(\epsilon)\cdot\frac{1}{\psi(\epsilon)}\cdot\frac{1}{\omega(\epsilon)}\right] = f'(z_0)\cdot\frac{1}{|f'(z_0)|}\cdot e^{i\theta}.$$

That is, for all $\theta \in \mathbb{R}$,

$$\lim_{\epsilon\to 0^+} \frac{f(z_0+\epsilon e^{i\theta}) - f(z_0)}{|f(z_0+\epsilon e^{i\theta}) - f(z_0)|} = \frac{f'(z_0)}{|f'(z_0)|}e^{i\theta},$$

where $\lambda = f'(z_0)/|f'(z_0)|$ is unimodular. Therefore f preserves angles at z_0. ■

By Theorem 8.22 it is immediate that a conformal mapping preserves angles at all points in its domain.

Example 8.23. The exponential function is clearly a conformal mapping on $\mathbb{C}$. Also, since Proposition 8.11 makes clear that a Möbius transformation T is injective on $\mathbb{C} \setminus P(T)$, by Proposition 8.20 it follows that T is a conformal mapping on $\mathbb{C} \setminus P(T)$; that is, Möbius transformations are conformal mappings on their region of analyticity in $\mathbb{C}$. ■

8.4 – Analytic Maps Between Discs

Proposition 8.24. *For any $a \in \mathbb{B}$ define $\varphi_a : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by*

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z},$$

with $\varphi_a(\infty) = -1/\bar{a}$ and $\varphi_a(-1/\bar{a}) = \infty$.[8] *Then the following hold.*

1. *φ_a is a Möbius transformation.*
2. *φ_a is analytic on a region containing $\overline{\mathbb{B}}$.*
3. *$\varphi_a^{-1} = \varphi_{-a}$.*
4. *$\varphi_a(\mathbb{B}) = \mathbb{B}$ and $\varphi_a(\partial\mathbb{B}) = \partial\mathbb{B}$.*
5. *For all $z \in \mathbb{C} \setminus \{1/\bar{a}\}$,*

$$\varphi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

The following proposition is a generalization of Schwarz's Lemma, which was introduced in §5.3 and is instrumental in the proof.

Theorem 8.25 (**Schwarz-Pick Theorem**)**.** *Let $f : \mathbb{B} \to \mathbb{B}$ be analytic. Then for any $a, z \in \mathbb{B}$*

$$\left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le |\varphi_a(z)| \tag{8.4}$$

and

$$|f'(a)| \le \frac{1-|f(a)|^2}{1-|a|^2}. \tag{8.5}$$

If equality holds in (8.4) for some $z \ne a$, or if equality holds in (8.5), then f is a Möbius transformation; in either case, there exists some $\lambda \in \mathbb{S}$ such that $f = \varphi_{-f(a)} \circ \lambda\varphi_a$, and hence

$$f(z) = \frac{\lambda\varphi_a(z) + f(a)}{1 + \overline{f(a)}\lambda\varphi_a(z)}$$

for all $z \in \mathbb{B}$.

Theorem 8.26. *If $f : \mathbb{B} \to \mathbb{B}$ is an analytic bijection, then $f = \lambda\varphi_a$ for some $\lambda \in \partial\mathbb{B}$ and $a \in \mathbb{B}$.*

Certain aspects of the proof of the following theorem bear a resemblance to the proof of Proposition 5.7, though the latter includes somewhat more detail.

Theorem 8.27. *Suppose f is analytic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$, and $|f| = 1$ on $\partial\mathbb{B}$. Then either $f \equiv \lambda$ for some $\lambda \in \partial\mathbb{B}$, or there exist $a_1, \ldots, a_n \in \mathbb{B}$, $k_1, \ldots, k_n \in \mathbb{N}$, and $\lambda \in \partial\mathbb{B}$ such that $f = \lambda\varphi_{a_1}^{k_1} \cdots \varphi_{a_n}^{k_n}$; that is,*

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z-a_j}{1-\bar{a}_j z}\right)^{k_j} \tag{8.6}$$

for all $z \in \overline{\mathbb{B}}$.

[8] By convention we take $-1/\bar{a} = \infty$ if $a = 0$.

Proof. Let $Z(f) = \{z \in \mathbb{B} : f(z) = 0\}$, which in fact equals the set of zeros for f on $\overline{\mathbb{B}}$. Since f is continuous on $\overline{\mathbb{B}}$ and $f(z) \neq 0$ for every $z \in \partial\mathbb{B}$, it is clear that $Z(f)$ has no limit point in $\partial\mathbb{B}$ and f is not identically zero on $\mathbb{B}$. The latter observation, together with the Identity Theorem, implies $Z(f)$ has no limit point in $\mathbb{B}$, so that $Z(f)$ has no limit point on the compact set $\overline{\mathbb{B}}$ and therefore must be a finite set.

Suppose that $Z(f) = \varnothing$. By the Maximum Principle either $|f| < 1$ on $\mathbb{B}$ or f is constant, and by the Minimum Principle (since $f \neq 0$ on $\mathbb{B}$) either $|f| > 1$ on $\mathbb{B}$ or f is constant. The only conclusion is that f is constant on $\mathbb{B}$, and since f is continuous on $\overline{\mathbb{B}}$ and $|f| = 1$ on $\partial\mathbb{B}$, it follows that $f \equiv \lambda$ on $\overline{\mathbb{B}}$ for some $\lambda \in \partial\mathbb{B}$.

Suppose that $Z(f) = \{a_1, \ldots, a_n\}$ for some $n \geq 1$. Let $\operatorname{ord}(f, a_j) = k_j$ for each $1 \leq j \leq n$, where $k_j \in \mathbb{N}$ by Proposition 5.15. Define $\psi : \overline{\mathbb{B}} \to \mathbb{C}$ by

$$\psi(z) = \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j},$$

which is analytic on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$ since $|a_j| < 1$ implies $1/\overline{a}_j \notin \overline{\mathbb{B}}$ for each j. Now, f/ψ is analytic on $\mathbb{B} \setminus Z(f)$ and continuous on $\overline{\mathbb{B}} \setminus Z(f)$, with

$$(f/\psi)(z) = f(z) \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j} \right)^{k_j} = \frac{f(z)}{\varphi_{a_1}^{k_1}(z) \cdots \varphi_{a_n}^{k_n}(z)}$$

for all $z \in \overline{\mathbb{B}} \setminus Z(f)$. For each j there exists analytic $\psi_j : \mathbb{B} \to \mathbb{C}$ with $\psi_j(a_j) \neq 0$ such that $f(z) = (z - a_j)^{k_j} \psi_j(z)$ for all $z \in \mathbb{B}$, and so

$$\begin{aligned}
\lim_{z \to a_j} (f/\psi)(z) &= \lim_{z \to a_j} \left[(z - a_j)^{k_j} \psi_j(z) \prod_{\ell=1}^{n} \left(\frac{1 - \overline{a}_\ell z}{z - a_\ell} \right)^{k_\ell} \right] \\
&= \lim_{z \to a_j} \left[(1 - \overline{a}_j z)^{k_j} \psi_j(z) \prod_{\ell \neq j} \left(\frac{1 - \overline{a}_\ell z}{z - a_\ell} \right)^{k_\ell} \right] \\
&= \psi_j(a_j)(1 - |a_j|^2)^{k_j} \prod_{\ell \neq j} \left(\frac{1 - |a_\ell|^2}{a_j - a_\ell} \right)^{k_\ell} \in \mathbb{C}_*.
\end{aligned}$$

Defining

$$(f/\psi)(a_j) = \lim_{z \to a_j} (f/\psi)(z) \neq 0$$

for each j makes f/ψ continuous at each a_j—indeed continuous on $B_r(a_j)$ and analytic on $B'_r(a_j)$ for sufficiently small r—and thus f/ψ is analytic at each a_j by Corollary 4.22. Now f/ψ is continuous on $\overline{\mathbb{B}}$ and analytic on $\mathbb{B}$, and also

$$|(f/\psi)(z)| = \frac{|f(z)|}{|\varphi_{a_1}(z)|^{k_1} \cdots |\varphi_{a_n}(z)|^{k_n}} = 1$$

for any $z \in \partial\mathbb{B}$, since $|f| = 1$ on $\partial\mathbb{B}$ and $\varphi_{a_j} : \partial\mathbb{B} \to \partial\mathbb{B}$ by Proposition 8.24(4). By the Maximum Principle either $f/\psi < 1$ on $\mathbb{B}$ or f/ψ is constant. Also, since $|f/\psi| \neq 0$ on $\mathbb{B}$, by the Minimum Principle either $f/\psi > 1$ on $\mathbb{B}$ or f/ψ is constant. Therefore f/ψ is constant such that $|f/\psi| = 1$ on $\overline{\mathbb{B}}$; that is, there exists some $\lambda \in \partial\mathbb{B}$ such that $f/\psi \equiv \lambda$, whence (8.6) results. ∎

Proposition 8.28. *Suppose f is analytic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$, and nonvanishing on $\partial\mathbb{B}$. If $Z(f) = \{a_1, \ldots, a_n\} \neq \varnothing$ and $\operatorname{ord}(f, a_j) = k_j$ for each $1 \leq j \leq n$, then there is a function g that is analytic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$, and nonvanishing on $\overline{\mathbb{B}}$ such that*

$$f(z) = g(z) \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}$$

for all $z \in \overline{\mathbb{B}}$.

Proof. Define $\psi : \overline{\mathbb{B}} \to \mathbb{C}$ by

$$\psi(z) = \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}.$$

As shown in the proof of Theorem 8.27, the function $f/\psi : \overline{\mathbb{B}} \setminus Z(f) \to \mathbb{C}$ can be extended to include $Z(f)$ so as to construct a function $g : \overline{\mathbb{B}} \to \mathbb{C}$ that is analytic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$, and has no zeros on $\overline{\mathbb{B}}$. Specifically we define

$$g(z) = \begin{cases} (f/\psi)(z), & z \in \overline{\mathbb{B}} \setminus Z(f) \\ \lim\limits_{z \to a_j} (f/\psi)(z), & z = a_j. \end{cases}$$

If $z \in \overline{\mathbb{B}} \setminus Z(f)$, then

$$g(z)\psi(z) = \frac{f(z)}{\psi(z)} \cdot \psi(z) = f(z);$$

and if $z = a_j$ for any $1 \leq j \leq n$, then

$$g(a_j)\psi(a_j) = g(a_j) \cdot 0 = 0 = f(a_j).$$

Therefore $f = g\psi$ on $\overline{\mathbb{B}}$. ■

Exercise 8.29 (AN4.6.3). Show that if $f : \mathbb{B} \to \mathbb{B}$ is an analytic map with at least two fixed points, then $f(z) = z$ for all $\mathbb{B}$.

Solution. Suppose that $f : \mathbb{B} \to \mathbb{B}$ is analytic, and $a, b \in \mathbb{B}$ are such that $a \neq b$, $f(a) = a$, and $f(b) = b$. Since

$$\left| \frac{f(b) - f(a)}{1 - \overline{f(a)} f(b)} \right| = \left| \frac{b - a}{1 - \overline{a} b} \right| = |\varphi_a(b)|,$$

by the Schwarz-Pick Theorem there exists some $\lambda \in \partial\mathbb{B}$ such that

$$f = \varphi_{-f(a)} \circ \lambda\varphi_a = \varphi_{-a} \circ \lambda\varphi_a$$

on $\mathbb{B}$; that is, $\varphi_{-a} \circ \lambda\varphi_a$ is analytic on $\mathbb{B}$, and

$$f(z) = \frac{\lambda\varphi_a(z) + a}{1 + \overline{a}\lambda\varphi_a(z)} = \frac{\lambda(z - a) + a(1 - \overline{a}z)}{(1 - \overline{a}z) + \lambda\overline{a}(z - a)}$$

for all $z \in \mathbb{B}$. Now, since $f(b) = b$, we obtain

$$\frac{\lambda(b - a) + a(1 - \overline{a}b)}{(1 - \overline{a}b) + \lambda\overline{a}(b - a)} = b,$$

which with some algebra (including factoring by grouping) leads to

$$(\lambda - 1)(b - a)(1 - \overline{a}b) = 0.$$

Since $a \neq b$ and $|\overline{a}b| < 1$, it follows that $\lambda = 1$. Thus $f = \varphi_{-a} \circ \varphi_a$ on $\mathbb{B}$, and then by Proposition 8.24(3) we obtain $f = \varphi_a^{-1} \circ \varphi_a$ on $\mathbb{B}$. That is, $f(z) = \varphi_a^{-1}(\varphi_a(z)) = z$ for all $z \in \mathbb{B}$. ■

Exercise 8.30 (AN4.6.4a). Characterize the entire functions f such that $|f| = 1$ on $\partial\mathbb{B}$.

Solution. Suppose that f is a nonconstant entire function such that $|f| = 1$ on $\partial\mathbb{B}$. By Theorem 8.27 there exist $a_1, \ldots, a_n \in \mathbb{B}$, $k_1, \ldots, k_n \in \mathbb{N}$, and $\lambda \in \partial\mathbb{B}$ such that

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}$$

for all $z \in \overline{\mathbb{B}}$. Let $\Omega = \mathbb{C} \setminus \{1/\overline{a}_j : 1 \leq j \leq n\}$, and define $\omega : \Omega \to \mathbb{C}$ by

$$\omega(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}$$

for all $z \in \Omega$. Since ω is analytic on Ω, $\overline{\mathbb{B}} \subseteq \Omega$, and $f - \omega \equiv 0$ on $\overline{\mathbb{B}}$, by the Identity Theorem it follows that $f - \omega \equiv 0$ on Ω. That is,

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j} = \lambda \varphi_{a_1}^{k_1}(z) \cdots \varphi_{a_n}^{k_n}(z)$$

for all $z \in \Omega$, where there must be removable singularities at each $1/\overline{a}_j$.[9]

Fix $1 \leq j \leq n$. If $a_j \neq 0$,

$$\lim_{z \to 0} \varphi_{a_j}(1/z) = \lim_{z \to 0} \frac{1/z - a_j}{1 - \overline{a}_j/z} = \lim_{z \to 0} \frac{1 - a_j z}{z - \overline{a}_j} = -\frac{1}{\overline{a}_j} \in \mathbb{C}_*,$$

and if $a_j = 0$,

$$\lim_{z \to 0} |\varphi_{a_j}(1/z)| = \lim_{z \to 0} \frac{1}{|z|} = +\infty.$$

Hence either $\lim_{z \to 0} f(1/z) \in \mathbb{C}_*$ if $a_j \neq 0$ for all j, or $\lim_{z \to 0} |f(1/z)| = +\infty$ if $a_j = 0$ for some j. That is, f has either a removable singularity or a pole at ∞, and so f must be a polynomial function by Exercise 7.23. In order for

$$f(z) = \frac{\lambda(z - a_1)^{k_1} \cdots (z - a_n)^{k_n}}{(1 - \overline{a}_1 z)^{k_1} \cdots (1 - \overline{a}_n z)^{k_n}}$$

to be a polynomial on $\mathbb{C}$, the denominator must be a constant. In fact we must have

$$(1 - \overline{a}_1 z)^{k_1} \cdots (1 - \overline{a}_n z)^{k_n} = 1 + (\text{higher order terms}) \equiv 1,$$

which requires that $n = 1$ and $a_1 = 0$. Thus, if f is nonconstant, we have

$$f(z) = \lambda z^k,$$

where $k = k_1 \geq 1$. If f is constant, then $f \equiv \lambda$ for some unimodular λ since $|f| = 1$ on $\partial\mathbb{B}$.

[9]It might be noticed that this already presents difficulties if $a_j \neq 0$ for some j.

Therefore entire functions f such that $|f| = 1$ on $\partial\mathbb{B}$ may be characterized by the formula $f(z) = \lambda z^k$ for some $\lambda \in \partial\mathbb{B}$ and $k \geq 0$. ■

Exercise 8.31 (AN4.6.4b). Characterize the functions f that are meromorphic on $\mathbb{C}$ such that $|f| = 1$ on $\partial\mathbb{B}$.

Solution. Let $z_1, \ldots, z_n$ be the zeros of f in $\mathbb{B}$, with $k_j = \operatorname{ord}(f, z_j)$ for each $1 \leq j \leq n$. Also let $w_1, \ldots, w_m$ be the poles of f in $\mathbb{B}$, with $l_j = \operatorname{ord}(f, w_j)$ for each $1 \leq j \leq m$, so that

$$\lim_{z \to w_j} (z - w_j)^{l_j} f(z) = b_j \in \mathbb{C}_*.$$

It follows that $f\varphi_{w_j}^{l_j}$ has a removable singularity at w_j:

$$\lim_{z \to w_j} (f\varphi_{w_j}^{l_j})(z) = \lim_{z \to w_j} \frac{(z - w_j)^{l_j} f(z)}{(1 - \overline{w}_j z)^{l_j}} = \frac{b_j}{(1 - |w_j|^2)^{l_j}} \in \mathbb{C}_*,$$

where of course $|w_j| < 1$. Define $h : \overline{\mathbb{B}} \to \mathbb{C}$ by

$$h(z) = f(z) \prod_{j=1}^{m} \varphi_{w_j}^{l_j}(z)$$

for all $z \in \overline{\mathbb{B}}$, with

$$h(w_j) = \frac{b_j}{(1 - |w_j|^2)^{l_j}} \prod_{p \neq j} \varphi_{w_p}^{l_p}(w_j), \quad 1 \leq j \leq m,$$

in particular. Then h is continuous on $\overline{\mathbb{B}}$ and analytic on $\overline{\mathbb{B}} \setminus \{w_1, \ldots, w_m\}$, hence is analytic on $\mathbb{B}$ by Corollary 4.22. By inspection it is clear that the zeros of h in $\mathbb{B}$ are $z_1, \ldots, z_n$, with $\operatorname{ord}(h, z_j) = k_j$ for each $1 \leq j \leq n$. Since $\varphi_{w_j}(\partial\mathbb{B}) = \partial\mathbb{B}$ by Proposition 8.24(4), and $|f| = 1$ on $\partial\mathbb{B}$ by hypothesis, it follows that $|h| = 1$ on $\partial\mathbb{B}$ as well. Thus, by Theorem 8.27 and its proof, there exists some unimodular $\lambda \in \mathbb{C}$ such that

$$h(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right)^{k_j}$$

for all $z \in \overline{\mathbb{B}}$, and hence

$$f(z) \prod_{j=1}^{m} \left(\frac{z - w_j}{1 - \overline{w}_j z} \right)^{l_j} = \lambda \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right)^{k_j} \tag{8.7}$$

on $\overline{\mathbb{B}}$. Letting Ω be the region of analyticity for f, then by the Identity Theorem equation (8.7) holds on

$$\Omega' = \Omega \setminus \big[\{1/\overline{z}_j : 1 \leq j \leq n\} \cup \{1/\overline{w}_j : 1 \leq j \leq m\} \big],$$

and therefore

$$f(z) = \frac{\lambda \displaystyle\prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right)^{k_j}}{\displaystyle\prod_{j=1}^{m} \left(\frac{z - w_j}{1 - \overline{w}_j z} \right)^{l_j}}$$

for all $z \in \Omega'$. ■

Exercise 8.32 (AN4.6.6). Suppose $f : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$ is continuous on $\overline{\mathbb{B}}$ and analytic on $\mathbb{B}$. Let $Z(f) = \{z_1, \ldots, z_n\} \subseteq \mathbb{B}$ with $\operatorname{ord}(f, z_j) = k_j$ for $1 \le j \le n$. Show that

$$|f(z)| \le \prod_{j=1}^{n} \left| \frac{z - z_j}{1 - \overline{z}_j z} \right|^{k_j} \tag{8.8}$$

for all $z \in \overline{\mathbb{B}}$. Find a formula for $f(z)$ if equality holds in (8.8) for some $z \in \mathbb{B} \setminus Z(f)$.

Solution. Define $g : \overline{\mathbb{B}} \to \mathbb{C}$ by

$$g(z) = \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right)^{k_j},$$

which is analytic on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$. As established in the proof of Theorem 8.27, the function

$$(f/g)(z) = f(z) \prod_{j=1}^{n} \left(\frac{1 - \overline{z}_j z}{z - z_j} \right)^{k_j} = \frac{f(z)}{\varphi_{z_1}^{k_1}(z) \cdots \varphi_{z_n}^{k_n}(z)}$$

on $\overline{\mathbb{B}} \setminus Z(f)$ has a continuous extension to $\overline{\mathbb{B}}$ that is analytic on $\mathbb{B}$. For any $z \in \partial\mathbb{B}$,

$$|(f/g)(z)| = \frac{|f(z)|}{|\varphi_{z_1}(z)|^{k_1} \cdots |\varphi_{z_n}(z)|^{k_n}} \le |f(z)| \le 1,$$

since $\varphi_{z_j}(\partial\mathbb{B}) = \partial\mathbb{B}$ by Proposition 8.24(4). Now,

$$M = \max_{z \in \partial\mathbb{B}} |(f/g)(z)| \le 1,$$

so by the Maximum Principle

$$\max_{z \in \overline{\mathbb{B}}} |(f/g)(z)| = \max_{z \in \partial\mathbb{B}} |(f/g)(z)| = M \le 1,$$

and therefore

$$|f(z)| \le |g(z)| = \prod_{j=1}^{n} \left| \frac{z - z_j}{1 - \overline{z}_j z} \right|^{k_j}$$

for all $z \in \overline{\mathbb{B}}$.

Another consequence of the Maximum Principle is that either $|f/g| < M$ on $\mathbb{B}$ or f/g is constant on $\overline{\mathbb{B}}$. Thus, if there exists some $a \in \mathbb{B} \setminus Z(f)$ such that

$$|f(a)| = \prod_{j=1}^{n} \left| \frac{a - z_j}{1 - \overline{z}_j a} \right|^{k_j},$$

then $|f(a)| = |g(a)| \ne 0$, which implies $|(f/g)(a)| = 1 \ge M$ and we conclude that f/g is constant on $\overline{\mathbb{B}}$. That is, there exists some $\mu \in \mathbb{C}$ such that $|\mu| = M \le 1$ and $f/g \equiv \mu$ on $\overline{\mathbb{B}}$. Therefore

$$f(z) = \mu \prod_{j=1}^{n} \left(\frac{z - z_j}{1 - \overline{z}_j z} \right)^{k_j}$$

for all $z \in \overline{\mathbb{B}}$. ■

9

The Poisson Integral

9.1 – The Poisson Integral Formula

Theorem 9.1. *If f is continuous on $\overline{\mathbb{B}}$ and analytic on $\mathbb{B}$, then*

$$\oint_{\partial\mathbb{B}} f = 0, \tag{9.1}$$

and

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\mathbb{B}} \frac{f(w)}{w-z}\,dw \tag{9.2}$$

for all $z \in \mathbb{B}$.

Proof. By Cauchy's Theorem for Starlike Regions f has a primitive on $\mathbb{B}$, and thus

$$\oint_{C_r(0)} f = 0$$

for all $0 < r < 1$ by the Fundamental Theorem of Path Integrals. Let $\epsilon > 0$. Since f is uniformly continuous on $\overline{\mathbb{B}}$, there exists some $\delta > 0$ such that, for all $z, w \in \overline{\mathbb{B}}$, $|z - w| \leq \delta$ implies $|f(z) - f(w)| < \epsilon$. Let $1 - \delta < r < 1$, so that $0 < 1 - r < \delta$. Since $|e^{it} - re^{it}| = 1 - r < \delta$ for any $t \in [0, 2\pi]$,

$$\begin{aligned}\left|\oint_{\partial\mathbb{B}} f\right| &= \left|\oint_{\partial\mathbb{B}} f - \frac{1}{r}\oint_{C_r(0)} f\right| = \left|\int_0^{2\pi} \big[f(e^{it}) - f(re^{it})\big] ie^{it}dt\right| \\ &\leq \int_0^{2\pi} \big|f(e^{it}) - f(re^{it})\big|\,dt \leq \int_0^{2\pi} \epsilon\,dt = 2\pi\epsilon,\end{aligned}$$

and therefore (9.1) obtains.

Next, fix $z \in \mathbb{B}$, and define $g : \overline{\mathbb{B}} \to \mathbb{C}$ by

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(z), & w = z. \end{cases}$$

Then g is continuous on $\overline{\mathbb{B}}$ and analytic on $\mathbb{B}$, so we apply (9.1) and Theorem 6.22 to obtain

$$0 = \oint_{\partial\mathbb{B}} g = \oint_{\partial\mathbb{B}} \frac{f(w)}{w-z}dw - f(z)\oint_{\partial\mathbb{B}} \frac{1}{w-z}dw = \oint_{\partial\mathbb{B}} \frac{f(w)}{w-z}dw - 2\pi i f(z),$$

which gives (9.2). ■

Definition 9.2. *The function $P : \mathbb{B} \times \mathbb{R} \to \mathbb{R}$ given by*

$$P_z(t) = \frac{1-|z|^2}{|e^{it}-z|^2}$$

for $(z,t) \in \mathbb{B} \times \mathbb{R}$ is called the ***Poisson kernel****, and the function $Q : \mathbb{B} \times \mathbb{R} \to \mathbb{C}$ given by*

$$Q_z(t) = \frac{e^{it}+z}{e^{it}-z}$$

for $(z,t) \in \mathbb{B} \times \mathbb{R}$ is called the ***Cauchy kernel****.*

As the notation suggests, the Poisson kernel (resp. Cauchy kernel) may be regarded as a family of functions $P_z : \mathbb{R} \to \mathbb{R}$ (resp. $C_z : \mathbb{R} \to \mathbb{C}$) indexed by $z \in \mathbb{B}$.

It is straightforward to verify that

$$P_z(t) = \mathrm{Re}[Q_z(t)],$$

and also

$$P_{re^{i\theta}}(t) = P_r(t-\theta),$$

and

$$P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} \tag{9.3}$$

for any $\theta \in \mathbb{R}$ and $r \in [0,1)$. Equation (9.3) in particular makes it clear that $P_r(t-\theta) = P_r(\theta-t)$ in general, so that each P_r is seen to be an even function on $\mathbb{R}$, and by extension P_z is even for each $z \in \mathbb{B}$. Very generally, a **Poisson integral** is any integral with a Poisson kernel appearing as a factor in the (complex-valued) integrand:

$$\int_a^b P_z(t)F(t)dt.$$

In particular, a Poisson integral is an integral of the kind appearing in the Poisson Integral Formula, which we first present on $\mathbb{B}$, and then give on arbitrary discs $B_r(z_0)$.

Theorem 9.3 (**Poisson Integral Formula**). *If $f : \overline{\mathbb{B}} \to \mathbb{C}$ is continuous on $\overline{\mathbb{B}}$ and analytic on $\mathbb{B}$, then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)f(e^{it})dt$$

for all $z \in \mathbb{B}$.

Corollary 9.4. *For all $z \in \mathbb{B}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(t)dt = 1.$$

The Poisson Integral Formula naturally generalizes to an arbitrary disc in $\mathbb{C}$ with radius $r > 0$ and center z_0 as follows.

Theorem 9.5. *If f is continuous on $\overline{B}_r(z_0)$ and analytic on $B_r(z_0)$, then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/r}(t) f(z_0 + re^{it})dt$$

for all $z \in B_r(z_0)$.

Exercise 9.6 (AN4.7.1). Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} Q_z(t)dt = 1.$$

Solution. If $z = 0$, then

$$\frac{1}{2\pi}\int_0^{2\pi} Q_0(t)dt = \frac{1}{2\pi}\int_0^{2\pi} dt = 1$$

obtains immediately. Suppose that $z \in \mathbb{B}'$. Then, applying partial fraction decomposition,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}dt &= \frac{1}{2\pi}\left(\int_0^{2\pi} \frac{e^{it}}{e^{it}-z}dt + \int_0^{2\pi} \frac{z}{e^{it}-z}dt\right) \\ &= \frac{1}{2\pi i}\oint_{\partial\mathbb{B}} \frac{1}{w-z}dw + \frac{1}{2\pi i}\oint_{\partial\mathbb{B}} \frac{z}{w(w-z)}dw \\ &= \frac{1}{2\pi i}\oint_{\partial\mathbb{B}} \frac{2}{w-z}dw - \frac{1}{2\pi i}\oint_{\partial\mathbb{B}} \frac{1}{w}dw\end{aligned}$$

By Theorem 6.22,

$$\frac{1}{2\pi}\int_0^{2\pi} Q_z(t)dt = 2\,\mathrm{wn}(\partial\mathbb{B}, z) - \mathrm{wn}(\partial\mathbb{B}, 0) = 2(1) - 1 = 1,$$

as desired. ∎

Exercise 9.7 (AN4.7.5). Let Ω be a bounded open set, let $\gamma : [a, b] \to \mathbb{C}$ be a closed path such that $\gamma^* = \partial\Omega$, and for some fixed $z_0 \in \Omega$ let $\gamma_\delta = z_0 + \delta(\gamma - z_0)$ for any real δ. Show that if $\gamma_\delta^* \subseteq \Omega$ for all $\delta \in [0, 1)$, f is continuous on $\overline{\Omega}$, and f is analytic on Ω, then

$$\oint_\gamma f = 0, \tag{9.4}$$

and

$$\mathrm{wn}(\gamma, z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw \tag{9.5}$$

for all $z \in \Omega$.

Solution. First we show that Ω is starlike with star center z_0. Let $z \in \Omega$, and consider the open ray $R = (z_0, z, \infty)$. Since $\overline{\Omega} = \Omega \cup \partial\Omega$,

$$R = (R \cap \Omega) \cup (R \cap \partial\Omega) \cup (R \cap \mathbb{C} \setminus \overline{\Omega}),$$

where $S_1 = R \cap \Omega$ and $S_2 = R \cap \mathbb{C} \setminus \overline{\Omega}$ are nonempty disjoint open sets in the topological subspace $R \subseteq \mathbb{C}$. (In particular $S_2 \neq \varnothing$ since Ω is given to be bounded.) However, $S_1 \cup S_2 \neq R$ since R is a connected set, and therefore $R \cap \partial\Omega \neq \varnothing$. Thus there exists some $\beta \in R$ such that $\beta \in \partial\Omega = \gamma^*$. By hypothesis $z_0 + \delta(\beta - z_0) \in \Omega$ for all $\delta \in [0, 1)$, which implies that $[z_0, \beta) \subseteq \Omega$.

Now, on the ray R, either β is between z_0 and z, or z is between z_0 and β. Suppose the former is the case, so that $\beta \in (z_0, z) \subseteq R$. Let $R_\beta = (\beta, z, \infty)$, so that $R_\beta \subseteq R$ and

$$R_\beta = (R_\beta \cap \Omega) \cup (R_\beta \cap \partial\Omega) \cup (R_\beta \cap \mathbb{C} \setminus \overline{\Omega}).$$

The connectedness of R_β implies that there exists some $\beta \neq b \in R_\beta$ such that $b \in \partial\Omega$, where $[z_0, b) \subseteq \Omega$ since $z_0 + \delta(b - z_0) \in \Omega$ for all $\delta \in [0, 1)$. However, $[z_0, \beta) \subseteq \Omega$ and $[z_0, b) \subseteq \Omega$ implies that either $[\beta, b) \subseteq \Omega$ or $[b, \beta) \subseteq \Omega$, depending on which of β or b is closer to z_0. But either $\beta \in \Omega$ or $b \in \Omega$ contradicts $\beta, b \in \partial\Omega$. We conclude that $z \in (z_0, \beta)$, so $[z_0, z] \subseteq [z_0, \beta) \subseteq \Omega$. Since $[z_0, z] \subseteq \Omega$ for all $z \in \Omega$, it follows that Ω is starlike with star center z_0.

By Cauchy's Theorem for Starlike Regions f has a primitive on Ω, and thus

$$\oint_{\gamma_\delta} f = 0$$

for all $0 \leq \delta < 1$ by the Fundamental Theorem of Path Integrals. Set

$$M = \max_{t \in [a,b]} |\gamma'(t)| \quad \text{and} \quad N = \max_{t \in [a,b]} |\gamma(t) - z_0|.$$

Fix $\epsilon > 0$. Since f is uniformly continuous on $\overline{\Omega}$, there exists some $0 < \alpha < N$ such that, for all $z, w \in \overline{\Omega}$, $|z - w| < \alpha$ implies

$$|f(z) - f(w)| < \frac{\epsilon}{M(b - a)}.$$

Let $1 - \alpha/N < \delta < 1$, so that

$$0 < 1 - \delta < \frac{\alpha}{N}.$$

Then for any $t \in [0, 2\pi]$,

$$|\gamma(t) - \gamma_\delta(t)| = |(1 - \delta)(\gamma(t) - z_0)| = (1 - \delta)|\gamma(t) - z_0| < \frac{\alpha}{N} \cdot N = \alpha,$$

and so

$$\begin{aligned} \left|\oint_\gamma f\right| &= \left|\oint_\gamma f - \frac{1}{\delta}\oint_{\gamma_\delta} f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt - \frac{1}{\delta}\int_a^b f(\gamma_\delta(t)) \cdot \delta\gamma'(t)dt\right| \\ &= \left|\int_a^b \big[f(\gamma(t)) - f(\gamma_\delta(t))\big]\gamma'(t)dt\right| \leq \int_a^b \big|f(\gamma(t)) - f(\gamma_\delta(t))\big||\gamma'(t)|dt \\ &\leq \int_a^b \left(\frac{\epsilon}{M(b-a)} \cdot M\right)dt = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that (9.4) holds.

Next, fix $z \in \Omega$, and define $g : \overline{\Omega} \to \mathbb{C}$ by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z}, & w \neq z \\ f'(z), & w = z. \end{cases}$$

Then g is continuous on $\overline{\Omega}$ and analytic on Ω, so we apply (9.4) and Theorem 6.22 to obtain

$$0 = \oint_\gamma g = \oint_\gamma \frac{f(w)}{w-z}\,dw - f(z)\oint_\gamma \frac{1}{w-z}\,dw = \oint_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i f(z),$$

which gives (9.5). ■

9.2 – The Dirichlet Problem

Recall from §3.2 that if a function $u : \Omega \to \mathbb{R}$ having continuous first- and second-order partial derivatives on Ω satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ on Ω, then u is said to be harmonic on Ω.

The **Dirichlet problem** is a type of boundary value problem that arises in different guises in many areas of mathematics. In complex analysis the problem can be stated as follows on a homologically simply connected bounded region $\Omega \subseteq \mathbb{C}$: "Find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ that is harmonic on Ω and satisfies a continuous **boundary condition** $u_0 : \partial\Omega \to \mathbb{R}$." In this section we will consider the Dirichlet problem for a disc, wherein $\Omega = B_r(z_0)$ for some $r > 0$ and $z_0 \in \mathbb{C}$. First, however, we establish that the solution of a given Dirichlet problem is unique.

Proposition 9.8. *Let Ω be a homologically simply connected bounded region. If $u_1, u_2 : \overline{\Omega} \to \mathbb{R}$ are each solutions to the Dirichlet problem*

$$u_{xx} + u_{yy} = 0, \quad u_0 : \partial\Omega \to \mathbb{R}, \tag{9.6}$$

then $u_1 \equiv u_2$.

Proof. Suppose $u : \overline{\Omega} \to \mathbb{R}$ satisfies the boundary value problem $u_{xx} + u_{yy} = 0$, $u_0 \equiv 0$ on $\partial\Omega$. Then u is harmonic on Ω, continuous on $\overline{\Omega}$, and $u(z) = u_0(z) = 0$ for all $z \in \partial\Omega$, so that by the Maximum and Minimum Principles for Harmonic Functions given in §5.4 we have

$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in\partial\Omega} u(z) = 0 \quad \text{and} \quad \min_{z\in\overline{\Omega}} u(z) = \min_{z\in\partial\Omega} u(z) = 0,$$

and hence $u \equiv 0$ on $\overline{\Omega}$.

Now suppose $u_1, u_2 : \overline{\Omega} \to \mathbb{R}$ satisfy the boundary value problem (9.6), so that u_1 and u_2 are harmonic on Ω, continuous on $\overline{\Omega}$, and $u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = u_0$. It follows that $u_1 - u_2$ is harmonic on Ω, continuous on $\overline{\Omega}$, and $u_1 - u_2 \equiv 0$ on $\partial\Omega$. Hence $u_1 - u_2$ satisfies the boundary value problem $u_{xx} + u_{yy} = 0$, $u_0 \equiv 0$ on $\partial\Omega$, so that $u_1 - u_2 \equiv 0$ on $\overline{\Omega}$. That is, $u_1 \equiv u_2$ on $\overline{\Omega}$. ■

In examining the Dirichlet problem for a disc $B_r(z_0)$, we start by formulating a solution in the case when $z_0 = 0$ and $r = 1$. The solution to the Dirichlet problem for an arbitrary disc then easily follows.

Theorem 9.9. *Suppose $u_0 : \partial\mathbb{B} \to \mathbb{R}$ is continuous, and define $u : \overline{\mathbb{B}} \to \mathbb{R}$ by*

$$u(z) = \begin{cases} u_0(z), & z \in \partial\mathbb{B} \\ \frac{1}{2\pi}\int_0^{2\pi} P_z(t)u_0(e^{it})dt, & z \in \mathbb{B}. \end{cases}$$

Then u is continuous on $\overline{\mathbb{B}}$ and harmonic on $\mathbb{B}$.

Proof. Let $f : \mathbb{B} \to \mathbb{C}$ be given by

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} Q_z(t)u_0(e^{it})dt.$$

It is straightforward to show that $u(z) = \operatorname{Re} f(z)$ for $z \in \mathbb{B}$. Since f is analytic on $\mathbb{B}$ by Lemma 6.26, and the real and imaginary parts of an analytic function are known to be harmonic functions, we conclude that u is harmonic on $\mathbb{B}$. It follows immediately that u is continuous on $\mathbb{B}$ as well, and it remains only to show that u is continuous on $\partial\mathbb{B}$.

Let $\epsilon > 0$. Let M be the maximum value of $|u_0(z)|$ for $z \in \partial\mathbb{B}$. Since u_0 is uniformly continuous on $\partial\mathbb{B}$, there exists some $\delta \in (0, \pi)$ such that

$$\left|u_0(e^{i\theta_1}) - u_0(e^{i\theta_2})\right| < \frac{\epsilon}{2} \tag{9.7}$$

for any $\theta_1, \theta_2 \in \mathbb{R}$ with $|\theta_1 - \theta_2| \leq \delta$. Now,

$$P_r(\delta) = \frac{1 - r^2}{1 - 2r\cos\delta + r^2}$$

for any $r \in (0, 1)$, so $P_r(\delta) \to 0$ as $r \to 1^-$, and there exists some $s \in (0, 1)$ such that $P_r(\delta) < \epsilon/4M$ for all $r \in (s, 1)$.

Let $r \in (s, 1)$ and $\theta \in \mathbb{R}$ be arbitrary. It follows from the Poisson Integral Formula that

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(t - \theta)dt = 1,$$

and since $re^{i\theta} \in \mathbb{B}$ with $P_{re^{i\theta}}(t) = P_r(t - \theta)$,

$$\begin{aligned}
u(re^{i\theta}) - u_0(e^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)u_0(e^{it})dt - u_0(e^{it}) \cdot \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)\big[u_0(e^{it}) - u_0(e^{i\theta})\big]dt \\
&= \frac{1}{2\pi}\int_{-\theta}^{2\pi-\theta} P_r(\tau)\big[u_0(e^{i(\tau+\theta)}) - u_0(e^{i\theta})\big]d\tau \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\tau)\big[u_0(e^{i(\tau+\theta)}) - u_0(e^{i\theta})\big]d\tau,
\end{aligned}$$

where the third equality follows from the substitution $\tau = t - \theta$, and the fourth equality follows from the fact that the integrand

$$g(\tau) = P_r(\tau)\big[u_0(e^{i(\tau+\theta)}) - u_0(e^{i\theta})\big]$$

is a periodic function with period 2π. Now,

$$\left|u(re^{i\theta}) - u_0(e^{i\theta})\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{-\delta} g + \int_{-\delta}^{\delta} g + \int_{\delta}^{\pi} g\right| \leq \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} |g| + \int_{-\delta}^{\delta} |g| + \int_{\delta}^{\pi} |g|\right). \tag{9.8}$$

Since P_r is increasing on $[-\pi, -\delta]$ and decreasing on $[\delta, \pi]$, and $P_r(-\delta) = P_r(\delta)$, we have

$$\int_{-\pi}^{-\delta} |g| = \int_{-\pi}^{-\delta} P_r(\tau)\big|u_0(e^{i(\tau+\theta)}) - u_0(e^{i\theta})\big|d\tau \leq \int_{-\pi}^{-\delta} 2P(-\delta)M\, d\tau = 2M(\pi - \delta)P(\delta),$$

and similarly

$$\int_{\delta}^{\pi} |g| \leq \int_{\delta}^{\pi} 2P(\delta)M\, d\tau = 2M(\pi - \delta)P(\delta).$$

Also, recalling (9.7) and our choice for δ,

$$\int_{-\delta}^{\delta} |g| = \int_{-\delta}^{\delta} P_r(\tau)\big|u_0(e^{i(\tau+\theta)}) - u_0(e^{i\theta})\big|d\tau \leq \int_{-\delta}^{\delta} \frac{\epsilon}{2}P_r(\tau)\, d\tau < \frac{\epsilon}{2}\int_0^{2\pi} P_r(\tau)\, d\tau = \pi\epsilon.$$

Returning to (9.8), and recalling that $P_r(\delta) < \epsilon/4M$,

$$\left|u(re^{i\theta}) - u_0(e^{i\theta})\right| < \frac{1}{2\pi}\left[4M(\pi-\delta)P_r(\delta) + \pi\epsilon\right] < \frac{1}{2\pi}\left[(\pi-\delta)\epsilon + \pi\epsilon\right] < \epsilon.$$

We have now shown that

$$\forall \epsilon > 0\ \exists s \in (0,1)\ \forall r \in (s,1)\ \forall \theta \in \mathbb{R}\left(\left|u(re^{i\theta}) - u_0(e^{i\theta})\right| < \epsilon\right).$$

Fix $w = e^{i\varphi} \in \partial\mathbb{B}$. Choose $\delta_1 > 0$ such that

$$\left|u_0(e^{i\theta_1}) - u_0(e^{i\theta_2})\right| < \frac{\epsilon}{2}$$

for any $\theta_1, \theta_2 \in \mathbb{R}$ with $|\theta_1 - \theta_2| \le \delta_1$. Now, there exists some $s \in (0,1)$ such that

$$\forall r \in (s,1)\ \forall \theta \in \mathbb{R}\left(\left|u(re^{i\theta}) - u_0(e^{i\theta})\right| < \frac{\epsilon}{2}\right).$$

Choose $\delta_2 > 0$ such that $1 - \delta_2 > s$, and set $\delta = \min\{\delta_1, \delta_2\}$. Let $z \in B_\delta(w) \cap \overline{\mathbb{B}}$. Then $z = re^{i\theta}$ for some $r \in (s,1]$ and $\theta \in \mathbb{R}$. If $r = 1$, then

$$|u(z) - u(w)| = |u_0(e^{i\theta}) - u_0(e^{i\varphi})| < \frac{\epsilon}{2}$$

since $|\theta - \varphi| < \delta \le \delta_1$. If $r \ne 1$, then

$$\left|u(z) - u_0(e^{i\theta})\right| = \left|u(re^{i\theta}) - u_0(e^{i\theta})\right| < \frac{\epsilon}{2}$$

since

$$re^{i\theta} \in B_\delta(w) \cap \mathbb{B} \;\;\Rightarrow\;\; 1 > r > 1 - \delta \ge 1 - \delta_2 > s \;\;\Rightarrow\;\; r \in (s,1),$$

and thus

$$|u(z) - u(w)| \le \left|u(re^{i\theta}) - u_0(e^{i\theta})\right| + \left|u_0(e^{i\theta}) - u_0(e^{i\varphi})\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We conclude that, for all $\epsilon > 0$, there exists some $\delta > 0$ such that $z \in B_\delta(w) \cap \overline{\mathbb{B}}$ implies $|u(z) - u(w)| < \epsilon$. Thus u is continuous at w, and since $w \in \partial\mathbb{B}$ is arbitrary it follows that $u : \overline{\mathbb{B}} \to \mathbb{R}$ is continuous on $\partial\mathbb{B}$. Therefore u is continuous on $\overline{\mathbb{B}}$. ∎

We now are in a position to present and prove a Poisson integral formula for harmonic functions.

Theorem 9.10. *If $u : \overline{\mathbb{B}} \to \mathbb{R}$ is continuous on $\overline{\mathbb{B}}$ and harmonic on $\mathbb{B}$, then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)u(e^{it})dt$$

for all $z \in \mathbb{B}$.

More generally if u is continuous on $\overline{B}_r(z_0)$ and harmonic on $B_r(z_0)$, then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/r}(t)u(z_0 + re^{it})dt$$

for all $z \in B_r(z_0)$.

Corollary 9.11 (**Mean Value Property for Harmonic Functions**). *If $u : \Omega \to \mathbb{R}$ is harmonic, $z_0 \in \Omega$, and $\overline{B}_r(z_0) \subseteq \Omega$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{\pi} u(z_0 + re^{it})dt.$$

Exercise 9.12 (**AN4.7.2**). Prove **Harnack's Inequality**: If $u : \overline{\mathbb{B}} \to \mathbb{R}$ is continuous on $\overline{\mathbb{B}}$, nonnegative on $\overline{\mathbb{B}}$, and harmonic on $\mathbb{B}$, then

$$\frac{1-r}{1+r} u(0) \le u(re^{i\theta}) \le \frac{1+r}{1-r} u(0)$$

for all $r \in [0,1)$ and $\theta \in \mathbb{R}$.

Solution. Suppose $u : \overline{\mathbb{B}} \to \mathbb{R}$ is continuous on $\overline{\mathbb{B}}$, nonnegative on $\overline{\mathbb{B}}$, and harmonic on $\mathbb{B}$. Fix $0 \le r < 1$ and $\theta \in \mathbb{R}$. On $[0, 2\pi]$ we have

$$P_r(t-\theta) \le \frac{1-r^2}{1-2r+r^2} = \frac{1+r}{1-r}$$

by (9.3), and also

$$P_r(t-\theta) \ge \frac{1-r^2}{1+2r+r^2} = \frac{1-r}{1+r}.$$

Thus, since

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} P_0(t) u(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt$$

by Theorem 9.10, and $u(0) \ge 0$ by hypothesis, we obtain

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} P_{re^{it}}(t) u(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) u(e^{it}) dt \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \frac{1+r}{1-r} u(e^{it}) dt = \frac{1+r}{1-r} u(0), \end{aligned}$$

and also

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) u(e^{it}) dt \ge \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r}{1+r} u(e^{it}) dt = \frac{1-r}{1+r} u(0).$$

Combining these results yields Harnack's Inequality. ■

Exercise 9.13 (**AN4.7.4**). Prove that if $u : \Omega \to \mathbb{R}$ is harmonic on Ω, then u has a local harmonic conjugate at each point of Ω.

Solution. Fix $z_0 \in \Omega$ and let $r > 0$ such that $\overline{B}_r(z_0) \subseteq \Omega$. Then u is continuous on $\overline{B}_r(z_0)$ and harmonic on $B_r(z_0)$, and so Theorem 9.10 implies that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_{(z-z_0)/r}(t) u(z_0 + re^{it}) dt$$

for all $z \in B_r(z_0)$. Define $f : B_r(z_0) \to \mathbb{C}$ by

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} Q_{(z-z_0)/r}(t) u(z_0 + re^{it}) dt,$$

which is analytic on $B_r(z_0)$ by Lemma 6.26, and moreover

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}[Q_{(z-z_0)/r}(t)]u(z_0 + re^{it})dt + \frac{i}{2\pi}\int_0^{2\pi} \mathrm{Im}[Q_{(z-z_0)/r}(t)]u(z_0 + re^{it})dt$$

makes plain that $\mathrm{Re}[f(z)] = u(z)$ since $\mathrm{Re}\, Q_w = P_w$ for any $w \in \mathbb{B}$. Define $v : B_r(z_0) \to \mathbb{R}$ by $v(z) = \mathrm{Im}[f(z)]$, so that $f = u + iv$. By Theorem 4.28 conclude that v is harmonic on $B_r(z_0)$, and therefore v is a harmonic conjugate of u on $B_r(z_0)$. ■

Exercise 9.14 (AN4.7.6). Poisson Integral Formula for a Half Plane. Let f be analytic on $\mathbb{H} = \{z : \mathrm{Im}\, z > 0\}$ and continuous on $\overline{\mathbb{H}}$. If $u = \mathrm{Re}\, f$, and there exists $R > 0$ and $p < 1$ such that $|f(z)| \le |z|^p$ for all $z \in \overline{\mathbb{H}} \cap A_{R,\infty}(0)$, then

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(t, 0)}{(t - x)^2 + y^2} dt$$

for all $(x, y) = x + iy$ with $y > 0$.

Solution. Note that $x + iy$ is being denoted by (x, y) in the formula, and t by $(t, 0)$, owing to tradition and also because the formula holds in $\mathbb{R}^2$ under appropriately adjusted hypotheses.

For $r > 0$ define $\gamma_1 : [-r, r] \to \mathbb{C}$ by $\gamma_1(t) = t$, define $\gamma_2 : [0, \pi] \to \mathbb{C}$ by $\gamma_2(t) = re^{it}$, and let $\gamma : [0, 1] \to \mathbb{C}$ be the concatenation $\gamma = \gamma_1 * \gamma_2$. The path γ is closed, and by Proposition 3.28 we have

$$\oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Let $\Omega_r = B_r(0) \cap \mathbb{H}$, a bounded open set shown in Figure 23. Then $\partial\Omega_r = \gamma^*$, and f is continuous on $\overline{\Omega}_r$ and analytic on Ω_r. By Exercise 9.7,

$$f(z) = \mathrm{wn}(\gamma, z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z} dw \tag{9.9}$$

for any $z \in \Omega_r$. Also, for any $z \in \Omega_r$, the function

$$\varphi(w) = \frac{f(w)}{w - \bar{z}}$$

is continuous on $\overline{\Omega}_r$ and analytic on Ω_r, so that

$$\oint_\gamma \varphi = 0$$

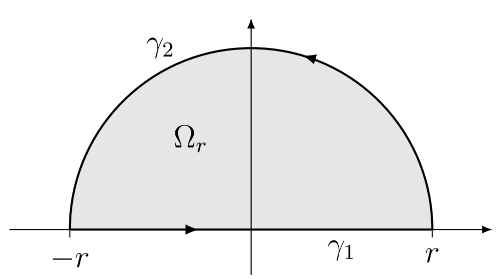

FIGURE 23.

by Exercise 9.7, and hence

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-\overline{z}}\,dw = 0 \tag{9.10}$$

for all $z \in \Omega_r$.

Now, from (9.9) and (9.10) we obtain, for all $z = x+iy \in \Omega_r$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-\overline{z}}\,dw = \frac{1}{2\pi i}\left[\oint_\gamma \frac{f(w)}{w-z} - \frac{f(w)}{w-\overline{z}}\right]dw \\ &= \frac{1}{\pi}\oint_\gamma \frac{yf(w)}{(w-z)(w-\overline{z})}\,dw = \frac{1}{\pi}\int_{\gamma_1} g(w)\,dw + \frac{1}{\pi}\int_{\gamma_2} g(w)\,dw, \end{aligned}$$

where

$$g(w) = \frac{yf(w)}{(w-z)(w-\overline{z})}.$$

Fix $z = x+iy \in \mathbb{H}$. Let $r > \max\{R, 2|z|\}$ be arbitrary. We have $z \in \Omega_r$, $|f(w)| \le |w|^p$ for all $w \in \gamma_2^*$, and also $|w-z| > r$ and $|w-\overline{z}| > r$ for all $w \in \gamma_2^*$. Thus

$$\left|\int_{\gamma_2} g\right| \le \pi r \sup_{w\in\gamma_2^*} \frac{y|f(w)|}{|w-z||w-\overline{z}|} \le \pi r \cdot \frac{yr^p}{r^2} = \frac{\pi y}{r^{1-p}}, \tag{9.11}$$

and since $p < 1$ it follows that

$$\lim_{r\to\infty}\int_{\gamma_2} g = 0,$$

and hence

$$\lim_{r\to\infty} \operatorname{Re}\left(\int_{\gamma_2} g\right) = 0 \tag{9.12}$$

Next,

$$\int_{\gamma_1} g = \int_{-r}^{r} \frac{yf(t)}{(t-z)(t-\overline{z})}\,dt = \int_{-r}^{r} \frac{yu(t,0)}{(t-x)^2+y^2}\,dt + i\int_{-r}^{r} \frac{yv(t,0)}{(t-x)^2+y^2}\,dt,$$

and since

$$\left|\frac{yu(t,0)}{(t-x)^2+y^2}\right|, \left|\frac{yv(t,0)}{(t-x)^2+y^2}\right| \le \frac{y|f(t)|}{(t-x)^2+y^2} \le \frac{yt^p}{(t-x)^2+y^2} \le \frac{yt^p}{(t-x)^2}$$

for all $t \in (-\infty, -\rho] \cup [\rho, \infty)$ for sufficiently large $\rho > \max\{|x|, R\}$, where the integrals

$$\int_\rho^\infty \frac{yt^p}{(t-x)^2}\,dt \quad \text{and} \quad \int_{-\infty}^{\rho} \frac{yt^p}{(t-x)^2}\,dt$$

converge by the p-Test for Integrals in §8.8 of the *Calculus Notes*, it follows by the Comparison Test for Integrals and Proposition 8.36 in the same §8.8 that

$$\int_0^\infty \frac{yu(t,0)}{(t-x)^2+y^2}\,dt, \quad \int_{-\infty}^0 \frac{yu(t,0)}{(t-x)^2+y^2}\,dt, \quad \int_0^\infty \frac{yv(t,0)}{(t-x)^2+y^2}\,dt, \quad \int_{-\infty}^0 \frac{yv(t,0)}{(t-x)^2+y^2}\,dt$$

are convergent. Therefore, in particular,

$$\lim_{r\to\infty} \operatorname{Re}\left(\int_{\gamma_1} g\right) = \lim_{r\to\infty}\int_{-r}^{r} \frac{yu(t,0)}{(t-x)^2+y^2}\,dt = \int_{-\infty}^{\infty} \frac{yu(t,0)}{(t-x)^2+y^2}\,dt \in \mathbb{R}. \tag{9.13}$$

From (9.12) and (9.13) we now have

$$\begin{aligned} u(x,y) &= \mathrm{Re}[f(z)] = \lim_{r\to\infty} \mathrm{Re}[f(z)] = \lim_{r\to\infty}\left[\mathrm{Re}\left(\frac{1}{\pi}\int_{\gamma_1} g\right) + \mathrm{Re}\left(\frac{1}{\pi}\int_{\gamma_2} g\right)\right] \\ &= \frac{1}{\pi}\lim_{r\to\infty} \mathrm{Re}\left(\int_{\gamma_1} g\right) + \frac{1}{\pi}\lim_{r\to\infty} \mathrm{Re}\left(\int_{\gamma_2} g\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(t,0)}{(t-x)^2+y^2}dt, \end{aligned}$$

as desired. ■

9.3 – The Poisson-Jensen Formula

Lemma 9.15. *If f is analytic on $B_r(0)$, and continuous and nonvanishing on $\overline{B}_r(0)$, then*

$$\ln|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} P_{z/r}(t)\ln\left|f(re^{it})\right| dt$$

for all $z \in B_r(0)$.

Proof. Suppose f is analytic on $B_r(0)$, and continuous and nonvanishing on $\overline{B}_r(0)$. Then f'/f is analytic on $B_r(0)$, so by Cauchy's Theorem for Starlike Regions it follows that f'/f has a primitive on $B_r(0)$, and therefore f'/f has an analytic logarithm on $B_r(0)$ by Theorem 6.11. Let $g : B_r(0) \to \mathbb{C}$ such that $f = \exp(g)$ on $B_r(0)$. By Theorem 4.42(6)

$$|f(z)| = |\exp(g(z))| = e^{\operatorname{Re} g(z)}$$

and so $\ln|f| = \operatorname{Re} g$. The function $\operatorname{Re} g : B_r(0) \to \mathbb{R}$ is harmonic by Theorem 4.28, and must also be continuous on $\overline{B}_r(0)$, so that by Theorem 9.10

$$\ln|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} P_{z/r}(t)\ln\left|f(re^{it})\right| dt$$

for all $z \in B_r(0)$. ∎

Theorem 9.16 (**Poisson-Jensen Formula**). *Suppose $f : \overline{B}_r(0) \to \mathbb{C}$ is analytic on $B_r(0)$, continuous on $\overline{B}_r(0)$, and nonvanishing on $C_r(0)$. If $Z(f) = \{\zeta_1, \dots, \zeta_n\}$ with $\operatorname{ord}(f, \zeta_j) = k_j$ for $1 \le j \le n$, then*

$$\ln|f(z)| = \sum_{j=1}^{n} k_j \ln\left|\frac{(z-\zeta_j)r}{r^2 - \overline{\zeta}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/r}(t)\ln\left|f(re^{it})\right| dt$$

for all $z \in B_r(0) \setminus Z(f)$.

The following result, known as Jensen's Formula, obtains easily from the Poisson-Jensen Formula simply by letting $z = 0$.

Corollary 9.17 (**Jensen's Formula**). *If f satisfies the hypotheses of the Poisson-Jensen Formula and $f(0) \neq 0$, then*

$$\ln|f(0)| = \sum_{j=1}^{n} k_j \ln\left|\frac{\zeta_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(re^{it})\right| dt.$$

Theorem 9.18. *Suppose f is analytic and nonvanishing on $\overline{B}_r(0)$. If $Z(f, B_r(0)) = \{\zeta_1, \dots, \zeta_n\}$ with $\operatorname{ord}(f, \zeta_j) = k_j$ for $1 \le j \le n$, then*

$$\ln|f(z)| = \sum_{j=1}^{n} k_j \ln\left|\frac{(z-\zeta_j)r}{r^2 - \overline{\zeta}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/r}(t)\ln\left|f(re^{it})\right| dt$$

for all $z \in B_r(0) \setminus Z(f, B_r(0))$.

Theorem 9.19 (**Generalized Jensen's Formula**). *Let f be analytic and not identically zero on $B_R(0)$, with $m = \operatorname{ord}(f,0)$. If $(\zeta_j)_{j=1}$ is a sequence consisting of all zeros of f in $B'_R(0)$ such that $\zeta_j \neq \zeta_k$ and $|\zeta_j| \leq |\zeta_k|$ whenever $j < k$, then*

$$m \ln r + \ln\left|\frac{f^{(m)}(0)}{m!}\right| = \sum_{j=1}^{N(r)} k_j \ln\left|\frac{\zeta_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(re^{it})\right| dt \tag{9.14}$$

for any $r \in (0,R)$, where $k_j = \operatorname{ord}(f,\zeta_j)$ and $N(r) = \max\{j : \zeta_j \in B_r(0)\}$.

Proof. Fix $r \in (0,R)$, so that $\overline{B}_r(0) \subseteq B_R(0)$. The set $S = Z(f) \cap \overline{B}_r(0)$ must be finite, since otherwise S must have a limit point in the compact set $\overline{B}_r(0)$, and hence in $B_R(0)$, and then the Identity Theorem would imply that $f \equiv 0$ on $B_R(0)$. Now, the finiteness of S implies the finiteness of $Z(f) \cap B_r(0)$, and therefore $N(r) \in \mathbb{N}$. In particular,

$$Z(f) \cap B'_r(0) = \{\zeta_1, \ldots, \zeta_{N(r)}\}.$$

By Proposition 5.11, $f^m(0) \neq 0$, and $f^k(0) = 0$ for all $1 \leq k \leq m-1$, so that

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{9.15}$$

for all $z \in B_R(0)$ by Theorem 4.29. Also there exists some function g that is analytic on $B_R(0)$, with $g(0) \neq 0$, and such that

$$f(z) = z^m g(z) \tag{9.16}$$

for all $z \in B_R(0)$. From (9.15) and (9.16) we obtain, for $z \in B'_R(0)$,

$$g(z) = z^{-m} f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n-m} = \frac{f^{(m)}(0)}{m!} + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n-m},$$

where we must have

$$g(0) = \frac{f^{(m)}(0)}{m!}$$

since the analyticity of g on $B_R(0)$ implies its continuity at 0. We have $Z(g) = Z(f) \setminus \{0\}$ by Proposition 5.8, with $\operatorname{ord}(g,\zeta_j) = \operatorname{ord}(f,\zeta_j) = k_j$ for all $j \geq 1$.

We now see that g is analytic and nonvanishing on $\overline{B}_r(0)$, with

$$Z(g, B_r(0)) = Z(g) \cap B_r(0) = \{\zeta_1, \ldots, \zeta_{N(r)}\}$$

and $\operatorname{ord}(g,\zeta_j) = k_j$ for each $1 \leq j \leq N(r)$. By Theorem 9.18,

$$\ln|g(z)| = \sum_{j=1}^{N(r)} k_j \ln\left|\frac{(z-\zeta_j)r}{r^2 - \bar{\zeta}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/r}(t) \ln\left|g(re^{it})\right| dt,$$

and since $P_0 \equiv 1$,

$$\ln|g(0)| = \sum_{j=1}^{N(r)} k_j \ln\left|\frac{\zeta_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|g(re^{it})\right| dt.$$

Now, for $t \in [0, 2\pi]$,

$$\ln\left|g(re^{it})\right| = \ln\left|(re^{it})^{-m} f(re^{it})\right| = \ln\!\big(r^{-m}\left|f(re^{it})\right|\big) = -m\ln r + \ln\left|f(re^{it})\right|$$

so that

$$\ln\left|\frac{f^{(m)}(0)}{m!}\right| = \sum_{j=1}^{N(r)} k_j \ln\left|\frac{\zeta_j}{r}\right| - m\ln r + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(re^{it})\right| dt,$$

whence (9.14) results. ■

Exercise 9.20 (AN4.8.2). Let f have set of poles $P(f) = \{p_1, \ldots, p_m\} \subseteq B_r'(0)$, with $\operatorname{ord}(f, p_j) = \ell_j$. Suppose f is analytic on $B_r(0) \setminus P(f)$, continuous on $\overline{B}_r(0) \setminus P(f)$, and nonvanishing on $C_r(0)$. If $Z(f) = \{\zeta_1, \ldots, \zeta_n\} \subseteq B_r'(0)$ with $\operatorname{ord}(f, \zeta_j) = k_j$, then

$$\ln|f(0)| = \sum_{j=1}^{n} k_j \ln\left|\frac{\zeta_j}{r}\right| - \sum_{j=1}^{m} \ell_j \ln\left|\frac{p_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(re^{it})\right| dt.$$

Solution. First suppose that $r = 1$. The function

$$g(z) = \prod_{j=1}^{m} (z - p_j)^{\ell_j} f(z)$$

has a removable singularity at each p_j by Theorem 5.10(1), so that g has continuous extension to $\overline{\mathbb{B}}$ which, by Corollary 4.22, is analytic on $\mathbb{B}$. Theorem 7.10(1) also indicates that the extended function is nonvanishing on $P(f)$. Then, since $1/\overline{p}_j \notin \overline{\mathbb{B}}$ for each $1 \le j \le m$, we conclude that

$$h(z) = \prod_{j=1}^{m} \frac{g(z)}{(1 - \overline{p}_j z)^{\ell_j}} = \prod_{j=1}^{m} \left(\frac{z - p_j}{1 - \overline{p}_j z}\right)^{\ell_j} f(z)$$

is continuous on $\overline{\mathbb{B}}$, analytic on $\mathbb{B}$, and nonvanishing on $\partial\mathbb{B}$. Also we have $Z(h) = Z(f)$ with $\operatorname{ord}(h, \zeta_j) = \operatorname{ord}(f, \zeta_j) = k_j$ for each $1 \le j \le n$, and $h(0) \neq 0$. Therefore, by the Poisson-Jensen Formula,

$$\ln|h(z)| = \sum_{j=1}^{n} k_j \ln\left|\frac{z - \zeta_j}{1 - \overline{\zeta}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_z(t) \ln\left|h(e^{it})\right| dt$$

for all $z \in \mathbb{B} \setminus Z(h)$, and thus

$$\ln|h(0)| = \sum_{j=1}^{n} k_j \ln|\zeta_j| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|h(e^{it})\right| dt. \tag{9.17}$$

By Proposition 8.24(4), for each $1 \le j \le m$,

$$\left|\frac{z - p_j}{1 - \overline{p}_j z}\right| = \left|\varphi_{p_j}(z)\right| = 1$$

for all $z \in \partial\mathbb{B}$, and so for all $t \in [0, 2\pi]$,

$$\ln\left|h(e^{it})\right| = \ln\left|f(e^{it})\right|.$$

Now,

$$\ln|h(0)| = \ln\left|f(0)\prod_{j=1}^{m} p_j^{\ell_j}\right| = \ln|f(0)| + \ln\left(\prod_{j=1}^{m}|p_j|^{\ell_j}\right) = \ln|f(0)| + \sum_{j=1}^{m}\ell_j\ln|p_j|,$$

and so from (9.17) we obtain

$$\ln|f(0)| = \sum_{j=1}^{n} k_j\ln|\zeta_j| - \sum_{j=1}^{m}\ell_j\ln|p_j| + \frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(e^{it})\right|dt. \tag{9.18}$$

This proves the statement of the theorem in the case when $r = 1$.

Next, let the radius $r > 0$ be arbitrary, let $z_j = \zeta_j/r$ for each $1 \le j \le n$, and let $q_j = p_j/r$ for each $1 \le j \le m$, so that $\{q_1, \dots, q_m\} \subseteq \mathbb{B}$. Define $F : \overline{\mathbb{B}} \setminus \{q_1, \dots, q_m\} \to \mathbb{C}$ by $F(z) = f(rz)$. Then

$$P(F) = \{q_1, \dots, q_m\} \subseteq \mathbb{B} \setminus \{0\}$$

with $\operatorname{ord}(F, q_j) = \operatorname{ord}(f, p_j) = \ell_j$ for each $1 \le j \le m$. Moreover F is analytic on $\mathbb{B} \setminus P(F)$, continuous on $\overline{\mathbb{B}} \setminus P(F)$, nonvanishing on $\partial\mathbb{B}$, and

$$Z(F) = \{z_1, \dots, z_n\} \subseteq \mathbb{B} \setminus \{0\}$$

with $\operatorname{ord}(F, z_j) = \operatorname{ord}(f, \zeta_j) = k_j$. All necessary hypotheses are satisfied to permit the employment of (9.18) to obtain

$$\ln|F(0)| = \sum_{j=1}^{n} k_j\ln|z_j| - \sum_{j=1}^{m}\ell_j\ln|q_j| + \frac{1}{2\pi}\int_0^{2\pi}\ln\left|F(e^{it})\right|dt,$$

and therefore

$$\ln|f(0)| = \sum_{j=1}^{n} k_j\ln\left|\frac{\zeta_j}{r}\right| - \sum_{j=1}^{m}\ell_j\ln\left|\frac{p_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(re^{it})\right|dt$$

as desired. ■

Exercise 9.21 (AN4.8.3). Let $N(r)$ be defined as in Theorem 9.19, the Generalized Jensen's Formula, and let

$$n(r) = \sum_{j=1}^{N(r)} k_j,$$

the sum of the multiplicities of the zeros of f in $B_r'(0)$, where we take $n(r) = 0$ if f has no zeros in $B_r'(0)$.[10] Show that

$$\int_0^r \frac{n(t)}{t}dt = \sum_{j=1}^{N(r)} k_j\ln\frac{r}{|\zeta_j|}.$$

[10] This is the same definition of $n(r)$ that is given by [AN].

Solution. Assume $N(r) \geq 2$, let $S = \{1, \ldots, N(r) - 1\}$, and define $A \subseteq S$ by

$$A = \{p \in S : |\zeta_p| < |\zeta_{p+1}|\} = \{p_1, \ldots, p_s\},$$

where we take $p_k < p_\ell$ whenever $k < \ell$. Thus,

$$0 < |\zeta_1| = \cdots = |\zeta_{p_1}| < |\zeta_{p_1+1}| = \cdots = |\zeta_{p_2}| < \cdots < |\zeta_{p_s+1}| = \cdots = |\zeta_{N(r)}| < r$$

Let $r_0 = 0$, $r_\ell = |\zeta_{p_\ell}|$ for each $1 \leq \ell \leq s$, $r_{s+1} = |\zeta_{p_s+1}|$, and $r_{s+2} = r$. Also let $p_{s+1} = N(r)$. Then

$$n(t) = \begin{cases} 0, & t \in [0, r_1] \\ \sum_{j=1}^{p_1} k_j, & t \in (r_1, r_2] \\ \quad\vdots & \quad\vdots \\ \sum_{j=1}^{p_{s+1}} k_j, & t \in (r_{s+1}, r_{s+2}], \end{cases}$$

and so

$$\int_0^r \frac{n(t)}{t} dt = \sum_{\ell=0}^{s+1} \int_{r_\ell}^{r_{\ell+1}} \frac{n(t)}{t} dt = \sum_{\ell=1}^{s+1} \int_{r_\ell}^{r_{\ell+1}} \frac{n(t)}{t} dt, \tag{9.19}$$

the last equality owing to

$$\int_0^{r_1} \frac{n(t)}{t} dt = \lim_{b \to 0^+} \int_b^{r_1} \frac{n(t)}{t} dt = \lim_{b \to 0^+} \int_b^{r_1} (0) dt = 0.$$

Now,

$$\begin{aligned}
\sum_{\ell=1}^{s+1} \int_{r_\ell}^{r_{\ell+1}} \frac{n(t)}{t} dt &= \sum_{\ell=1}^{s+1} \int_{r_\ell}^{r_{\ell+1}} \left(\frac{1}{t} \sum_{j=1}^{p_\ell} k_j \right) dt = \sum_{\ell=1}^{s+1} \left(\sum_{j=1}^{p_\ell} k_j \ln \frac{r_{\ell+1}}{r_\ell} \right) \\
&= \sum_{j=1}^{p_1} k_j \ln \frac{r_2}{r_1} + \sum_{j=1}^{p_2} k_j \ln \frac{r_3}{r_2} + \sum_{j=1}^{p_3} k_j \ln \frac{r_4}{r_3} + \cdots + \sum_{j=1}^{p_{s+1}} k_j \ln \frac{r_{s+2}}{r_{s+1}} \\
&= \left(\ln \frac{r_2}{r_1} + \ln \frac{r_3}{r_2} + \cdots + \ln \frac{r_{s+2}}{r_{s+1}} \right) \sum_{j=1}^{p_1} k_j + \left(\ln \frac{r_3}{r_2} + \ln \frac{r_4}{r_3} + \cdots + \ln \frac{r_{s+2}}{r_{s+1}} \right) \sum_{j=p_1+1}^{p_2} k_j \\
&\quad + \left(\ln \frac{r_4}{r_3} + \ln \frac{r_5}{r_4} + \cdots + \ln \frac{r_{s+2}}{r_{s+1}} \right) \sum_{j=p_2+1}^{p_3} k_j + \cdots + \ln \frac{r_{s+2}}{r_{s+1}} \sum_{j=p_s+1}^{p_{s+1}} k_j \\
&= \ln \frac{r_{s+2}}{r_1} \sum_{j=1}^{p_1} k_j + \ln \frac{r_{s+2}}{r_2} \sum_{j=p_1+1}^{p_2} k_j + \ln \frac{r_{s+2}}{r_3} \sum_{j=p_2+1}^{p_3} k_j + \cdots + \ln \frac{r_{s+2}}{r_{s+1}} \sum_{j=p_s+1}^{p_{s+1}} k_j \\
&= \sum_{j=1}^{p_1} k_j \ln \frac{r_{s+2}}{|\zeta_j|} + \sum_{j=p_1+1}^{p_2} k_j \ln \frac{r_{s+2}}{|\zeta_j|} + \sum_{j=p_2+1}^{p_3} k_j \ln \frac{r_{s+2}}{|\zeta_j|} + \cdots + \sum_{j=p_s+1}^{p_{s+1}} k_j \ln \frac{r_{s+2}}{|\zeta_j|} \\
&= \sum_{j=1}^{p_{s+1}} k_j \ln \frac{r_{s+2}}{|\zeta_j|} = \sum_{j=1}^{N(r)} k_j \ln \frac{r}{|\zeta_j|},
\end{aligned}$$

which together with (9.19) gives the desired result. ■

Exercise 9.22 (AN4.8.4). Let f be as in Theorem 9.19, the Generalized Jensen's Formula, and fix $r \in (0, R)$. Show that if

$$M(r) = \max\{|f(z)| : z \in C_r(0)\},$$

then

$$\int_0^r \frac{n(t)}{t} dt \leq \ln\left(\frac{M(r)m!}{r^m |f^{(m)}(0)|}\right) \tag{9.20}$$

Solution. First we observe that $0 < |\zeta_j| < r$ implies $r/|\zeta_j| > 1$, and hence

$$\ln \frac{r}{|\zeta_j|} > 0$$

for each j. Now, from the Generalized Jensen's Formula,

$$-\sum_{j=1}^{N(r)} k_j \ln \left|\frac{\zeta_j}{r}\right| = -m \ln r - \ln \left|\frac{f^{(m)}(0)}{m!}\right| + \frac{1}{2\pi} \int_0^{2\pi} \ln \left|f(re^{it})\right| dt,$$

and hence

$$\begin{aligned}
\sum_{j=1}^{N(r)} k_j \ln \frac{r}{|\zeta_j|} &= \ln \frac{1}{r^m} + \ln \left|\frac{m!}{f^{(m)}(0)}\right| + \frac{1}{2\pi} \int_0^{2\pi} \ln \left|f(re^{it})\right| dt \\
&\leq \ln \frac{1}{r^m} + \ln \left|\frac{m!}{f^{(m)}(0)}\right| + \frac{1}{2\pi} \int_0^{2\pi} \ln[M(r)]\, dt \\
&= \ln \frac{1}{r^m} + \ln \frac{m!}{|f^{(m)}(0)|} + \ln[M(r)] = \ln\left(\frac{M(r)m!}{r^m |f^{(m)}(0)|}\right).
\end{aligned}$$

This result, together with the result of Exercise 9.21, delivers (9.20). ■

10

Analytic Continuation

10.1 – Natural Boundaries

If f is analytic on Ω, then an **analytic extension** of f to an open set $\Omega' \supseteq \Omega$ is an analytic function $g : \Omega' \to \mathbb{C}$ such that $f \equiv g$ on Ω.

Definition 10.1. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{10.1}$$

have radius of convergence $r \in (0, \infty)$. A point $z^* \in C_r(z_0)$ is a ***singular point*** *of f if f does not have an analytic extension to an open set $\Omega \supseteq B_r(z_0) \cup \{z^*\}$. If every point on $C_r(z_0)$ is a singular point of f, then $C_r(z_0)$ is called the* ***natural boundary*** *for f.*

Lemma 10.2. *Let f given by (10.1) have radius of convergence $r \in (0, \infty)$, and let $r(t)$ be the radius of convergence of the power series representation of f about*

$$z_t = (1 - t)z_0 + tz^*$$

for each $t \in (0, 1)$. Then $r(t) \geq (1 - t)r$ for all $t \in (0, 1)$.

Proof. Let $0 < t < 1$ be arbitrary. Since $B_{(1-t)r}(z_t) \subseteq B_r(z_0)$ and f is analytic on $B_r(z_0)$ by Proposition 4.31(1), it follows that f is analytic on $B_{(1-t)r}(z_t)$. By Theorem 4.29 f is representable by power series in $B_{(1-t)r}(z_t)$, and in particular

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_t)}{n!} (z - z_t)^n \tag{10.2}$$

for all $z \in B_{(1-t)r}(z_t)$, with the series converging absolutely on $B_{(1-t)r}(z_t)$. Since the series at right in (10.2) is the power series representation of f about z_t, Theorem 4.2(1) implies that $r(t) \geq (1 - t)r$. ∎

Proposition 10.3. *Let f given by (10.1) have radius of convergence $r \in (0, \infty)$, let $z^* \in C_r(z_0)$, and let $r(t)$ be the radius of convergence of the power series representation of f about*

$$z_t = (1 - t)z_0 + tz^*$$

for each $t \in (0,1)$. Then the following statements are equivalent.
1. *z^* is a singular point of f.*
2. *$r(\tau) = (1-\tau)r$ for some $\tau \in (0,1)$.*
3. *$r(t) = (1-t)r$ for all $t \in (0,1)$.*

Proof.
(1) → (3): Suppose there exists some $\tau \in (0,1)$ such that $r(\tau) \neq (1-\tau)r$. Then $r(\tau) > (1-\tau)r$ by Lemma 10.2, so there exists some $\epsilon > 0$ such that $r(\tau) = (1-\tau)r + \epsilon$. That is, the power series representation of f about z_τ,

$$\sum_{n=0}^{\infty} b_n(z - z_\tau)^n,$$

has radius of convergence $r(\tau) = (1-\tau)r + \epsilon$, implying that the series converges absolutely on $B_{r(\tau)}(z_\tau)$. Let $\Omega = B_r(z_0) \cup B_\epsilon(z^*)$, and note that $B_\epsilon(z^*) \subseteq B_{r(\tau)}(z_\tau)$ since

$$|z^* - z_\tau| = \left|z^* - (1-\tau)z_0 - \tau z^*\right| = (1-\tau)|z^* - z_0| = (1-\tau)r,$$

and so for any $z \in B_\epsilon(z^*)$,

$$|z - z_\tau| \leq |z^* - z_\tau| + |z - z^*| < (1-\tau)r + \epsilon = r(\tau).$$

Define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \begin{cases} \sum_{n=0}^{\infty} a_n(z-z_0)^n, & z \in B_r(z_0) \\ \sum_{n=0}^{\infty} b_n(z-z_\tau)^n, & z \in B_\epsilon(z^*) \end{cases}$$

The function g is well-defined on $S = B_\epsilon(z^*) \cap B_r(z_0) \subseteq B_{r(\tau)}(z_\tau)$, since for each $z \in S$,

$$\sum_{n=0}^{\infty} b_n(z - z_\tau)^n = f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Also g is analytic on Ω since, by Proposition 3.14(1), it is analytic on $B_\epsilon(z^*)$ and $B_r(z_0)$. Finally, it is clear that $f \equiv g$ on $B_r(z_0)$. Therefore g is an analytic extension of f to an open set containing $B_r(z_0) \cup \{z^*\}$, and we conclude that z^* is not a singular point of f.

(3) → (2): This is obvious.

(2) → (1): Suppose $z^* \in C_r(z_0)$ is not a singular point of f. Then there exists an analytic function $g : \Omega \to \mathbb{C}$ such that $\Omega \supseteq B_r(z_0) \cup \{z^*\}$ and $f \equiv g$ on $B_r(z_0)$. Fix $t \in (0,1)$. Since Ω is open, there exists some $\delta > 0$ such that $B_\delta(z^*) \subseteq \Omega$, and consequently there is some $\epsilon > 0$ sufficiently small that $B = B_{(1-t)r+\epsilon}(z_t) \subseteq \Omega$. By Theorem 4.29 there exists a sequence $(c_n)_{n=0}^{\infty}$ in $\mathbb{C}$ such that

$$g(z) = \sum_{n=0}^{\infty} c_n(z - z_t)^n$$

for all $z \in B$. Let $B' = B_{(1-t)r}(z_t)$. Since $B' \subseteq B_r(z_0)$, $f \equiv g$ on B'; so since $B' \subseteq B$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_t)^n$$

for all $z \in B'$. Thus $\sum c_n(z-z_t)^n$ is the power series representation of f about z_t, and its radius of convergence is seen to be

$$r(t) \geq (1-t)r + \epsilon > (1-t)r.$$

Therefore if z^* is not a singular point of f, then $r(t) > (1-t)r$ for all $t \in (0,1)$. The contrapositive states that if $r(\tau) \leq (1-\tau)r$ for some $\tau \in (0,1)$, then z^* is a singular point of f. Lemma 10.2, of course, makes clear that $r(\tau) < (1-\tau)r$ is never possible. ■

Theorem 10.4. *Let f given by* (10.1) *have radius of convergence $r \in (0,\infty)$. Then f has at least one singular point on $C_r(z_0)$.*

Proof. Suppose that f has no singular point on $C_r(z_0)$. Then for each $z \in C_r(z_0)$ there exists some $\delta_z > 0$ and some function g_z analytic on $\Omega_z = B_r(z_0) \cup B_{\delta_z}(z)$ such that $g_z \equiv f$ on $B_r(z_0)$. Let $B_z = B_{\delta_z}(z)$ for each z. Now, $\{B_z : z \in C_r(z_0)\}$ is an open cover for the compact set $C_r(z_0)$, and so there can be found $z_1, \ldots, z_m \in C_r(z_0)$ such that $\{B_{z_k} : 1 \leq k \leq m\}$ is a finite subcover. For each $1 \leq k \leq m$ we have analytic $g_{z_k} : \Omega_{z_k} \to \mathbb{C}$ such that $g_{z_k} \equiv f$ on $B_r(z_0)$.

Let

$$\Omega = \bigcup_{k=1}^{m} \Omega_{z_k} = B_r(z_0) \cup \left(\bigcup_{k=1}^{m} B_{z_k} \right),$$

and define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z), & z \in B_r(z_0) \\ g_{z_k}(z), & z \in B_{z_k}. \end{cases}$$

For any $1 \leq k \leq m$ and $z \in B_r(z_0) \cap B_{z_k}$ we have $g_{z_k}(z) = f(z)$, and so it is clear that g is well-defined at least on $B_r(z_0)$. Suppose $1 \leq j < k \leq m$ are such that $B_{z_j} \cap B_{z_k} \neq \varnothing$. For any

$$z \in S_{jk} := B_{z_j} \cap B_{z_k} \cap B_r(z_0)$$

we have

$$g_{z_j}(z) = f(z) = g_{z_k}(z),$$

and since S_{jk} is a nonempty open set in $B_{z_j} \cap B_{z_k}$, we see that $Z(g_{z_j} - g_{z_k})$ has a limit point in $B_{z_j} \cap B_{z_k}$, and therefore $g_{z_j} - g_{z_k} \equiv 0$ on $B_{z_j} \cap B_{z_k}$ by the Identity Theorem. That is, $g_{z_j}(z) = g_{z_k}(z)$ for all $z \in B_{z_j} \cap B_{z_k}$, and we conclude that g is well-defined throughout Ω.

Now, $\overline{B}_r(z_0) \subseteq \Omega$ since $C_r(z_0)$ is covered by $\{B_{z_k} : 1 \leq k \leq m\}$, and g is analytic on Ω such that $g \equiv f$ on $B_r(z_0)$. By Exercise 2.47 there exists some $\epsilon > 0$ such that $B_{r+\epsilon}(z_0) \subseteq \Omega$, and since g is analytic on $B_{r+\epsilon}(z_0)$ there is a sequence of complex numbers $(b_n)_{n=0}^{\infty}$ such that

$$g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

for all $z \in B_{r+\epsilon}(z_0)$. However,

$$\sum_{n=0}^{\infty} b_n(z-z_0)^n = g(z) = f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for all $z \in B_r(z_0)$, so that $a_n = b_n$ for all n by Corollary 4.33, and therefore

$$g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in B_{r+\epsilon}(z_0)$. This implies that $\sum a_n(z - z_0)^n$ has radius of convergence $\rho \geq r + \delta$, which is a contradiction.

Therefore f must have at least one singular point on $C_r(z_0)$. ■

Lemma 10.5. *Let $n \in \mathbb{N}$. If*

$$\varphi(z) = \frac{z^n + z^{n+1}}{2}$$

for all $z \in \mathbb{C}$, then the following hold.

1. $\varphi(\mathbb{B}) \subseteq \mathbb{B}$.
2. *For any $\epsilon > 0$ there exists some $r > 1$ such that $\varphi(B_r(0)) \subseteq \mathbb{B} \cup B_\epsilon(1)$.*

Theorem 10.6 (Ostrowski-Hadamard Gap Theorem). *Let $s > 1$, let $(n_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that $n_{k+1}/n_k \geq s$ for all k, and let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{C}$. If*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

and the radius of convergence of the series is 1, then $\partial\mathbb{B}$ is the natural boundary for f.

Proof. Suppose 1 is not a singular point of f, so there exists some $\epsilon > 0$ and analytic function $g : \mathbb{B} \cup B_\epsilon(1) \to \mathbb{C}$ such that $g \equiv f$ on $\mathbb{B}$. Let $p \in \mathbb{N}$ be such that $1 < (p+1)/p < s$, and let φ be defined as in Lemma 10.5. Then $\varphi(\mathbb{B}) \subseteq \mathbb{B}$ by the same lemma, and there exists some $r > 1$ such that $\varphi(B_r(0)) \subseteq \mathbb{B} \cup B_\epsilon(1)$. By the Chain Rule the function $h = g \circ \varphi$ is analytic on $B_r(0)$, and for any $z \in \mathbb{B}$,

$$\begin{aligned} h(z) = g(\varphi(z)) = f(\varphi(z)) &= \sum_{k=1}^{\infty} a_k[\varphi(z)]^{n_k} = \sum_{k=1}^{\infty} a_k\left(\frac{z^p + z^{p+1}}{2}\right)^{n_k} \\ &= \sum_{k=1}^{\infty} a_k 2^{-n_k}\left(z^p + z^{p+1}\right)^{n_k} = \sum_{k=1}^{\infty}\left[a_k 2^{-n_k}\sum_{n=0}^{n_k}\binom{n_k}{n} z^{p(n_k-n)} z^{(p+1)n}\right] \\ &= \sum_{k=1}^{\infty}\left[a_k 2^{-n_k}\sum_{n=0}^{n_k}\binom{n_k}{n} z^{pn_k+n}\right], \end{aligned} \tag{10.3}$$

since $\varphi(z) \in \mathbb{B}$. We see that the series (10.3) converges on $\mathbb{B}$, and therefore is absolutely convergent on $\mathbb{B}$ by Theorem 4.2.

Now, for any $k \in \mathbb{N}$,

$$\frac{p+1}{p} < s \leq \frac{n_{k+1}}{n_k} \quad\Rightarrow\quad pn_k + n_k < pn_{k+1},$$

and so because the kth term in (10.3) is

$$a_k 2^{-n_k}\left[\binom{n_k}{0} z^{pn_k} + \binom{n_k}{1} z^{pn_k+1} + \cdots + \binom{n_k}{n_k} z^{pn_k+n_k}\right]$$

whilst the $(k+1)$st term is

$$a_{k+1}2^{-n_{k+1}}\left[\binom{n_{k+1}}{0}z^{pn_{k+1}}+\binom{n_{k+1}}{1}z^{pn_{k+1}+1}+\cdots+\binom{n_{k+1}}{n_{k+1}}z^{pn_{k+1}+n_{k+1}}\right],$$

it follows that each power of z that appears in (10.3) does so only once. Thus, since (10.3) is absolutely convergent on $\mathbb{B}$, it can be rearranged as a power series $\sum b_n z^n$. On the other hand, by Theorem 4.29, the analyticity of h on $B_r(0)$ implies there is a sequence of complex numbers $(c_n)_{n=0}^{\infty}$ such that

$$h(z)=\sum_{n=0}^{\infty}c_n z^n$$

for all $z\in B_r(0)$. But then $\sum b_n z^n=\sum c_n z^n=h(z)$ for all $z\in\mathbb{B}$, so by Corollary 4.16 we conclude that $b_n=c_n$ for all $n\geq 0$, and thus $\sum c_n z^n$ is the series (10.3) rearranged as a power series. This leads us to conclude that (10.3) converges absolutely on $B_r(0)$; that is,

$$\sum_{n=0}^{\infty}|c_n||z|^n\in\mathbb{C}$$

for all $z\in B_r(0)$. Reversing the rearrangement then yields

$$\sum_{k=1}^{\infty}\left[|a_k|2^{-n_k}\sum_{n=0}^{n_k}\binom{n_k}{n}|z|^{pn_k+n}\right]\in\mathbb{C},$$

which, recalling that z^n appears at most once in (10.3) for each $n\geq 0$, finally gives

$$\sum_{k=1}^{\infty}|a_k|2^{-n_k}\left(|z|^p+|z|^{p+1}\right)^{n_k}\in\mathbb{C}$$

for all $z\in B_r(0)$.

Fix $w\in B_r(0)\setminus\mathbb{B}$, so $1<|w|<r$, and also

$$|w|^p>1\quad\text{and}\quad\frac{1+|w|}{2}>1.$$

Now,

$$2^{-n_k}\left(|w|^p+|w|^{p+1}\right)^{n_k}=\left(\frac{|w|^p+|w|^{p+1}}{2}\right)^{n_k}=\left[|w|^p\left(\frac{1+|w|}{2}\right)\right]^{n_k}>1$$

for each $k\geq 1$, and if we let $z\in\mathbb{C}$ be such that

$$|z|=|w|^p\left(\frac{1+|w|}{2}\right),$$

then

$$\sum_{k=1}^{\infty}|a_k||z|^{n_k}=\sum_{k=1}^{\infty}|a_k|2^{-n_k}\left(|w|^p+|w|^{p+1}\right)^{n_k}\in\mathbb{C};$$

that is, $\sum a_k z^{n_k}$ converges for some z such that $|z|>1$, contradicting the hypothesis that $\sum a_k z^{n_k}$ has radius of convergence 1. Therefore 1 must be a singular point of f.

Now suppose that some arbitrary $z^*\in\partial\mathbb{B}$ is not a singular point of f, so there exists some $\epsilon>0$ and analytic function $g:\mathbb{B}\cup B_\epsilon(e^{it})\to\mathbb{C}$ such that $g\equiv f$ on $\mathbb{B}$. Let $t\in[0,2\pi)$ be such that $z^*=e^{it}$, and define $\rho:\mathbb{C}\to\mathbb{C}$ to be the rotation $\rho(z)=e^{it}z$.

Since $\rho(\mathbb{B}) \subseteq \mathbb{B}$, we may define $F = f \circ \rho$, where

$$F(z) = f(\rho(z)) = \sum_{k=1}^{\infty} a_k(\rho(z))^{n_k} = \sum_{k=1}^{\infty} a_k(e^{it}z)^{n_k} = \sum_{k=1}^{\infty} a_k e^{itn_k} z^{n_k} = \sum_{k=1}^{\infty} \alpha_k z^{n_k} \tag{10.4}$$

for each $z \in \mathbb{B}$, setting $\alpha_k = a_k e^{itn_k}$ for each k. Note that $|\alpha_k| = |a_k|$ implies that the radius of convergence of $\sum \alpha_k z^{n_k}$ is 1.

Since $\rho(\mathbb{B} \cup B_\epsilon(1)) = \mathbb{B} \cup B_\epsilon(e^{it})$, we may define $G = g \circ \rho$, which is analytic on $\mathbb{B} \cup B_\epsilon(1)$ by the Chain Rule. For $z \in \mathbb{B}$ we have

$$G(z) = g(\rho(z)) = f(\rho(z)) = F(z),$$

since $\rho(z) \in \mathbb{B}$ and $g \equiv f$ on $\mathbb{B}$. It follows that G is an analytic continuation of F to an open set containing $\mathbb{B} \cup \{1\}$, which is to say 1 is not a singular point for F as given by (10.4), contradicting the foregoing argument involving f and $\sum a_k z^{n_k}$. Therefore z^* must be a singular point for f, and we conclude that f has no singular points on $\partial\mathbb{B}$. ■

Exercise 10.7 (AN4.9.3). Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{10.5}$$

have radius of convergence $r \in (0, \infty)$. If $z_1 \in B_r(z_0)$, then for some sufficiently small $\rho > 0$ the series (10.5) may be rearranged to become a power series with center z_1 that is in fact the power series representation of f at z_1. Specifically we obtain

$$f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} \binom{n}{k} a_n (z_1 - z_0)^{n-k} \right] (z - z_1)^k \tag{10.6}$$

for all $z \in B_\rho(z_1)$. Confirm this, and make an argument for why, if (10.6) converges at some $z \notin \overline{B}_r(z_0)$, it does not then follow that (10.5) converges at z in contradiction to Theorem 4.2(1) which states that (10.5) must diverge outside of $\overline{B}_r(z_0)$.

Solution. To start, we observe that ρ must be such that $|z_1 - z_0| + \rho < r$. Then for any $z \in B_\rho(z_1)$ we have

$$|z - z_0| \le |z - z_1| + |z_1 - z_0| < \rho + (r - \rho) = r,$$

and hence $B_\rho(z_1) \subseteq B_r(z_0)$. Moreover, since (10.5) converges absolutely on $B_r(z_0)$ by Theorem 4.2, and $|z - z_1| + |z_1 - z_0| < r$, it follows that

$$\sum_{n=0}^{\infty} |a_n| \big(|z - z_1| + |z_1 - z_0|\big)^n < +\infty.$$

By the Binomial Theorem,

$$\big(|z - z_1| + |z_1 - z_0|\big)^n = \sum_{k=0}^{n} \binom{n}{k} |z_1 - z_0|^{n-k} |z - z_1|^k,$$

and so
$$\sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{n}\binom{n}{k}|z_1-z_0|^{n-k}|z-z_1|^k=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}|a_n||z_1-z_0|^{n-k}|z-z_1|^k<+\infty.$$
We then obtain
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}|a_n||z_1-z_0|^{n-k}|z-z_1|^k<+\infty$$
since $\binom{n}{k}=0$ for all $k>n$, so the double series
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}(z-z_1)^k \tag{10.7}$$
is absolutely convergent on $B_\rho(z_1)$, and therefore interchanging the order of summation will not alter its value. That is,
$$\begin{aligned} f(z)&=\sum_{n=0}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}a_n\big[(z-z_1)+(z_1-z_0)\big]^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a_n(z_1-z_0)^{n-k}(z-z_1)^k=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}(z-z_1)^k\\ &=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}(z-z_1)^k=\sum_{k=0}^{\infty}\left[\sum_{n=0}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}\right](z-z_1)^k\\ &=\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}\right](z-z_1)^k \end{aligned}$$
for all $z\in B_\rho(z_1)$, which confirms (10.6).

Suppose now that (10.6) converges for some $z\notin\overline{B}_r(z_0)$, so that $z\notin B_\rho(z_1)$. If the convergence is conditional, then the interchange of the order of summation done above (the fifth equality) may not hold since the value of a conditionally convergent series is not necessarily invariant under rearrangements, and thus we cannot conclude that $\sum a_n(z-z_0)^n$ is convergent. If the convergence is absolute, then
$$\sum_{k=0}^{\infty}\left|\sum_{n=k}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}\right||z-z_1|^k<+\infty;$$
but this does not imply that
$$\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}\binom{n}{k}|a_n||z_1-z_0|^{n-k}\right]|z-z_1|^k<+\infty,$$
which is to say the double series (10.7) is not necessarily absolutely convergent, and once again interchanging the order of summation to obtain (10.5) cannot be justified. ∎

10.2 – Analytic Continuation Along a Curve

Definition 10.8. *A **function element** is an ordered pair (f, B), where B is an open disc and f is an analytic function on B. If $B \subseteq \Omega$, then (f, B) is called a **function element in** Ω; and if $z \in B$, then (f, B) is called a **function element at** z. Two function elements (f, B) and (g, D) are **direct analytic continuations** of each other if $B \cap D \neq \varnothing$ and $f \equiv g$ on $B \cap D$, in which case we write*

$$(f, B) \smile (g, D).$$

*A **chain** is a finite sequence of function elements $\mathcal{C} = (f_k, B_k)_{k=1}^n$ such that*

$$(f_k, B_k) \smile (f_{k+1}, B_{k+1})$$

*for each $1 \leq k \leq n-1$, in which case we say (f_n, B_n) is the **analytic continuation of** (f_1, B_1) **along** $\mathcal{C}$. In general, two function elements are **analytic continuations** of each other if there exists a chain that starts with one and ends with the other.*

*A **chain in** Ω is a chain consisting entirely of function elements in Ω. Two function elements are **analytic Ω-continuations** of each other if there exists a chain in Ω that starts with one and ends with the other.*

If (f_1, B_1) and (f_2, B_2) are direct analytic continuations of each other, then the function $F : B_1 \cup B_2 \to \mathbb{C}$ given by

$$F(z) = \begin{cases} f_1(z), & z \in B_1 \\ f_2(z), & z \in B_2 \end{cases}$$

is in fact an analytic continuation of f_1 from B_1 to $B_1 \cup B_2$, and also an analytic continuation of f_2 from B_2 to $B_1 \cup B_2$. In contrast, if for some $n \geq 3$ the function element (f_n, B_n) is given to be an analytic continuation of (f_1, B_1) along a chain $\mathcal{C}$, and $B = \bigcup_{k=1}^n B_k$, it is not necessarily the case that $F : B \to \mathbb{C}$ given by

$$F(z) = \begin{cases} f_1(z), & z \in B_1 \\ \quad\vdots & \\ f_n(z), & z \in B_n \end{cases}$$

is an analytic extension of f_1 to B, even if $B_1 \cap B_n \neq \varnothing$.[11]

Definition 10.9. *Let (f, B) and (g, D) be function elements, and let $\gamma : [a, b] \to \mathbb{C}$ be a curve such that $\gamma(a) \in B$ and $\gamma(b) \in D$. If there is a chain $\mathcal{C} = (f_k, B_k)_{k=1}^n$ with $(f_1, B_1) = (f, B)$ and $(f_n, B_n) = (g, D)$, and a partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

*with $\gamma([t_{k-1}, t_k]) \subseteq B_k$ for each $1 \leq k \leq n$, then (g, D) is said to be an **analytic continuation of** (f, B) **along** γ.*

*If (f, B) and (g, D), the curve γ, and the chain $\mathcal{C}$ all lie in Ω, then (g, D) is called an **analytic Ω-continuation of** (f, B) **along** γ.*

[11] See page 324 of Rudin's *Real and Complex Analysis*.

An "analytic continuation of (f, B) along γ" may be referred to as a "continuation of (f, B) along γ" for brevity.

Given a function element (f, B) and a curve $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) \in B$, we say that (f, B) can be **continued along** γ if there exists a function element (g, D) at $\gamma(b)$ that is an analytic continuation of (f, B) along γ. If (f, B) and (g, D), the curve γ, and the relevant chain $\mathcal{C}$ all lie in Ω, then we say (f, B) can be **Ω-continued along** γ.

Theorem 10.10. *If (f_n, B_n) and (g_m, D_m) are two analytic continuations of (f_1, B_1) along the same curve $\gamma : [a, b] \to \mathbb{C}$, then $(f_n, B_n) \smile (g_m, D_m)$.*

Definition 10.11. *Let Ω be a region. Function elements (f, B) and (g, D) in Ω are **Ω-equivalent** if they are analytic Ω-continuations of each other, and we write*

$$(f, B) \overset{\Omega}{\sim} (g, D).$$

The relation $\overset{\Omega}{\sim}$ is an equivalence relation, and the equivalence class corresponding to a function element (f, B) is denoted by $[f, B]$. That is,

$$[f, B] = \big\{(g, D) : (g, D) \overset{\Omega}{\sim} (f, B)\big\}.$$

*If for every $z \in \Omega$ there is some $(g, D) \in [f, B]$ with $z \in D$, then $[f, B]$ is a **generalized analytic function** on Ω.*

Definition 10.12. *Let $S \subseteq \mathbb{C}$, and let $\gamma_0, \gamma_1 : [a, b] \to S$ be two curves in S such that $\gamma_0(a) = \gamma_1(a) = \alpha$ and $\gamma_0(b) = \gamma_1(b) = \beta$. Then γ_0 and γ_1 are **homotopic in** S (or S-**homotopic**) if there exists a continuous transformation $H : [a, b] \times [0, 1] \to S$ such that*

$$H(\cdot, 0) \equiv \gamma_0, \quad H(\cdot, 1) \equiv \gamma_1, \quad H(a, \cdot) \equiv \alpha, \quad H(b, \cdot) \equiv \beta.$$

*We call H the **homotopy of** γ_0 **and** γ_1.*

By the definition of continuity, if H is a homotopy of γ_0 and γ_1, then for each (s_0, t_0) in $[a, b] \times [0, 1]$ the following holds: For each $\epsilon > 0$ there exists some $\delta > 0$ such that, for all $(s, t) \in [a, b] \times [0, 1]$,

$$\sqrt{(s - s_0)^2 + (t - t_0)^2} < \delta \quad \Rightarrow \quad |H(s, t) - H(s_0, t_0)| < \epsilon.$$

Of course H is also uniformly continuous on its domain, a fact used in the proof of the next theorem.

Theorem 10.13. *Let Ω be a region, and let γ_0, γ_1 be curves that are homotopic in Ω. Let (f, B) be a function element in Ω at $\gamma_0(a)$ that can be Ω-continued along all curves in Ω with initial point $\gamma_0(a)$. If (g_0, D_0) is an Ω-continuation of (f, B) along γ_0, and (g_1, D_1) is an Ω-continuation of (f, B) along γ_1, then $(g_0, D_0) \smile (g_1, D_1)$.*

Proof. Let $\alpha = \gamma_0(a)$, $\beta = \gamma_0(b)$ and $R = [a, b] \times [0, 1]$. Suppose (g_0, D_0) and (g_1, D_1) are Ω-continuations of (f, B) along γ_0 and γ_1, respectively. Let $H : R \to \Omega$ be a homotopy of γ_0 and γ_1. For any $t \in [0, 1]$ the function element (f, B) can be Ω-continued along the curve $\gamma_t = H(\cdot, t)$, which is to say there exists a function element (g_t, D_t) in Ω at β that is an Ω-continuation of (f, B) along γ_t.

Fix $t \in [0,1]$, and let $\mathcal{C} = (h_k, E_k)_{k=1}^n$ be a chain in Ω with $(h_1, E_1) = (f, B)$ and $(h_n, E_n) = (g_t, D_t)$, and let

$$a = s_0 < s_1 < \cdots < s_n = b$$

be a partition of $[a,b]$ with $\Gamma_k := \gamma_t([s_{k-1}, s_k]) \subseteq E_k$ for each $1 \le k \le n$. Each Γ_k is compact since $[s_{k-1}, s_k]$ is compact and γ_t continuous, and so since $\Gamma_k \cap E_k^c = \varnothing$ it follows by Theorem 2.45 that $d(\Gamma_k, E_k^c) > 0$ for each k, and hence

$$\epsilon_t = \min_{1 \le k \le n} d(\Gamma_k, E_k^c) > 0.$$

Now, H is uniformly continuous on R, so there exists $\delta_t > 0$ such that, for all $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in R$,

$$\sqrt{(\sigma_1 - \sigma_2)^2 + (\tau_1 - \tau_2)^2} < \delta_t \quad \Rightarrow \quad |H(\sigma_1, \tau_1) - H(\sigma_2, \tau_2)| < \epsilon_t.$$

Thus, for any $s \in [a,b]$,

$$\tau \in (t - \delta_t, t + \delta_t) \cap [0,1] \quad \Rightarrow \quad |\gamma_t(s) - \gamma_\tau(s)| = |H(s,t) - H(s,\tau)| < \epsilon_t.$$

Fix $\tau \in (t - \delta_t, t + \delta_t) \cap [0,1]$. The function element (g_τ, D_τ) is the Ω-continuation of (f, B) along the curve γ_τ. On the other hand, for any $s \in [s_{k-1}, s_k]$ we have $|\gamma_t(s) - \gamma_\tau(s)| < \epsilon_t$ and $\gamma_t(s) \in \Gamma_k \subseteq E_k$, and hence $\gamma_\tau(s) \in E_k$. That is, for each $1 \le k \le n$ we have $\gamma_\tau([s_{k-1}, s_k]) \subseteq E_k$, which shows that (g_t, D_t) is an Ω-continuation of (f, B) along γ_τ (by way of the chain $\mathcal{C}$), and so $(g_\tau, D_\tau) \smile (g_t, D_t)$ by Theorem 10.10. Therefore for each $t \in [0,1]$ there exists an interval $(t - \delta_t, t + \delta_t)$ such that $g_\tau \equiv g_t$ on $D_\tau \cap D_t$ for every $\tau \in (t - \delta_t, t + \delta_t) \cap [0,1]$. The collection of sets

$$\{(t - \delta_t, t + \delta_t) : t \in [0,1]\}$$

is an open cover for $[0,1]$, and so there exists a finite subcover

$$\left\{ I_j = (t_j - \delta_{t_j}, t_j + \delta_{t_j}) : 1 \le j \le m \right\},$$

arranged so that $0 \in I_1$ and $1 \in I_m$, and $I_j \cap I_{j+1} \neq \varnothing$ for each $1 \le j \le m-1$.

Now, for each $1 \le j \le m$ we have $g_\tau \equiv g_{t_j}$ on $D_\tau \cap D_{t_j}$ for every $\tau \in I_j$, and (g_{t_j}, D_{t_j}) is a function element in Ω at β that is an Ω-continuation of (f, B) along γ_{t_j}. For each $1 \le j \le m-1$ choose some $\tau \in I_j \cap I_{j+1}$. Then $g_\tau \equiv g_{t_j}$ on $D_\tau \cap D_{t_j}$ and $g_\tau \equiv g_{t_{j+1}}$ on $D_\tau \cap D_{t_{j+1}}$, and hence $g_{t_j} \equiv g_{t_{j+1}}$ on $D_\tau \cap D_{t_j} \cap D_{t_{j+1}}$, a nonempty open subset of $D_{t_j} \cap D_{t_{j+1}}$ since it contains β. It follows that $Z(g_{t_j} - g_{t_{j+1}})$ has a limit point in $D_{t_j} \cap D_{t_{j+1}}$, and therefore $g_{t_j} \equiv g_{t_{j+1}}$ on $D_{t_j} \cap D_{t_{j+1}}$ by the Identity Theorem. Also, since $0 \in I_1$ we have $g_0 \equiv g_{t_1}$ on $D_0 \cap D_{t_1}$, and since $1 \in I_m$ we have $g_1 \equiv g_{t_m}$ on $D_1 \cap D_{t_m}$. So on

$$D = D_0 \cap D_1 \cap \left(\bigcap_{j=1}^m D_{t_j} \right)$$

we have

$$g_0 \equiv g_{t_1} \equiv g_{t_2} \equiv \cdots \equiv g_{t_m} \equiv g_1.$$

But D is nonempty since $\beta \in D$, and it is also an open subset of $D_0 \cap D_1$. Since $g_0 \equiv g_1$ on $D \subseteq D_0 \cap D_1$, we find that $Z(g_0 - g_1)$ has a limit point in $D_0 \cap D_1$, and we conclude that $g_0 - g_1 \equiv 0$ on $D_0 \cap D_1$ by the Identity Theorem. That is, $D_0 \cap D_1 \neq \varnothing$ and $g_0 \equiv g_1$ on $D_0 \cap D_1$, and therefore $(g_0, D_0) \smile (g_1, D_1)$. ■

Exercise 10.14 (AN4.9.2). Let $B = B_1(1)$, and let

$$[\mathrm{Log}, B] = \left\{(g, D) : (g, D) \overset{\mathbb{C}_*}{\sim} (\mathrm{Log}, B)\right\},$$

the class of all function elements in $\mathbb{C}_*$ that are analytic $\mathbb{C}_*$-continuations of (Log, B).

(a) Show that $[\mathrm{Log}, B]$ is a generalized analytic function on $\mathbb{C}_*$; that is, for all $z \in \mathbb{C}_*$ there is some $(g, D) \in [\mathrm{Log}, B]$ such that $z \in D$.

(b) Show that there is no function h analytic on $\mathbb{C}_*$ such that for every $(g, D) \in [\mathrm{Log}, B]$ we have $h \equiv g$ on D.

Solution.
(a) By Proposition 6.5 the principal logarithm Log is analytic on $\mathbb{C} \setminus R_{-\pi}$. Let $z \in \mathbb{C} \setminus R_{-\pi}$. The line segment $[1, z]$ is a compact set that does not contain 0, and so $d(0, [1, z]) = r > 0$. We may now construct a chain $\mathcal{C} = (\mathrm{Log}, D_k)_{k=1}^n$ with $D_1 = B$, $D_n = B_r(z)$, and D_k an open disc of radius r and center at some point along $[1, z]$ so that $D_{k-1} \cap D_k \neq \varnothing$ for $2 \leq k \leq n-1$. Thus (Log, D_n) is an analytic $\mathbb{C}_*$-continuation of (Log, B), which is to say $(\mathrm{Log}, D_n) \in [\mathrm{Log}, B]$, where $z \in D_n$.

Now suppose that $z \in R_{-\pi}$ with $z \neq 0$, so that $z = re^{-i\pi} = -r$ for some $r > 0$. We have $\log_0 = \mathbb{C}_* \to H_0$ analytic on $\mathbb{C} \setminus R_0$, and $\log_{-\pi} : \mathbb{C}_* \to H_{-\pi}$ analytic on $\mathbb{C} \setminus R_{-\pi}$, and hence $\log_0$ and $\log_{-\pi}$ are both analytic on $\mathbb{C} \setminus (R_0 \cup R_{-\pi})$. In fact (see Figure 11 in §6.1) $\log_0 \equiv \log_{-\pi} := \mathrm{Log}$ on $\overline{\mathbb{H}} \setminus \{0\}$, with analyticity on $\mathbb{H}$. Let $\mathcal{C} = (g_k, B_k)_{k=1}^3$ such that

$$(g_1, D_1) = (\mathrm{Log}, B), \quad (g_2, D_2) = (\mathrm{Log}, B_1(i)), \quad (g_3, D_3) = (\log_0, B_r(z)).$$

Then $D_{k-1} \cap D_k \neq \varnothing$ for $k = 2, 3$, and since $\mathrm{Log} \equiv \log_0$ on $B_1(i) \subseteq \mathbb{H}$, we have $g_1 \equiv g_2$ on $D_1 \cap D_2$, and $g_2 \equiv g_3$ on $D_2 \cap D_3$. Thus $(\log_0, B_r(z))$ is an analytic $\mathbb{C}_*$-continuation of (Log, B), which is to say $(\log_0, B_r(z)) \in [\mathrm{Log}, B]$, where $z \in B_r(z)$.

At this juncture is has been shown that, for each $z \in \mathbb{C}_*$, there exists some $(g, D) \in [f, B]$ with $z \in D$. Therefore $[f, B]$ is a generalized analytic function on $\mathbb{C}_*$.

(b) Suppose h is analytic on $\mathbb{C}_*$ such that for every $(g, D) \in [\mathrm{Log}, B]$ we have $h \equiv g$ on D. In part (a) we found that, for any $z \in \mathbb{C} \setminus R_{-\pi}$, there is a chain $(\mathrm{Log}, D_k)_{k=1}^n$ in $\mathbb{C} \setminus R_{-\pi}$ with $z \in D_n$ and $(\mathrm{Log}, D_n) \in [\mathrm{Log}, B]$, and thus $h \equiv \mathrm{Log}$ on D_n. This implies that $h \equiv \mathrm{Log}$ on $\mathbb{C} \setminus R_{-\pi}$. Also, for each $x \in \mathbb{C} \setminus R_{-\pi}$ we have

$$h(x) = \lim_{y \to 0^-} h(x + iy) = \lim_{y \to 0^-} \mathrm{Log}(x + iy) = \mathrm{Log}(x),$$

the first equality due to the continuity of h on $\mathbb{C}_*$, and the last equality due to the "one-sided" continuity of Log on the negative real axis. Thus $h \equiv \mathrm{Log}$ on $\mathbb{C}_*$, and we conclude that Log is analytic on $\mathbb{C}_*$, which is a contradiction. Therefore the hypothesized function h cannot exist. ∎

Exercise 10.15 (AN4.9.4). Let $F : \mathbb{C}^{k+1} \to \mathbb{C}$ be a function for which all first-order partial derivatives are everywhere continuous.[12] Let $f_1, \ldots, f_k$ be analytic on an open disc $D \subseteq \mathbb{C}$, and assume that

$$F(z, f_1(z), \ldots, f_k(z)) = 0$$

[12]We do not need the hypothesis that F is itself continuous.

for all $z \in D$. For each $1 \leq i \leq k$ let $(f_{ij}, D_j)_{j=1}^n$ be a chain such that $(f_{i1}, D_1) = (f_i, D)$, so (f_{in}, D_n) is an analytic continuation of (f_i, D). Show that

$$F(z, f_{1n}(z), \ldots, f_{kn}(z)) = 0 \tag{10.8}$$

for all $z \in D_n$.

Solution. Suppose that $G(z_1, \ldots, z_m) = w$ is a function $\mathbb{C}^m \to \mathbb{C}$ with first-order partial derivatives that are continuous on $\mathbb{C}^m$. By Theorem 14.34 of the *Calculus Notes*, easily seen to be applicable to complex-valued variables, it follows that G is differentiable on $\mathbb{C}^m$. Adopting the notation of the *Calculus Notes*, let $\Omega \subseteq \mathbb{C}$ be open, suppose $g_1, \ldots, g_m : \Omega \to \mathbb{C}$ are analytic functions, and define $\mathbf{r} : \Omega \to \mathbb{C}^m$ by

$$\mathbf{r}(z) = \langle g_1(z), \ldots, g_m(z) \rangle$$

for all $z \in \Omega$. Since

$$\mathbf{r}'(z) = \langle g_1'(z), \ldots, g_m'(z) \rangle,$$

it is clear that $\mathbf{r}$ is differentiable on Ω. Thus, for each $w \in \Omega$, $\mathbf{r}$ is differentiable at w and $\mathbf{r}(w) \in \mathbb{C}^m$, so since G is differentiable on $\mathbb{C}^m$, by Theorem 14.38 of the *Calculus Notes* (also easily extended to complex variables) it follows that $G \circ \mathbf{r}$, which is the map $\Omega \to \mathbb{C}$ given by

$$z \mapsto G(g_1(z), \ldots, g_m(z)),$$

is differentiable at w, and therefore $G \circ \mathbf{r}$ is analytic on Ω.

For each $1 \leq j \leq n$ we have that f_{ij} is analytic on D_j for all $1 \leq i \leq k$, and so, recalling that F has continuous first partials, the same argument as above can be employed to conclude that $h_j : D_j \to \mathbb{C}$ given by

$$h_j(z) = F(z, f_{1j}(z), \ldots, f_{kj}(z))$$

is analytic on D_j. By hypothesis $h_1 \equiv 0$ on D_1, and since $f_{i1} \equiv f_{i2}$ on $D_1 \cap D_2$ for each $1 \leq i \leq k$, we have

$$h_1(z) = F(z, f_{11}(z), \ldots, f_{k1}(z)) = F(z, f_{12}(z), \ldots, f_{k2}(z)) = h_2(z)$$

for all $z \in D_1 \cap D_2$. That is $h_2 \equiv h_1 \equiv 0$ on $D_1 \cap D_2 \neq \varnothing$, and so $h_2 \equiv 0$ on D_2 by the Identity Theorem. Continuing in this fashion we ultimately obtain $h_n \equiv h_{n-1} \equiv 0$ on $D_{n-1} \cap D_n \neq \varnothing$, and so $h_n \equiv 0$ on D_n. But $h_n(z) = 0$ is precisely equation (10.8), and so we are done. ■

Exercise 10.16 (AN4.9.5). Let (g, D) be an analytic continuation of (f, B). Show that (g', D) is an analytic continuation of (f', B).

Solution. Since (g, D) is a continuation of (f, B), there exists a chain $(h_k, B_k)_{k=1}^n$ such that $(h_1, B_1) = (f, B)$ and $(h_n, B_n) = (g, D)$. Since $h_k \equiv h_{k+1}$ on $B_k \cap B_{k+1}$ implies that $h_k' \equiv h_{k+1}'$ on $B_k \cap B_{k+1}$ for all $1 \leq k \leq n-1$, it is immediate that $(h_k', B_k)_{k=1}^n$ is a chain such that $(h_1', B_1) = (f', B)$ and $(h_n', B_n) = (g', D)$, and therefore (g', D) is an analytic continuation of (f', B). ■

10.3 – Monodromy Theorem

Definition 10.17. *An open set $\Omega \subseteq \mathbb{C}$ is* ***homotopically simply connected*** *if every closed curve in Ω is Ω-homotopic to a point.*

Thus, an open set Ω is homotopically simply connected if and only if every closed curve $\gamma_0 : [a, b] \to \Omega$ is Ω-homotopic to the constant curve $\gamma_1 \equiv \gamma_0(a)$, where of course $\gamma_0(a)$ is a point in Ω. It will be found in the next chapter that an open set in $\mathbb{C}$ is homotopically simply connected if and only if it is homologically simply connected.

Theorem 10.18 (**Monodromy Theorem**). *Let Ω be a homotopically simply connected region in $\mathbb{C}$. If Φ is a generalized analytic function on Ω such that each element of Φ can be Ω-continued along all curves in Ω, then there exists an analytic function $g : \Omega \to \mathbb{C}$ such that for all $(f, B) \in \Phi$ we have $g \equiv f$ on B.*

Proof. For each $z \in \Omega$ there exists some $(f, B) \in \Phi$ such that $z \in B$, and so set $g(z) = f(z)$. It must be shown that g is a well-defined function. That is, for each $z \in \Omega$, if $(f, B), (\varphi, D) \in \Phi$ are such that $z \in B$ and $z \in D$, then $f(z) = \varphi(z)$.

Fix $z \in \Omega$. Since Φ is an equivalence class, we have $(f, B) \overset{\Omega}{\sim} (\varphi, D)$, which is to say (φ, D) is an analytic Ω-continuation of (f, B). Thus there exists a chain $(h_k, B_k)_{k=1}^n$ such that $(h_1, B_1) = (f, B)$ and $(h_n, B_n) = (\varphi, D)$. Let $z_0 = z_n = z$, and for each $1 \leq k \leq n-1$ choose some $z_k \in B_k \cap B_{k+1}$. Define $\gamma_1 : [0, 1] \to \mathbb{C}$ to be the polygonal path $[z_0, \ldots, z_n]$, which is a closed curve with base point z. For $1 \leq k \leq n$ we have $z_{k-1}, z_k \in B_k$, and since B_k is a convex set it follows that $[z_{k-1}, z_k] \subseteq B_k$. Letting

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition such that $\gamma_1([t_{k-1}, t_k]) = [z_{k-1}, z_k]$ for each $1 \leq k \leq n$, we secure (φ, D) as an analytic Ω-continuation of (f, B) along γ_1 in particular. Also (f, B) is an analytic Ω-continuation of itself along the constant curve $\gamma_0 \equiv z$. Since γ_1 is homotopic in Ω to γ_0, Theorem 10.13 implies that $(f, B) \smile (\varphi, D)$, and hence $f \equiv \varphi$ on $B \cap D$. Therefore $f(z) = \varphi(z)$.

We now see that g is a well-defined function on Ω, with $g \equiv f$ on B for each $(f, B) \in \Phi$. Moreover g is analytic on Ω since $(f, B) \in \Phi$ implies that f is analytic on B. ■

Corollary 10.19. *Let Ω be a homotopically simply connected region. If (f, B) is a function element in Ω that can be Ω-continued along all curves in Ω with initial point in B, then there exists an analytic function $g : \Omega \to \mathbb{C}$ such that $f \equiv g$ on B.*

Theorem 10.20. *If Ω is a homotopically simply connected region, then every harmonic function on Ω has a harmonic conjugate.*

11

Families of Analytic Functions

11.1 – Spaces of Analytic and Continuous Functions

Let $\Omega \subseteq \mathbb{C}$ be open. Define

$$\mathcal{C}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous}\},$$

and

$$\mathcal{A}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is analytic}\}.$$

Both $\mathcal{C}(\Omega)$ and $\mathcal{A}(\Omega)$ are easily verified to be linear spaces (i.e. vector spaces) under the usual operations of function addition and scalar multiplication,

$$(f+g)(z) = f(z) + g(z) \quad \text{and} \quad (cf)(z) = cf(z),$$

and in fact $\mathcal{A}(\Omega)$ is a subspace of $\mathcal{C}(\Omega)$. Recall the uniform metric $\|\cdot\|$ on $\mathcal{B}(\Omega)$, the set of bounded functions $\Omega \to \mathbb{C}$, introduced in §2.1. Given $f, g \in (\mathcal{B}(\Omega), \|\cdot\|)$, the distance between f and g is taken to be

$$\|f - g\|_\Omega = \sup\{|f(z) - g(z)| : z \in \Omega\},$$

usually denoted simply by $\|f - g\|$. As noted in §2.6, a sequence of functions $(f_n : \Omega \to \mathbb{C})$ converges to $f : \Omega \to \mathbb{C}$ on Ω with respect to the uniform metric if and only if $f_n \xrightarrow{\text{u}} f$ on Ω. Now, because Ω is open, not all members of $\mathcal{A}(\Omega)$ or $\mathcal{C}(\Omega)$ will be bounded functions, and hence neither $(\mathcal{A}(\Omega), \|\cdot\|)$ nor $(\mathcal{C}(\Omega), \|\cdot\|)$ are in fact metric spaces! If we do not wish to restrict ourselves to only bounded continuous functions on Ω, then it remains still to find a viable metric for $\mathcal{C}(\Omega)$.

Let $(K_n)_{n=1}^\infty$ be the sequence of sets such that

$$K_n = \overline{B}_n(0) \cap \{z : |z - w| \geq 1/n \text{ for all } w \in \mathbb{C} \setminus \Omega\} \tag{11.1}$$

for each $n \in \mathbb{N}$. For each $f \in C(\Omega)$ define

$$|||f||| = \sum_{n=1}^\infty \frac{2^{-n}\|f\|_{K_n}}{1 + \|f\|_{K_n}},$$

where

$$\|f\|_{K_n} = \begin{cases} \sup\{|f(z)| : z \in K_n\}, & K_n \neq \varnothing \\ 0, & K_n = \varnothing \end{cases}$$

As will be verified in Theorem 11.2, $\|\!|\cdot|\!\|$ defines a norm on $\mathcal{C}(\Omega)$, and the norm $\|\!|\cdot|\!\|$ induces a metric d on $\mathcal{C}(\Omega)$ in the usual way: for each $f, g \in \mathcal{C}(\Omega)$ define the **distance** between f and g to be

$$d(f,g) = \|\!|f-g|\!\|.$$

The metric space $(\mathcal{C}(\Omega), d)$ is taken to have the topology induced by the metric d.

Lemma 11.1. *The sequence $(K_n)_{n=1}^{\infty}$ has the following properties.*

1. *K_n is compact for each n.*
2. *$K_n \subseteq K_{n+1}^{\circ}$.*
3. *If $K \subseteq \Omega$ is compact, then there exists some $m \geq 1$ such that $K \subseteq K_n$ for all $n \geq m$.*

Let $\mathcal{X}(\Omega)$ represent any collection of functions $\Omega \to \mathbb{C}$ for which $(\mathcal{X}(\Omega), d)$ is a metric space. A sequence (f_n) in $(\mathcal{X}(\Omega), d)$ is said to be ***d*-convergent on** Ω (or simply ***d*-convergent**) if (f_n) is convergent in $\mathcal{X}(\Omega)$ with respect to the metric d, which is to say there exists some $f \in \mathcal{X}(\Omega)$ for which the following holds: for every $\epsilon > 0$, there exists some k such that

$$d(f_n, f) < \epsilon$$

for all $n \geq k$. If such a function f exists, then we may say (f_n) is d-convergent (on Ω) to f and write $f_n \xrightarrow{\mathrm{d}} f$ or d-lim $f_n = f$.[13] Another common phrase is to say (f_n) is d-convergent in $\mathcal{X}(\Omega)$. Finally, a sequence (f_n) in $(\mathcal{X}(\Omega), d)$ is said to be ***d*-Cauchy on** Ω (or simply ***d*-Cauchy**) if (f_n) is a Cauchy sequence with respect to the metric d.

Theorem 11.2. *Let $\mathcal{X}$ represent either $\mathcal{C}$ or $\mathcal{A}$. The function $f \mapsto \|\!|f|\!\|$ is a norm on $\mathcal{X}(\Omega)$, which induces the metric d. Let (f_n) be a sequence in the metric space $(\mathcal{X}(\Omega), d)$.*

1. *(f_n) is d-Cauchy on Ω iff it is uniformly Cauchy on compact subsets of Ω.*
2. *(f_n) is d-convergent on Ω to $f \in \mathcal{X}(\Omega)$ iff it is uniformly convergent on compact subsets of Ω to $f \in \mathcal{X}(\Omega)$.*
3. *$(\mathcal{X}(\Omega), d)$ is a complete metric space.*

Proof.

Proof of Part (3). Let (f_n) be a d-Cauchy sequence in $\mathcal{A}(\Omega)$. It follows from Part (1) that (f_n) is uniformly Cauchy on compact subsets of Ω, and so (f_n) is uniformly convergent on compact subsets of Ω by Theorem 2.52. Since uniform convergence on compact subsets of Ω immediately implies pointwise convergence on Ω, we may define $f : \Omega \to \mathbb{C}$ by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for each $z \in \Omega$. Thus (f_n) converges uniformly to f on compact subsets of Ω (as is demonstrated in the proof of Theorem 2.37), and Theorem 4.30 implies that f is analytic on Ω. Finally, by Part (2) we find that (f_n) is d-convergent on Ω to the same function f, and since $f \in \mathcal{A}(\Omega)$, we conclude that $(\mathcal{A}(\Omega), d)$ is a complete metric space.

[13] As ever, the synonymous symbols $f_n \to f$ and $\lim f_n = f$ are reserved for pointwise convergence in $\mathbb{C}$.

Next, let (f_n) be a d-Cauchy sequence in $\mathcal{C}(\Omega)$. By Part (1), (f_n) is uniformly Cauchy on compact subsets of Ω, and so by Theorem 2.52 and the same argument as before, there is a function $f : \Omega \to \mathbb{C}$ such that (f_n) converges uniformly to f on compact subsets of Ω. Theorem 2.54 implies that f is continuous on Ω. Finally, by Part (2) and the details of its proof, we find that (f_n) is d-convergent on Ω to the same function f, and since $f \in \mathcal{C}(\Omega)$, we conclude that $(\mathcal{C}(\Omega), d)$ is a complete metric space. ■

It shall henceforth be assumed that the symbols $\mathcal{A}(\Omega)$ and $\mathcal{C}(\Omega)$ denote the metric spaces $(\mathcal{A}(\Omega), d)$ and $(\mathcal{C}(\Omega), d)$, respectively, unless said otherwise.

Theorem 11.3 (Hurwitz's Theorem). *Suppose (f_n) is a sequence in $\mathcal{A}(\Omega)$ that converges uniformly to $f : \Omega \to \mathbb{C}$ on compact subsets of Ω. Let $\overline{B}_r(z_0) \subseteq \Omega$, with f nonvanishing on $C_r(z_0)$. Then there exists some $k \in \mathbb{N}$ such that, for all $n \geq k$, f_n and f have the same number of zeros in $B_r(z_0)$ counting multiplicities.*

Proof. Observe that f is not identically zero on Ω since it is nonvanishing on $C_r(z_0)$. Theorem 4.30 implies that f is analytic on Ω. Let

$$\epsilon = \min\{|f(z)| : z \in C_r(z_0)\},$$

which can be shown to exist in $(0, \infty)$ by applying the Extreme Value Theorem to $-|f|$. Let

$$\gamma(t) = z_0 + re^{it}$$

for $t \in [0, 2\pi]$, so $\gamma^* = C_r(z_0)$ and $\operatorname{wn}(\gamma, z) = 0$ for $z \notin \Omega$. Now, since $\overline{B}_r(z_0)$ is compact, there exists some $k \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon$$

for all $z \in \overline{B}_r(z_0)$ and $n \geq k$. Thus we have

$$|f - f_n| = |f_n - f| < \epsilon \leq |f|$$

on γ^* for all $n \geq k$, giving

$$|f + (-f_n)| < |f| + |-f_n|.$$

By Rouché's Theorem,

$$\sum_{z \in Z(f)} \operatorname{ord}(f, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(-f_n)} \operatorname{ord}(-f_n, z) \operatorname{wn}(\gamma, z) = \sum_{z \in Z(f_n)} \operatorname{ord}(f_n, z) \operatorname{wn}(\gamma, z)$$

for all $n \geq k$, the last equality owing to $Z(-f_n) = Z(f_n)$ and $\operatorname{ord}(-f_n, z) = \operatorname{ord}(f_n, z)$. Since $\operatorname{wn}(\gamma, z) = 1$ for all $z \in B_r(z_0)$ and $\operatorname{wn}(\gamma, z) = 0$ for all $z \notin \overline{B}_r(z_0)$, we conclude that, for all $n \geq k$, the functions f_n and f have the same number of zeros in $B_r(z_0)$ counting multiplicities. ■

Theorem 11.4. *Let Ω be a region, and let (f_n) be a sequence in $\mathcal{A}(\Omega)$ that converges uniformly to $f : \Omega \to \mathbb{C}$ on compact subsets of Ω.*

1. *If f_n is nonvanishing on Ω for infinitely many n, then either f is nonvanishing or identically zero on Ω.*
2. *If f_n is injective on Ω for all n, then either f is injective or constant on Ω.*

Proof.
Proof of Part (1). Suppose f_n is nonvanishing on Ω for infinitely many n, and suppose also that f is not identically zero on Ω. Fix $z \in \Omega$. By Theorem 4.30 the function f is analytic on Ω, and so by the Identity Theorem the set $Z(f)$ has no limit points in Ω, and so there exists some $r > 0$ sufficiently small that $\overline{B}_r(z) \subseteq \Omega$ and $\overline{B}'_r(z) \cap Z(f) = \varnothing$. In particular f is nonvanishing on $C_r(z)$. By Hurwitz's Theorem there exists some k such that, for all $n \geq k$, f_n and f have the same number of zeros in $B_r(z)$ counting multiplicities. On the other hand there exists some $m \geq k$ such that f_m is nonvanishing on Ω, so that f_m has no zeros in $B_r(z)$, and hence f also has no zeros in $B_r(z)$. Therefore $f(z) \neq 0$, and we conclude that f is nonvanishing on Ω.

Proof of Part (2). Suppose f_n is injective on Ω for all n, and suppose also that f is nonconstant on Ω. Fix $z_0 \in \Omega$. Let $g_n = f_n - f_n(z_0)$ for all n, and let $\Omega' = \Omega \setminus \{z_0\}$. Since Ω' is a region, (g_n) is a sequence in $\mathcal{A}(\Omega')$ that converges uniformly to $g = f - f(z_0)$ on compact subsets of Ω', and g_n is nonvanishing on Ω' for infinitely many n (in fact for all n), by Part (1) it follows that g is either nonvanishing or identically zero on Ω'. But g identically zero on Ω' implies that $f \equiv f(z_0)$ on Ω, which contradicts the hypothesis that f is nonconstant. Hence g must be nonvanishing on Ω'; that is, $f(z) \neq f(z_0)$ for all $z \in \Omega \setminus \{z_0\}$. Since $z_0 \in \Omega$ is arbitrary, we conclude that f is injective on Ω. ∎

Proposition 11.5. *The following functions are continuous on their domains.*

1. *The function $D : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ given by $D(f) = f'$.*
2. *For fixed $z \in \Omega$, the function $E_z : \mathcal{C}(\Omega) \to \mathbb{C}$ given by $E_z(f) = f(z)$.*

Proof.
Proof of Part (1). Fix $f \in \mathcal{A}(\Omega)$. Let (f_n) be any sequence in $\mathcal{A}(\Omega)$ such that $\text{d-lim} f_n = f$, which is to say (f_n) is d-convergent on Ω to f. By Theorem 11.2, (f_n) is uniformly convergent to f on compact subsets of Ω, and thus (f'_n) is uniformly convergent to f' on compact subsets of Ω by Theorem 4.30. Since (f'_n) is itself a sequence in $\mathcal{A}(\Omega)$, Theorem 11.2 implies that (f'_n) is d-convergent on Ω to f', which is to say $\text{d-lim} f'_n = f'$, and hence $\text{d-lim} D(f_n) = D(f)$. By Theorem 2.20 we conclude that D is continuous at f.

Proof of Part (2). Fix $f \in \mathcal{C}(\Omega)$. Let (f_n) be any sequence in $\mathcal{C}(\Omega)$ such that $f_n \xrightarrow{\text{d}} f$. Then $f_n \xrightarrow{\text{u}} f$ on compact subsets of Ω by Theorem 11.2, which implies that $f_n(z) \to f(z)$ since $\{z\} \subseteq \Omega$ is compact, and hence $E_z(f_n) \to E_z(f)$. By Theorem 2.20 we conclude that E_z is continuous at f. ∎

Proposition 11.6. *Suppose $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is nonempty and compact. For each $z \in \Omega$ there exists some $h \in \mathcal{F}$ such that $|h'(z)| \geq |f'(z)|$ for all $f \in \mathcal{F}$.*

Proof. Fix $z \in \Omega$, and let D and E_z be the functions defined in Proposition 11.5. The euclidian norm $z \mapsto |z|$ is of course a continuous function $\mathbb{C} \to \mathbb{R}$, and so if we let $\Phi = |E_z \circ D|$, then by Proposition 11.5 and Theorem 2.21 it follows that $\Phi : \mathcal{A}(\Omega) \to \mathbb{R}$ is continuous on $\mathcal{A}(\Omega)$. Hence Φ is continuous on the compact set $\mathcal{F}$, and then the Extreme Value Theorem implies that Φ attains a maximum on $\mathcal{F}$; that is, there exists some $h \in \mathcal{F}$ such that $\Phi(h) \geq \Phi(f)$ for all $f \in \mathcal{F}$. Since

$$\Phi(f) = |E_z \circ D|(f) = |(E_z \circ D)(f)| = |E_z(D(f))| = |E_z(f')| = |f'(z)|$$

for any $f \in \mathcal{F}$, the desired conclusion obtains. ■

Exercise 11.7 (AN11.1.3a). If $\mathcal{F} \subseteq \mathcal{C}(\Omega)$, show that $\mathcal{F}$ is precompact if and only if each sequence in $\mathcal{F}$ has a subsequence that is d-convergent in $\mathcal{C}(\Omega)$, in which case $\mathcal{F}$ is in some texts referred to as **normal**.

Solution. This is nothing more than a restatement of Proposition 2.46 with the metric space $(\mathcal{C}(\Omega), d)$ being (X, d) and the set $\mathcal{F}$ being S. ■

Exercise 11.8 (AN11.1.8). Let $L : \mathcal{A}(\Omega) \to \mathbb{C}$ be a multiplicative linear functional, which is to say for all $a, b \in \mathbb{C}$ and $f, g \in \mathcal{A}(\Omega)$ we have

$$L(af + bg) = aL(f) + bL(g) \quad \text{and} \quad L(fg) = L(f)L(g).$$

If $L \not\equiv 0$, then show that there exists some $w \in \Omega$ such that $L(f) = f(w)$ for all $f \in \mathcal{A}(\Omega)$.

Solution. Let κ_1 denote the constant function $z \mapsto 1$ on Ω, so that $\kappa_1^2 = 1$. Then

$$L(\kappa_1) = L(\kappa_1^2) = L(\kappa_1)L(\kappa_1) = [L(\kappa_1)]^2,$$

which shows that $L(\kappa_1) \in \{0, 1\}$. If $L(\kappa_1) = 0$, then for any $g \in \mathcal{A}(\Omega)$ we have

$$L(g) = L(\kappa_1 g) = L(\kappa_1)L(g) = 0 \cdot L(g) = 0,$$

and thus we arrive at the contradiction $L \equiv 0$. Therefore $L(\kappa_1) = 1$.

Next, for any $c \in \mathbb{C}$ let κ_c denote the constant function $z \mapsto c$ on Ω. Then $\kappa_c = c\kappa_1$, and so

$$L(\kappa_c) = L(c\kappa_1) = cL(\kappa_1) = c \cdot 1 = c.$$

We make use of these results presently.

Let ι denote the identity function $z \mapsto z$ on Ω. Suppose that $L(\iota) = w \notin \Omega$. Then

$$g(z) = \frac{1}{z - w}$$

is a function in $\mathcal{A}(\Omega)$. Now,

$$\big(g(\iota - \kappa_w)\big)(z) = g(z) \cdot (\iota - \kappa_w)(z) = \frac{1}{z - w} \cdot (z - w) = 1 = \kappa_1(z)$$

for all $z \in \Omega$, and so $g(\iota - \kappa_w) = 1$ and we obtain

$$L\big(g(I - \kappa_w)\big) = L(\kappa_1) = 1. \tag{11.2}$$

On the other hand, since

$$L(\iota - \kappa_w) = L(\iota) - L(\kappa_w) = w - w = 0,$$

we obtain

$$L\big(g(\iota - \kappa_w)\big) = L\left(\frac{1}{z - w} \cdot (\iota - \kappa_w)\right) = L\left(\frac{1}{z - w}\right)L(\iota - \kappa_w) = 0. \tag{11.3}$$

From equations (11.2) and (11.3) we have a contradiction, and therefore $L(\iota) = w \in \Omega$ must be the case.

Finally, let $f \in \mathcal{A}(\Omega)$ be arbitrary, and define $h \in \mathcal{A}(\Omega)$ by

$$h(z) = \begin{cases} \dfrac{f(z) - f(w)}{z - w}, & z \neq w \\ f'(w), & z = w \end{cases}$$

The analyticity of h on Ω follows from Corollary 4.22. Now,

$$\big(h(\iota - \kappa_w)\big)(z) = h(z) \cdot (\iota - \kappa_w)(z) = h(z) \cdot (z - w) = f(z) - f(w)$$

for all $z \in \Omega$, including w. Hence $h(\iota - \kappa_w) = f - \kappa_{f(w)}$, so that

$$L(f) - f(w) = L(f - \kappa_{f(w)}) = L\big(h(\iota - \kappa_w)\big) = L(h)L(\iota - \kappa_w) = L(h)(w - w) = 0,$$

and therefore $L(f) = f(w)$ as desired. ∎

11.2 – Equicontinuity and Boundedness

Definition 11.9. *Let $\mathcal{F}$ be a family of functions $\Omega \to \mathbb{C}$. If $z \in \Omega$, then $\mathcal{F}$ is* ***equicontinuous at*** *z if for any $\epsilon > 0$ there exists some $\delta > 0$ such that*

$$|f(w) - f(z)| < \epsilon$$

for all $w \in B_\delta(z)$ and $f \in \mathcal{F}$. If $\mathcal{F}$ is equicontinuous at each $z \in \Omega$, then $\mathcal{F}$ is ***equicontinuous on*** *Ω.*

Definition 11.10. *Let $\mathcal{F}$ be a family of functions $\Omega \to \mathbb{C}$. If $S \subseteq \Omega$, then $\mathcal{F}$ is* ***uniformly bounded*** *on S if*

$$\sup\{\|f\|_S : f \in \mathcal{F}\} \in \mathbb{R}.$$

$\mathcal{F}$ is ***bounded on*** *S if $\mathcal{F}$ is uniformly bounded on each compact $K \subseteq S$.*

A couple remarks. For a family of functions $\mathcal{F}$ to be bounded on Ω, it is necessary but not sufficient that $\mathcal{F} \subseteq \mathcal{B}(\Omega)$. If $S \subseteq \Omega$ is compact, then $\mathcal{F}$ is uniformly bounded on S if and only if it is bounded on S.

Henceforth, if $\mathcal{F}$ is given to be a family of functions with domain Ω, then to say $\mathcal{F}$ is "bounded" will be taken to mean "bounded on Ω."

Theorem 11.11. *If $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is bounded, then $\mathcal{F}$ is equicontinuous on Ω.*

Proof. Suppose $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is bounded. Let $z_0 \in \Omega$, and fix $\epsilon > 0$. Let $r > 0$ be such that $K = \overline{B}_r(z_0) \subseteq \Omega$. Let $M = \sup\{\|f\|_K : f \in \mathcal{F}\}$, so $M \in [0, \infty)$. If $M = 0$, then all members of $\mathcal{F}$ are identically zero on K, so that $\mathcal{F}$ can contain only the zero function on Ω by the Identity Theorem, and thus the equicontinuity of $\mathcal{F}$ on Ω follows trivially. Assume that $M > 0$.

By Cauchy's Integral Formula for a Circle, for any $z \in B_r(z_0)$ and $f \in \mathcal{F}$,

$$\begin{aligned}
|f(z) - f(z_0)| &= \frac{1}{2\pi}\left|\oint_{C_r(z_0)} \frac{f(w)}{w - z}\,dw - \oint_{C_r(z_0)} \frac{f(w)}{w - z_0}\,dw\right| \\
&= \frac{1}{2\pi}\left|\oint_{C_r(z_0)} \frac{(z - z_0)f(w)}{(w - z)(w - z_0)}\,dw\right| = r \sup_{w \in C_r(z_0)} \frac{|z - z_0||f(w)|}{|w - z||w - z_0|} \\
&= |z - z_0| \sup_{w \in C_r(z_0)} \frac{|f(w)|}{|w - z|} \le |z - z_0| \sup_{w \in C_r(z_0)} \frac{M}{|w - z|}.
\end{aligned}$$

Since for each $z \in B_{r/2}(z_0)$ we have $|w - z| > r/2$ for all $w \in C_r(z_0)$, and hence

$$\frac{1}{|w - z|} < \frac{2}{r},$$

we obtain

$$|f(z) - f(z_0)| \le \frac{2M}{r}|z - z_0|$$

for all $z \in B_{r/2}(z_0)$ and $f \in \mathcal{F}$. Choose

$$\delta = \min\left\{\frac{r}{2}, \frac{r\epsilon}{2M}\right\}.$$

Then for any $z \in B_\delta(z_0)$ and $f \in \mathcal{F}$,

$$|f(z) - f(z_0)| \le \frac{2M}{r}|z - z_0| < \frac{2M}{r}\delta \le \frac{2M}{r} \cdot \frac{r\epsilon}{2M} = \epsilon.$$

Therefore $\mathcal{F}$ is equicontinuous at z_0, and since $z_0 \in \Omega$ is arbitrary, we conclude that $\mathcal{F}$ is equicontinuous at each point in Ω. ■

Theorem 11.12. *Suppose $\mathcal{F} \subseteq \mathcal{C}(\Omega)$ is equicontinuous on Ω, and $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.*

1. *If $f_n \to f$ on Ω, then $f \in \mathcal{C}(\Omega)$ and $f_n \xrightarrow{\text{u}} f$ on compact subsets of Ω.*
2. *If (f_n) is pointwise convergent on a dense subset of Ω, then there exists some $f : \Omega \to \mathbb{C}$ such that $f_n \to f$ on Ω.*

Proof.

Proof of Part (1). Suppose $f_n \to f$ on Ω. Fix $z_0 \in \Omega$, and let $\epsilon > 0$. There exists some $\delta > 0$ such that

$$|f_n(z) - f(z_0)| < \frac{\epsilon}{3}$$

for all $z \in B_\delta(z_0)$ and $n \in \mathbb{N}$. Now, since there is some $k_0 \in \mathbb{N}$ such that

$$|f_n(z_0) - f(z_0)| < \frac{\epsilon}{3}$$

for all $n \ge k_0$, and also some $k_1 \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3},$$

for all $n \ge k_1$, we can choose any $n \ge \max\{k_0, k_1\}$ to obtain

$$|f(z) - f(z_0)| \le |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| < \epsilon.$$

for all $z \in B_\delta(z_0)$. Therefore f is continuous at z_0, and since $z_0 \in \Omega$ is arbitrary, we conclude that $f \in \mathcal{C}(\Omega)$.

Next, let $K \subseteq \Omega$ be compact. Again let $\epsilon > 0$. Since f is uniformly continuous on K, there exists some $\rho > 0$ such that

$$|z - w| < \rho \quad \Rightarrow \quad |f(z) - f(w)| < \frac{\epsilon}{3}$$

for all $z, w \in K$. Also, for each $z \in K$ there is a $0 < \delta_z < \rho$ such that

$$|f_n(w) - f_n(z)| < \frac{\epsilon}{3} \tag{11.4}$$

for all $w \in B_{\delta_z}(z)$ and $n \in \mathbb{N}$. Since $\{B_{\delta_z}(z) : z \in K\}$ is an open cover for K, there is a finite subcover $\{B_{\delta_j}(z_j) : 1 \le j \le m\}$, and for each $1 \le j \le m$ there exists some $k_j \in \mathbb{N}$ such that

$$|f_n(z_j) - f(z_j)| < \frac{\epsilon}{3}$$

for all $n \ge k_j$. Let $k = \max\{k_j : 1 \le j \le m\}$. Fix $n \ge k$ and $z \in K$. Then $z \in B_{\delta_p}(z_p)$ for some $1 \le p \le m$, so that

$$|f_n(z) - f_n(z_p)| < \frac{\epsilon}{3}$$

by (11.4), and hence

$$|f_n(z) - f(z)| \le |f_n(z) - f_n(z_p)| + |f_n(z_p) - f(z_p)| + |f(z_p) - f(z)| < \epsilon,$$

observing that $|z_p - z| < \delta_p < \rho$. Therefore (f_n) converges uniformly to f on K.

Proof of Part (2). Suppose that (f_n) is pointwise convergent on a set $S \subseteq \Omega$ that is dense in Ω. We must show that $(f_n(z))$ converges to a complex number w_z for every $z \in \Omega$, whereupon we may simply define $f : \Omega \to \mathbb{C}$ by $f(z) = w_z$. Fix $z_0 \in \Omega \setminus S$, and let $\epsilon > 0$. There exists some $\delta > 0$ such that

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$$

for all $z \in B_\delta(z_0)$ and $n \in \mathbb{N}$. Now, z_0 is a limit point for S since S is dense in Ω, and so there exists some $z_1 \in S$ such that $z_1 \in B'_\delta(z_0)$. The sequence $(f_n(z_1))_{n\in\mathbb{N}}$ converges in $\mathbb{C}$, and so is a Cauchy sequence. Let $k \in \mathbb{N}$ be such that

$$|f_m(z_1) - f_n(z_1)| < \frac{\epsilon}{3}$$

for all $m, n \geq k$. Fix $m, n \geq k$. Then

$$|f_m(z_0) - f_n(z_0)| \leq |f_m(z_0) - f_m(z_1)| + |f_m(z_1) - f_n(z_1)| + |f_n(z_1) - f_n(z_0)| < \epsilon,$$

which shows that $(f_n(z_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Hence $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $z \in \Omega$, and therefore converges in $\mathbb{C}$ for all $z \in \Omega$. Defining $f : \Omega \to \mathbb{C}$ by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for each $z \in \Omega$, it is immediate that $f_n \to f$ on Ω. ∎

Theorem 11.13 (**Montel's Theorem**). *If $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is bounded, then each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to some $f \in \mathcal{A}(\Omega)$ on compact subsets of Ω.*

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is bounded, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. Theorem 11.11 implies that $\mathcal{F}$ is equicontinuous at each point in Ω. Define

$$S = \{x + iy \in \Omega : x, y \in \mathbb{Q}\},$$

so S is dense in Ω, and also S is countable so that we may write $S = \{z_j : j \in \mathbb{N}\}$. The strategy will be to show that (f_n) converges pointwise on S.

Because $\mathcal{F}$ is bounded, for each $j \in \mathbb{N}$ we have

$$\sup\{|f_n(z_j)| : n \in \mathbb{N}\} = \sup\{\|f_n\|_{\{z_j\}} : n \in \mathbb{N}\} \in \mathbb{R},$$

which is to say the sequence $(f_n(z_j))_{n\in\mathbb{N}}$ is bounded for each j. In particular $(f_n(z_1))$ is bounded, and so there is a convergent subsequence $(f_{1n}(z_1))$; that is, (f_{1n}) converges at z_1. Next, $(f_{1n}(z_j))$ is bounded for each j, with $(f_n(z_2))$ bounded in particular, and so there is a convergent subsequence $(f_{2n}(z_2))$; that is, (f_{2n}) converges at z_1 and z_2. We continue in this fashion for each $m \in \mathbb{N}$, obtaining a subsequence $(f_{mn})_{n\in\mathbb{N}}$ of (f_n) that converges at $z_1, \ldots, z_m$. Now define the sequence $(f_{n_k})_{k\in\mathbb{N}}$ by $f_{n_k} = f_{kk}$. Then $(f_{n_k})_{k\in\mathbb{N}}$ is a subsequence of (f_n) that converges at each $z_j \in S$. By Theorem 11.12(2) there exists some $f : \Omega \to \mathbb{C}$ such that $f_{n_k} \to f$ on Ω, and then Theorem 11.12(1) implies that $f_{n_k} \xrightarrow{\text{u}} f$ on compact subsets of Ω. That $f \in \mathcal{A}(\Omega)$ is a consequence of Theorem 4.30. ∎

Proposition 11.14. *Let $\mathcal{F} \subseteq \mathcal{X}(\Omega)$. Then each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to some $f \in \mathcal{X}(\Omega)$ on compact subsets of Ω if and only if $\mathcal{F}$ is precompact.*

Proof. By Theorem 11.2(2), each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to $f \in \mathcal{X}(\Omega)$ on compact subsets of Ω iff each sequence in $\mathcal{F}$ has a subsequence that is d-convergent to some $f \in \mathcal{X}(\Omega)$. Now, by Proposition 2.46, each sequence in $\mathcal{F}$ has a subsequence that is d-convergent to some $f \in \mathcal{X}(\Omega)$ iff $\mathcal{F}$ is precompact. ■

Theorem 11.15 (Compactness Criterion). *Let $\mathcal{F} \subseteq \mathcal{A}(\Omega)$. Then*

1. *$\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed and bounded.*
2. *$\mathcal{F}$ is precompact if and only if $\mathcal{F}$ is bounded.*

Proof.
Proof of Part (1). The proof is straightforward in the case when $\mathcal{F}$ is finite, so assume that $\mathcal{F}$ is infinite. Suppose $\mathcal{F}$ is compact. Then $\mathcal{F}$ is closed by Proposition 2.38. Moreover, Theorem 2.39 implies that each sequence in $\mathcal{F}$ has a subsequence that is d-convergent on Ω to some $f \in \mathcal{F}$, and then by Theorem 11.2(2) it follows that each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to some $f \in \mathcal{F}$ on compact subsets of Ω.

Suppose there exists some compact $K \subseteq \Omega$ such that $\sup\{\|f\|_K : f \in \mathcal{F}\} \notin \mathbb{R}$. By the Completeness Axiom we conclude that for each $n \in \mathbb{N}$ there exists some $f_n \in \mathcal{F}$ such that $\|f_n\|_K > n$. Suppose $(f_{n_k})_{k\in\mathbb{N}}$ is a subsequence of (f_n) that is uniformly convergent to some function $f : \Omega \to \mathbb{C}$ on compact subsets of Ω. Then (f_{n_k}) is uniformly convergent to f on K in particular, so that there exists some j such that

$$|f_{n_k}(z) - f(z)| < 1$$

for all $k \geq j$ and $z \in K$, whence

$$\big||f(z)| - |f_{n_k}(z)|\big| < 1,$$

and finally

$$|f(z)| > |f_{n_k}(z)| - 1$$

for all $k \geq j$ and $z \in K$. But by construction we have $\|f_{n_k}\|_K > n_k$ for each k, so that for each k there exists some $z_k \in K$ such that $|f_{n_k}(z_k)| > n_k$, and hence

$$|f(z_k)| > |f_{n_k}(z_k)| - 1 > n_k - 1$$

for all $k \geq j$. We see that $|f(z_k)| \to \infty$ as $k \to \infty$, which shows that f is not a bounded function on K, and therefore f cannot be continuous on Ω. Thus, if (f_n) has any subsequence that is uniformly convergent to some $f : \Omega \to \mathbb{C}$ on compact subsets of Ω, then $f \notin \mathcal{C}(\Omega)$ must be the case, and hence $f \notin \mathcal{F}$. It follows that (f_n) is a sequence in $\mathcal{F}$ that has no subsequence that is uniformly convergent to some $f \in \mathcal{F}$ on compact subsets of Ω, which is a contradiction. Therefore $\sup\{\|f\|_K : f \in \mathcal{F}\} \in \mathbb{R}$ for all compact $K \subseteq \Omega$, and we conclude that $\mathcal{F}$ is bounded.

For the converse, suppose that $\mathcal{F}$ is closed and bounded. By Montel's Theorem each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to some $f \in \mathcal{A}(\Omega)$ on compact subsets of Ω, and thus $\mathcal{F}$ is precompact by Proposition 11.14. By definition this means that $\overline{\mathcal{F}}$ is compact, and since $\overline{\mathcal{F}} = \mathcal{F}$ on account of $\mathcal{F}$ being closed, we conclude that $\mathcal{F}$ is compact.

Proof of Part (2). Suppose $\mathcal{F}$ is precompact. Suppose there exists some compact $K \subseteq \Omega$ such that $\sup\{\|f\|_K : f \in \mathcal{F}\} \notin \mathbb{R}$. Then, as before, we can construct a sequence (f_n) in $\mathcal{F}$ having no subsequence that is uniformly convergent to some $f \in \mathcal{C}(\Omega)$. On the other hand, Proposition 11.14 implies that each sequence in $\mathcal{F}$ has a subsequence that is uniformly convergent to some $f \in \mathcal{A}(\Omega)$ on compact subsets of Ω. As this is a contradiction, we conclude that $\sup\{\|f\|_K : f \in \mathcal{F}\} \in \mathbb{R}$ for all compact $K \subseteq \Omega$, and therefore $\mathcal{F}$ is bounded.

For the converse, suppose that $\mathcal{F}$ is bounded. Then Montel's Theorem and Proposition 11.14 imply that $\mathcal{F}$ is precompact. ∎

Remark. A study of the proof of Theorem 11.15(2) shows that, even for $\mathcal{F} \subseteq \mathcal{C}(\Omega)$, the precompactness of $\mathcal{F}$ implies the boundedness of $\mathcal{F}$; however, the converse requires $\mathcal{F} \subseteq \mathcal{A}(\Omega)$.

Proposition 11.16. *Let Ω be a region, $z_0 \in \Omega$, and $\epsilon > 0$. Define*

$$\mathcal{F} = \{f \in \mathcal{A}(\Omega) : f \text{ is injective, } f(\Omega) \subseteq S, \text{ and } |f'(z_0)| \geq \epsilon\}.$$

If $S = \overline{\mathbb{B}}$ or $S = \mathbb{B}$, then $\mathcal{F}$ compact.

Proof. Suppose $S = \overline{\mathbb{B}}$. Let f be a limit point for $\mathcal{F}$. Then for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ such that $f_n \in B'_{1/n}(f)$, and in this fashion we obtain a sequence (f_n) in $\mathcal{F}$ such that d-lim $f_n = f$. It is immediate that $f : \Omega \to \mathbb{C}$, and also that (f_n) is a Cauchy sequence in $\mathcal{A}(\Omega)$. Since $(\mathcal{A}(\Omega), d)$ is a complete metric space by Theorem 11.2(3), it follows that $f \in \mathcal{A}(\Omega)$, and then Theorem 11.2(2) implies that (f_n) is uniformly convergent to f on compact subsets of Ω.

Now, f_n is injective on Ω for all n, so f is either injective or constant on Ω by Theorem 11.4(2). Suppose $|f'(z_0)| < \epsilon$, so that $|f'(z_0)| = \epsilon - \delta$ for some $\delta > 0$. The sequence (f'_n) converges uniformly to f' on compact subsets of Ω by Theorem 4.30, and since $\{z_0\} \subseteq \Omega$ is compact, there is some k such that $|f'_n(z_0) - f'(z_0)| < \delta/2$ for all $n \geq k$. However,

$$|f'_k(z_0) - f'(z_0)| < \frac{\delta}{2} \;\Rightarrow\; |f'_k(z_0)| < |f'(z_0)| + \frac{\delta}{2} \;\Rightarrow\; |f'_k(z_0)| < \epsilon - \frac{\delta}{2} < \epsilon,$$

which implies that $f_k \notin \mathcal{F}$, a contradiction. Therefore $|f'(z_0)| \geq \epsilon$ must be the case, which is to say $f'(z_0) \neq 0$ and f cannot be constant on Ω, so f must be injective on Ω.

It remains to verify that $f(\Omega) \subseteq \overline{\mathbb{B}}$, or in other words $|f(z)| \leq 1$ for all $z \in \Omega$. But if there were some $w \in \Omega$ such that $|f(w)| > 1$, then since $f_n(w) \to f(w)$ there would be some k such that $|f_n(w)| > 1$ for all $n \geq k$, implying in particular that $f_k \notin \mathcal{F}$ since $f_k(\Omega) \subseteq \overline{\mathbb{B}}$ would be false. As this is a contradiction, we conclude that $f(\Omega) \subseteq \overline{\mathbb{B}}$ must be the case, and hence $f \in \mathcal{F}$. Therefore $\mathcal{F}$ is closed since it contains all of its limit points.

That $\mathcal{F}$ is bounded is clear: for any compact $K \subseteq \Omega$ we have $|f(z)| \leq 1$ for all $z \in K$ and $f \in \mathcal{F}$, and thus

$$\sup\{\|f\|_K : f \in \mathcal{F}\} \in \mathbb{R}$$

since 1 is an upper bound for $\{\|f\|_K : f \in \mathcal{F}\}$. Since $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ is closed and bounded, the Compactness Criterion finally implies that $\mathcal{F}$ is compact.

Next, suppose $S = \mathbb{B}$. The proof is the same in every regard save the matter of showing that $f(\Omega) \subseteq \mathbb{B}$. For each $z \in \Omega$ we have $f_n(z) \to f(z)$, and since $|f_n(z)| < 1$ for all n, it must be that $|f(z)| \leq 1$. Now, since f is not constant on the region Ω, the Maximum Principle implies that $|f|$ has no local maximum at any point in Ω, and so there can exist no $z \in \Omega$ such that

$|f(z)| = 1$. This shows that $f(\Omega) \subseteq \mathbb{B}$, and therefore $f \in \mathcal{F}$. We conclude that $\mathcal{F}$ is closed if $S = \mathbb{B}$. ■

Theorem 11.17 (Vitali's Theorem). *Let Ω be a region, and let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be a bounded subset of $\mathcal{A}(\Omega)$. Suppose the sequence (f_n) converges pointwise on $S \subseteq \Omega$, and S has a limit point in Ω. Then (f_n) is uniformly convergent on compact subsets of Ω to some $f \in \mathcal{A}(\Omega)$.*

Proof. By Montel's Theorem there is some subsequence (g_n) of (f_n) that is compactly convergent to some $g \in \mathcal{A}(\Omega)$ on Ω. Suppose there exists some compact $K \subseteq \Omega$ on which (f_n) does not converge uniformly to g. Thus there exists $\delta > 0$ such that, for each n, there is some $k \geq n$ and $z \in K$ for which $|f_k(z) - g(z)| \geq \delta$. This implies we may construct a subsequence $(f_{n_k})_{k=1}^{\infty}$ of (f_n) such that, for every k, there is some $z \in K$ for which $|f_{n_k}(z) - g(z)| \geq \delta$. However, Montel's Theorem implies that (f_{n_k}) itself has a subsequence (h_n) that is compactly convergent to some $h \in \mathcal{A}(\Omega)$ on Ω, and by construction it must be that $h \neq g$ on Ω.

On the other hand since (g_n) and (h_n) are subsequences of (f_n), and (f_n) converges pointwise on $S \subseteq \Omega$, we have

$$g(z) = \lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} h_n(z) = h(z)$$

for all $z \in S$. Thus $Z(g-h) \supseteq S$, implying $Z(g-h)$ has a limit point in Ω, and therefore $g \equiv h$ on Ω by the Identity Theorem. As this is a contradiction, we conclude that (f_n) converges uniformly to g on compact subsets of Ω. ■

Exercise 11.18 (AN11.1.5a). Suppose $f \in \mathcal{A}(\Omega)$ and $\overline{B}_R(a) \subseteq \Omega$. Prove that

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left|f(a + re^{it})\right|^2 r\,dr\,dt.$$

Proof. Since $f^2 \in \mathcal{A}(\Omega)$ we have, for each $r \in [0, R]$,

$$f^2(a) = \frac{1}{2\pi} \int_0^{2\pi} f^2(a + re^{it})dt$$

by Cauchy's Integral Formula, and so

$$f^2(a)r = \frac{r}{2\pi} \int_0^{2\pi} f^2(a + re^{it})dt.$$

Integrating,

$$\int_0^R f^2(a)r\,dr = \int_0^R \left[\frac{r}{2\pi} \int_0^{2\pi} f^2(a + re^{it})dt\right] dr,$$

whence by Fubini's Theorem as given in §14.2 of the *Calculus Notes* we obtain

$$\frac{R^2}{2} f^2(a) = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} f^2(a + re^{it})r\,dt\,dr = \frac{1}{2\pi} \int_0^{2\pi} \int_0^R f^2(a + re^{it})r\,dr\,dt,$$

and then

$$f^2(a) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R f^2(a + re^{it})r\,dr\,dt.$$

Finally

$$|f(a)|^2 = |f^2(a)| \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left| f(a + re^{it}) \right|^2 r\, dr\, dt$$

as desired. ∎

Exercise 11.19 (AN11.1.5b). Let $M \in \mathbb{R}_+$ and define

$$\mathcal{F} = \left\{ f \in \mathcal{A}(\Omega) : \iint_\Omega |f|^2 \leq M \right\}.$$

(Letting $\varphi : \mathbb{C} \to \mathbb{R}^2$ be the isometry $\varphi(x + iy) = (x, y)$, we take the integral above to represent

$$\iint_{\varphi(\Omega)} |f \circ \varphi^{-1}|^2$$

in the interests of brevity.) Show that $\mathcal{F}$ is precompact.

Solution. Let $K \subseteq \Omega$ be a compact set. Since $\Omega^c = \mathbb{C} \setminus \Omega$ is closed and $K \cap \Omega^c = \varnothing$, by Theorem 2.45 there exists some $z_0 \in K$ and $w_0 \in \Omega^c$ such that $\operatorname{dist}(K, \Omega^c) = |z_0 - w_0| \in \mathbb{R}_+$. Let $\rho = |z_0 - w_0|/2$, and fix $w = a + ib \in K$. Then $\overline{B}_\rho(w) \subseteq \Omega$, and so for all $f \in \mathcal{F}$

$$|f(w)|^2 \leq \frac{1}{\pi \rho^2} \int_0^{2\pi} \int_0^\rho \left| f(w + re^{it}) \right|^2 r\, dr\, dt \tag{11.5}$$

by Exercise 11.18 (the integrand is taken to be a function of $r, t \in \mathbb{R}$). Defining $g(z) = |f(w+z)|^2$, letting I denote the integral in (11.5), and setting

$$S = \{(r, t) : 0 \leq t \leq 2\pi \text{ and } 0 \leq r \leq \rho\},$$

we apply Theorem 15.11 in §15.3 of the *Calculus Notes* to obtain

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\rho g(re^{it}) r\, dr\, dt = \int_0^{2\pi} \int_0^\rho (g \circ \varphi^{-1})(r \cos t, r \sin t) r\, dr\, dt \\ &= \iint_S (g \circ \varphi^{-1})(r \cos t, r \sin t) r\, dA = \iint_{\overline{B}_\rho(0)} (g \circ \varphi^{-1})(x, y) dA \\ &= \iint_{\overline{B}_\rho(a,b)} (g \circ \varphi^{-1})(x - a, y - b) dA = \iint_{\overline{B}_\rho(w)} \left| f(x + iy) \right|^2 dA. \end{aligned}$$

Returning to (11.5),

$$|f(w)|^2 \leq \frac{1}{\pi \rho^2} \iint_{\overline{B}_\rho(w)} \left| f(x + iy) \right|^2 dA \leq \frac{1}{\pi \rho^2} \iint_\Omega \left| f(x + iy) \right|^2 dA,$$

and hence

$$|f(w)| \leq \sqrt{\frac{1}{\pi \rho^2} \iint_\Omega |f|^2} \leq \sqrt{\frac{M}{\pi \rho^2}}$$

holds for all $f \in \mathcal{F}$. Since $w \in K$ is arbitrary, it follows that

$$\|f\|_K \leq \sqrt{\frac{M}{\pi \rho^2}}$$

for each $f \in \mathcal{F}$, and thus $\sup\{\|f\|_K : f \in \mathcal{F}\} \in \mathbb{R}_+$. Since K is an arbitrary compact subset of Ω, we conclude that $\mathcal{F}$ is bounded. Therefore $\mathcal{F}$ is precompact by the Compactness Criterion. ∎

The next exercise shows that the hypothesis that $\mathcal{F} \subseteq \mathcal{A}(\Omega)$ in the Compactness Criterion cannot be relaxed to admit $\mathcal{F} \subseteq \mathcal{C}(\Omega)$, for it is possible for a set in $\mathcal{C}(\Omega)$ to be closed and bounded but not compact.

Exercise 11.20 (AN11.1.6). Let $D = \overline{B}_R(a) \subseteq \Omega$, and define

$$\mathcal{F} = \{f \in \mathcal{C}(\Omega) : |f| \leq 1 \text{ on } D \text{ and } f \equiv 0 \text{ on } \Omega \setminus D\}.$$

Show that $\mathcal{F}$ is a closed and bounded subset of $\mathcal{C}(\Omega)$, and yet $\mathcal{F}$ is not compact.

Solution. Define $\Psi : \mathcal{F} \to \mathbb{R}$ by

$$\Psi(f) = \iint_D (1 - |f|),$$

where $1 - |f| \geq 0$ on D for any $f \in \mathcal{F}$. In general, if $g : D \to \mathbb{C}$ is continuous on D, then

$$\iint_D |g| = 0 \quad \Leftrightarrow \quad g \equiv 0.$$

Now, continuity of $f \in \mathcal{F}$ dictates that $f \equiv 0$ on ∂D (otherwise $f \equiv 0$ on $\Omega \setminus D$ is not possible), and thus $1 - |f|$ is not identically zero on D and we obtain

$$\iint_D (1 - |f|) > 0.$$

Fix $f \in \mathcal{F}$, and let (f_n) be a sequence in $\mathcal{F}$ such that $\text{d-lim}\, f_n = f$ on Ω. Then (f_n) converges uniformly to f on compact subsets of Ω by Theorem 11.2, and so in particular $\text{u-lim}\, f_n = f$ on D. Fix $\epsilon > 0$. Then there exists some k such that $\|f - f_n\|_D < \epsilon/\pi R^2$ for all $n \geq k$, which is to say $|f(z) - f_n(z)| < \epsilon/\pi R^2$ for all $z \in D$. Now,

$$\begin{aligned} |\Psi(f_n) - \Psi(f)| &= \left| \iint_D (1 - |f_n|) - \iint_D (1 - |f|) \right| = \left| \iint_D (|f| - |f_n|) \right| \\ &\leq \iint_D \big| |f| - |f_n| \big| \leq \iint_D |f - f_n| \leq \iint_D \frac{\epsilon}{\pi R^2} = \epsilon \end{aligned}$$

for each $n \geq k$, implying that $\lim \Psi(f_n) = \Psi(f)$ in $\mathbb{R}$ and therefore Ψ is continuous at f by Theorem 2.20. Define $\Phi : \mathcal{F} \to \mathbb{R}$ by $\Phi(f) = [\Psi(f)]^{-1}$. Since $\Psi(f) \neq 0$ for any $f \in \mathcal{F}$, the function Φ is indeed well-defined on $\mathcal{F}$, and moreover the continuity of Ψ on $\mathcal{F}$ implies that Φ is continuous on $\mathcal{F}$ as well.

For each $r \in (0, R)$ let $D_r = \overline{B}_r(a)$. Since $D_r \subseteq D^\circ$, the sets D_r and $\mathbb{C} \setminus D^\circ$ are disjoint closed sets, and so by Urysohn's Lemma there exists a continuous function $f_r : \mathbb{C} \to [0, 1]$ such that $f_r \equiv 1$ on D_r, $f_r \equiv 0$ on $\mathbb{C} \setminus D^\circ$, and $0 < f_r < 1$ on the open annulus $A_{r,R}(a)$. It follows that $f_r|_\Omega \in \mathcal{F}$, and

$$\iint_D |f_r| \geq \iint_{D_r} |f_r| = \iint_{D_r} (1) = \pi r^2.$$

Thus

$$\Psi(f_r) = \iint_D (1 - |f_r|) = \pi R^2 - \iint_D |f_r| \leq \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

We see that $\Psi(f_r) \to 0^+$ as $r \to R^-$, and thus $\Phi(f_r) \to +\infty$ as $r \to R^-$. That is, the set $\Phi(\mathcal{F})$ is unbounded in $\mathbb{R}$, and therefore cannot be compact. Since Φ is continuous we conclude by Theorem 2.41 that $\mathcal{F}$ itself is not compact in $\mathcal{C}(\Omega)$.

We now show that $\mathcal{F}$ is bounded and closed in $\mathcal{C}(\Omega)$. That $\mathcal{F}$ is bounded is clear: for any compact $K \subseteq \Omega$ we have $\|f\|_K \leq 1$ for each $f \in \mathcal{F}$, and thus

$$\sup\{\|f\|_K : f \in \mathcal{F}\} \leq 1.$$

Now, let f be a limit point for $\mathcal{F}$. It is immediate that $f \in \mathcal{C}(\Omega)$ since $(\mathcal{C}(\Omega), d)$ is a complete metric space. For each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ such that $f_n \in B'_{1/n}(f)$, and in this fashion we obtain a sequence (f_n) in $\mathcal{F}$ such that d-lim $f_n = f$. By Theorem 11.2(2) it follows that (f_n) is uniformly convergent to f on compact subsets of Ω. Since each $\{z\} \subseteq \Omega$ is compact, we find that for each $z \in \Omega$ and $\epsilon > 0$ there exists some k such that

$$|f_n(z) - f(z)| < \epsilon$$

for all $n \geq k$. Thus

$$|f(z)| < |f_n(z)| + \epsilon \leq 1 + \epsilon$$

for all $z \in D$ and $\epsilon > 0$, implying that $|f| \leq 1$ on D. Also, $|f(z)| < |f_n(z)| + \epsilon = \epsilon$ for all $z \in \Omega \setminus D$ and $\epsilon > 0$, implying that $f \equiv 0$ on $\Omega \setminus D$. Since $f \in \mathcal{C}(\Omega)$, $|f| \leq 1$ on D, and $f \equiv 0$ on $\Omega \setminus D$, we conclude that $f \in \mathcal{F}$. Therefore $\mathcal{F}$ contains all its limits points and must be a closed set. ■

The next proposition is a consequence of the Baire Category Theorem, and it will be used to complete the exercise that follows.

Proposition 11.21. *If (X, d) is a complete metric space and $(S_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X such that $X = \bigcup_{n=1}^\infty S_n$, then there is some $k \in \mathbb{N}$ such that S_k contains a nonempty open ball in X.*

Exercise 11.22 (AN11.1.9). Osgood's Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}(\Omega)$ that converges pointwise to $f : \Omega \to \mathbb{C}$ on Ω. Show there exists an open set $U \subseteq \Omega$ that is dense in Ω, and such that (f_n) converges uniformly to f on compact subsets of U.

Solution. For each $k \in \mathbb{N}$ let

$$A_j = \big\{z \in \Omega : \forall n \in \mathbb{N}\big(|f_n(z)| \leq j\big)\big\},$$

so that $(A_j)_{j\in\mathbb{N}}$ is a sequence of closed sets such that

$$\Omega = \bigcup_{j=1}^\infty A_j.$$

Fix $w \in \Omega$, and let $r > 0$ be such that $Z = \overline{B}_r(w) \subseteq \Omega$. The metric space $(Z, |\cdot|)$ is complete since Z is compact,[14] and since each $S_j = A_j \cap Z$ is closed in Z with

$$\bigcup_{j=1}^{\infty} S_j = Z \cap \bigcup_{j=1}^{\infty} A_j = Z \cap \Omega = Z,$$

Proposition 11.21 implies there exists some $k \in \mathbb{N}$ such that S_k contains a nonempty open ball in Z. Thus there is some $a \in Z$ and $\delta > 0$ such that

$$B = \{z \in Z : |z - a| < \delta\} \subseteq S_k.$$

Since we can obtain an open $\mathbb{C}$-ball $B_{\delta'}(a') \subseteq Z'$ by taking any point $a' \in Z'$ in the interior of Z and choosing a sufficiently small $\delta' > 0$, we can assume that $B = B_\delta(a)$. Thus $B \subseteq A_k \cap Z$. The set B is a region, $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{A}(B)$ that converges pointwise on B to f, and $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is a bounded set in $\mathcal{A}(B)$. The last claim follows from the fact that $B \subseteq A_k$: for all $n \in \mathbb{N}$ and $z \in B$ we have $|f_n(z)| \leq k$, whence $\|f_n\|_B \leq k$, and finally

$$\sup\{\|f_n\|_B : n \in \mathbb{N}\} \leq k.$$

By Vitali's Theorem (f_n) converges uniformly to f on compact subset of B.

Let U be the union of all possible open $\mathbb{C}$-balls $B \subseteq \Omega$ constructed in the manner above. Clearly U is an open set, but is it dense in Ω? Assuming $U \neq \Omega$, fix $z \in \Omega \setminus U$ and let $\epsilon > 0$. We may take ϵ to be sufficiently small that $\overline{B}_\epsilon(z) \subseteq \Omega$. Let $w \in B'_\epsilon(z)$, and let $r > 0$ be such that $Z = \overline{B}_r(w) \subseteq B'_\epsilon(z)$. Now, each set $S_j = A_j \cap Z$ is a closed subset of the complete metric space $(Z, |\cdot|)$, and since $\bigcup_{j=1}^{\infty} S_j = Z$, by Proposition 11.21 and the same argument as before there is some S_k that contains an open $\mathbb{C}$-ball B. The ball B is among those whose union forms U, so that $B \subseteq U$. Now,

$$B \subseteq S_k = A_k \cap Z \subseteq Z \subseteq B'_\epsilon(z)$$

makes clear that $B \cap B'_\epsilon(z) \neq \varnothing$, and then $B'_\epsilon(z) \cap U \neq \varnothing$. Thus for each $z \in \Omega \setminus U$ we have $B'_\epsilon(z) \cap U \neq \varnothing$ for all $\epsilon > 0$, which shows that every point in Ω is either in U or is a limit point of U, and therefore U is dense in Ω.

Finally, let $K \subseteq U$ be compact. Since U is a union of open $\mathbb{C}$-balls in Ω, a finite number of these balls, $B_1, \ldots, B_m$, must cover K. For each $1 \leq j \leq m$ there must be a compact set $K_j \subseteq B_j$ such that $K = \bigcup_{j=1}^{m} K_j$. Since (f_n) converges uniformly to f on each K_j, it follows that (f_n) converges uniformly to f on K as a whole. ■

One immediate consequence of Osgood's Theorem is that the function f is analytic on U by Theorem 4.30.

[14] It is worth recalling that the subspace topology on Z that is inherited from $(\mathbb{C}, |\cdot|)$ is equivalent to the topology on Z induced by the euclidian metric $|\cdot|$

11.3 – Riemann Mapping Theorem

Recall that an analytic nth root of $f \in \mathcal{A}(\Omega)$ is a function $g \in \mathcal{A}(\Omega)$ such that $g^n = f$ on Ω. In particular f is said to have an analytic square root on Ω if there exists some $g \in \mathcal{A}(\Omega)$ such that $f(z) = [g(z)]^2$ for all $z \in \Omega$.

Lemma 11.23. *Let $\Omega \notin \{\varnothing, \mathbb{C}\}$ be a region in $\mathbb{C}$ such that every nonvanishing function in $\mathcal{A}(\Omega)$ has an analytic square root on Ω. Then there exists an injective analytic function $\Omega \to \mathbb{B}$.*

Proof. Since $\Omega \neq \mathbb{C}$ there exists some $a \in \mathbb{C} \setminus \Omega$, so that $h(z) = z - a$ is a nonvanishing function in $\mathcal{A}(\Omega)$ and by hypothesis there exists some $g \in \mathcal{A}(\Omega)$ such that $g^2 = h$ on Ω. Since h (and hence g^2) is nonvanishing and injective, it is clear that g is also nonvanishing and injective, and in particular is nonconstant. By the Open Mapping Theorem $g(\Omega)$, and hence $-g(\Omega)$, is open in $\mathbb{C}$, with $-g(\Omega) \cap g(\Omega) = \varnothing$ since $0 \notin g(\Omega)$. Fix $w \in -g(\Omega)$, and set $r > 0$ such that $\overline{B}_r(w) \subseteq -g(\Omega)$, so $g(\Omega) \cap \overline{B}_r(w) = \varnothing$. Define $f : \Omega \to \mathbb{C}$ by

$$f(z) = \frac{r}{g(z) - w}.$$

Since $g(z) \neq w$ for all $z \in \Omega$, we have $f \in \mathcal{A}(\Omega)$, and also f is injective. Finally, for each $z \in \Omega$ we have $g(z) \in g(\Omega)$, where

$$g(z) \in g(\Omega) \quad \Rightarrow \quad g(z) \notin \overline{B}_r(w) \quad \Rightarrow \quad |g(z) - w| > r \quad \Rightarrow \quad \frac{r}{|g(z) - w|} < 1 \quad \Rightarrow \quad |f(z)| < 1,$$

and hence $f(z) \in \mathbb{B}$. Therefore f is an injective analytic function $\Omega \to \mathbb{B}$. ∎

Theorem 11.24 (**Riemann Mapping Theorem**)**.** *If $\Omega \notin \{\varnothing, \mathbb{C}\}$ is a homologically simply connected region in $\mathbb{C}$, then there exists an analytic bijection $\Omega \to \mathbb{B}$.*

Proof. Suppose $\Omega \notin \{\varnothing, \mathbb{C}\}$ is a homologically simply connected region in $\mathbb{C}$. By the Second Cauchy Theorem in §6.5 it follows that every nonvanishing function in $\mathcal{A}(\Omega)$ has an analytic square root on Ω. Let $\xi : \Omega \to \mathbb{B}$ be an injective analytic function, which exists by Lemma 11.23. Fix $z_0 \in \Omega$, and define

$$\mathcal{F} = \big\{ f \in \mathcal{A}(\Omega) : f \text{ is injective, } f(\Omega) \subseteq \mathbb{B}, \text{ and } |f'(z_0)| \geq |\xi'(z_0)| \big\},$$

where $|\xi'(z_0)| > 0$ by Proposition 8.20. That $\mathcal{F}$ is compact is an immediate consequence of Proposition 11.16, and because $\mathcal{F} \neq \varnothing$ on account of it containing ξ, by Proposition 11.6 there exists some $g \in \mathcal{F}$ such that $|g'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$.

It will be shown that $g(\Omega) = \mathbb{B}$. Toward that end, suppose there exists some $a \in \mathbb{B} \setminus g(\Omega)$, and define $\varphi_a : \mathbb{B} \to \mathbb{B}$ by

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

By Proposition 8.24 φ_a is injective and analytic, and thus $\varphi_a \circ g : \Omega \to \mathbb{B}$ is injective and analytic. Also, since $a \notin g(\Omega)$ and φ_a only vanishes at a, we find that $\varphi_a \circ g$ is nonvanishing on Ω. So $\varphi_a \circ g$ has an analytic square root; that is, there is some $h \in \mathcal{A}(\Omega)$ such that $h^2 = \varphi_a \circ g$, which immediately implies that h is injective. Moreover, since $[h(z)]^2 \in \mathbb{B}$ for each $z \in \Omega$, it is clear that $h(\Omega) \subseteq \mathbb{B}$.

Let $b = h(z_0)$, which is in $\mathbb{B}$, and so

$$\varphi_b(z) = \frac{z-b}{1-\bar{b}z}$$

is analytic on $\mathbb{B}$ by Proposition 8.24. Since $h : \Omega \to \mathbb{B}$ and $\varphi_b : \mathbb{B} \to \mathbb{B}$, we may define $f : \Omega \to \mathbb{B}$ by $f = \varphi_b \circ h$, which is analytic and also injective on Ω. Recalling from Proposition 8.24 that $\varphi_a^{-1} = \varphi_{-a}$ and $\varphi_b^{-1} = \varphi_{-b}$,

$$g = \varphi_{-a} \circ h^2 = \varphi_{-a} \circ (\varphi_{-b} \circ f)^2 = \varphi_{-a} \circ (\varphi_{-b}^2 \circ f) = (\varphi_{-a} \circ \varphi_{-b}^2) \circ f.$$

Now, since $f(z_0) = \varphi_b(b) = 0$, by the Chain Rule we have

$$g'(z_0) = (\varphi_{-a} \circ \varphi_{-b}^2)'(f(z_0))f'(z_0) = (\varphi_{-a} \circ \varphi_{-b}^2)'(0)f'(z_0), \tag{11.6}$$

and since $\varphi_{-a} \circ \varphi_{-b}^2 : \mathbb{B} \to \mathbb{B}$ is analytic,

$$\left|(\varphi_{-a} \circ \varphi_{-b}^2)'(0)\right| \leq 1 - \left|(\varphi_{-a} \circ \varphi_{-b}^2)^2(0)\right| \tag{11.7}$$

by the Schwarz-Pick Theorem (Theorem 8.24). If equality holds in (11.7), then the Schwarz-Pick Theorem implies that $\varphi_{-a} \circ \varphi_{-b}^2$ is a Möbius transformation, which in turn implies that $\varphi_{-a} \circ \varphi_{-b}^2$ is injective by Proposition 8.11. But this is not so, for since $\varphi_{-b}(\mathbb{B}) = \mathbb{B}$ by Proposition 8.24, for any nonzero $z \in \mathbb{B}$ there exist $w_1 \neq w_2$ in $\mathbb{B}$ such that $\varphi_{-b}(w_1) = z$ and $\varphi_{-b}(w_2) = -z$, whereupon we obtain

$$\begin{aligned}(\varphi_{-a} \circ \varphi_{-b}^2)(w_1) &= \varphi_{-a}(\varphi_{-b}^2(w_1)) = \varphi_{-a}(z^2) = \varphi_{-a}((-z)^2) \\ &= \varphi_{-a}(\varphi_{-b}^2(w_2)) = (\varphi_{-a} \circ \varphi_{-b}^2)(w_2),\end{aligned}$$

Therefore

$$\left|(\varphi_{-a} \circ \varphi_{-b}^2)'(0)\right| < 1 - \left|(\varphi_{-a} \circ \varphi_{-b}^2)^2(0)\right|, \tag{11.8}$$

and since $f'(z_0) \neq 0$ by Proposition 8.20, from (11.6) and (11.8) it follows that

$$|g'(z_0)| = \left|(\varphi_{-a} \circ \varphi_{-b}^2)'(0)\right||f'(z_0)| < \left(1 - \left|(\varphi_{-a} \circ \varphi_{-b}^2)^2(0)\right|\right)|f'(z_0)| \leq |f'(z_0)|. \tag{11.9}$$

However, since $f \in \mathcal{A}(\Omega)$ is injective, $f(\Omega) \subseteq \mathbb{B}$, and $|f'(z_0)| \geq |\xi'(z_0)|$ (this last because $|g'(z_0)| \geq |\xi'(z_0)|$), we have $f \in \mathcal{F}$, and hence (11.9) contradicts the property of $g \in \mathcal{F}$ that $|g'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$. Therefore there must exist no $a \in \mathbb{B} \setminus g(\Omega)$, which is to say $g(\Omega) = \mathbb{B}$ and $g : \Omega \to \mathbb{B}$ is an analytic bijection. ■

Two regions Ω_1 and Ω_2 in $\mathbb{C}$ are said to be **conformally equivalent** if there exists an analytic bijection $f : \Omega_1 \to \Omega_2$, in which case f is called a **conformal equivalence**. It is immediate from the Inverse Function Theorem that $f^{-1} : \Omega_2 \to \Omega_1$ is likewise a conformal equivalence. Moreover, because f is injective, Proposition 8.20 makes clear that $f'(z) \neq 0$ for all $z \in \Omega_1$, and thus a conformal equivalence is also a conformal map. By Theorem 8.22 a conformal equivalence is also everywhere angle preserving.

Proposition 11.25. *Let Ω be as in the Riemann Mapping Theorem, let $z_0 \in \Omega$, and let $\xi : \Omega \to \mathbb{B}$ be an injective analytic function. Define*

$$\mathcal{F} = \left\{f \in \mathcal{A}(\Omega) : f \text{ is injective, } f(\Omega) \subseteq \mathbb{B}, \text{ and } |f'(z_0)| \geq |\xi'(z_0)|\right\}.$$

Then the following hold.

1. *If $g \in \mathcal{F}$ is such that*
$$|g'(z_0)| = \max\{|f'(z_0)| : f \in \mathcal{F}\}, \tag{11.10}$$
then $g(z_0) = 0$.
2. *If $f, h : \Omega \to \mathbb{B}$ are analytic bijections such that $f(z_0) = h(z_0) = 0$ and $f'(z_0) = h'(z_0)$, then $f = h$.*
3. *If $f : \Omega \to \mathbb{B}$ is analytic with $f(z_0) = 0$, and $g \in \mathcal{F}$ satisfies* (11.10)*, then $|f'(z_0)| \leq |g'(z_0)|$. Moreover $|f'(z_0)| = |g'(z_0)|$ if and only if $f = \lambda g$ for some $\lambda \in \partial\mathbb{B}$.*
4. *There exists a unique analytic bijection $g : \Omega \to \mathbb{B}$ such that $g(z_0) = 0$ and $g'(z_0) \in \mathbb{R}_+$.*

Proposition 11.26. *Let $\Omega_1, \Omega_2 \notin \{\varnothing, \mathbb{C}\}$ each be a region in $\mathbb{C}$ on which every nonvanishing analytic function has an analytic square root, and let $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$. Then there is a unique analytic bijection $f : \Omega_1 \to \Omega_2$ such that $f(z_1) = z_2$ and $f'(z_1) \in \mathbb{R}_+$.*

Proof. By Proposition 11.25(4) there are unique analytic bijections $f_1 : \Omega_1 \to \mathbb{B}$ and $f_2 : \Omega_2 \to \mathbb{B}$ such that $f_1(z_1) = f_2(z_2) = 0$ and $f_1'(z_1), f_2'(z_2) \in \mathbb{R}_+$. By the Inverse Function Theorem $f_2^{-1} : \mathbb{B} \to \Omega_2$ is analytic, and thus $f : \Omega_1 \to \Omega_2$ given by $f = f_2^{-1} \circ f_1$ is analytic. It is easily verified that $f(z_1) = z_2$, and by the Chain Rule and another application of the Inverse Function Theorem,
$$f'(z_1) = (f_2^{-1})'(f_1(z_1))f_1'(z_1) = (f_2^{-1})'(0)f_1'(z_1) = \frac{1}{f_2'(f_2^{-1}(0))}f_1'(z_1) = \frac{f_1'(z_1)}{f_2'(z_2)},$$
and thus $f'(z_1) \in \mathbb{R}_+$. This proves existence.

To prove uniqueness, suppose $g : \Omega_1 \to \Omega_2$ is an analytic bijection such that $g(z_1) = z_2$ and $g'(z_1) \in \mathbb{R}_+$. There exists an analytic bijection $g_2 : \Omega_2 \to \mathbb{B}$ such that $g_2(z_2) = 0$, and thus $g_2^{-1} : \mathbb{B} \to \Omega_2$ is analytic with $g_2^{-1}(0) = z_2$. Define $g_1 : \Omega_1 \to \mathbb{B}$ by $g_1 = g_2 \circ g$, also analytic, with
$$g_1(z_1) = g_2(g(z_1)) = g_2(z_2) = 0.$$
Now we have $g = g_2^{-1} \circ g_1$, and by the Chain Rule and Inverse Function Theorem
$$g'(z_1) = \frac{g_1'(z_1)}{g_2'(z_2)}.$$
Since $g'(z_1) > 0$, either $g_1'(z_1), g_2'(z_2) > 0$ or $g_1'(z_1), g_2'(z_2) < 0$. Suppose $g_1'(z_1), g_2'(z_2) > 0$, so that for $k \in \{1, 2\}$ we have $g_k : \Omega_k \to \mathbb{B}$ with $g_k(z_k) = 0$ and $g_k'(z_k) \in \mathbb{R}_+$. Then by the uniqueness of f_k we must have $g_k = f_k$, and thus
$$g = g_2^{-1} \circ g_1 = f_2^{-1} \circ f_1 = f.$$
Now suppose $g_1'(z_1), g_2'(z_2) < 0$. Then $-g_k : \Omega_k \to \mathbb{B}$ with $-g_k(z_k) = 0$ and $-g_k'(z_k) \in \mathbb{R}_+$, so that $-g_k = f_k$ and we have $g = (-f_2)^{-1} \circ (-f_1)$. For any $z \in \Omega_1$,
$$w = g(z) = \big((-f_2)^{-1} \circ (-f_1)\big)(z) = (-f_2)^{-1}(-f_1(z)),$$
and so $-f_1(z) = -f_2(w)$, whence
$$f_1(z) = f_2(w) \quad \Rightarrow \quad w = f_2^{-1}(f_1(z)) = (f_2^{-1} \circ f_1)(z) = f(z),$$
and thus $g(z) = f(z)$ and we once again conclude that $g = f$. This proves uniqueness. ∎

The next proposition, and the one that follows, are called upon in the course of proving much of Theorem 11.30 below.

Proposition 11.27. *Let γ_0 and γ_1 be closed curves in Ω. If γ_0 and γ_1 are Ω-homotopic, then they are Ω-homologous.*

Proof. Suppose $\gamma_0, \gamma_1 : [a,b] \to \mathbb{C}$ are Ω-homotopic, and let $H : R \to \Omega$ be a homotopy of γ_0 and γ_1, where $R = [a,b] \times [0,1]$. Fix $z \in \mathbb{C} \setminus \Omega$. Then $H - z : R \to \mathbb{C}_*$ is continuous, and so has a continuous logarithm on R by the remark following Exercise 6.16, whereupon Proposition 6.8(1) implies that there is a continuous argument $\alpha : R \to \mathbb{R}$ of $H - z$. Thus

$$(H - z)(s,t) = |(H - z)(s,t)|e^{i\alpha(s,t)} = |H(s,t) - z|e^{i\alpha(s,t)}$$

for all $(s,t) \in R$, and for each $t \in [0,1]$ the function $\alpha_t = \alpha(\cdot, t) : [a,b] \to \mathbb{R}$ is a continuous argument of the closed curve $\gamma_t = H(\cdot, t) - z$ on $[a,b]$. By Definition 6.19

$$\operatorname{wn}(\gamma_t, z) = \frac{\alpha_t(b) - \alpha_t(a)}{2\pi} = \frac{\alpha(b,t) - \alpha(a,t)}{2\pi},$$

and so the function $t \mapsto \operatorname{wn}(\gamma_t, z)$ is continuous on $[0,1]$ since α is continuous on R. However, $\operatorname{wn}(\gamma_t, z) \in \mathbb{Z}$ for all $t \in [0,1]$ by Proposition 6.7(1), and so $t \mapsto \operatorname{wn}(\gamma_t, z)$ must be a constant function. Therefore $\operatorname{wn}(\gamma_0, z) = \operatorname{wn}(\gamma_1, z)$, and since $z \in \mathbb{C} \setminus \Omega$ is arbitrary, we conclude that $\operatorname{wn}(\gamma_0, z) = \operatorname{wn}(\gamma_1, z)$ for all $z \in \mathbb{C} \setminus \Omega$, which is to say γ_0 and γ_1 are Ω-homologous. ■

Exercise 11.28 (AN5.2.3). Let Ω be a convex open set, and let $\gamma : [a,b] \to \Omega$ be a closed curve. Prove that $H : [a,b] \times [0,1] \to \mathbb{C}$ given by

$$H(s,t) = t\gamma(a) + (1-t)\gamma(s)$$

is an Ω-homotopy of γ to the point $\gamma(a)$.

Solution. Since Ω is convex and $\gamma(s) \in \Omega$ for each $s \in [a,b]$, the line segment $[\gamma(a), \gamma(s)]$ is a subset of Ω for each $s \in [a,b]$. By definition

$$[\gamma(a), \gamma(s)] = \big\{ t\gamma(a) + (1-t)\gamma(s) : t \in [0,1] \big\},$$

and thus it is clear that $\operatorname{Ran}(H) \subseteq \Omega$. Moreover, the function $(s,t) \mapsto \gamma(s)$ is continuous since γ is continuous on $[a,b]$, and since $(s,t) \mapsto t\gamma(a)$ and $(s,t) \mapsto 1 - t$ are also continuous, it follows readily that H is continuous on its domain. Finally, $H(\cdot, 0) = \gamma$, $H(\cdot, 1) = \gamma(a)$, and $H(a,t) = H(b,t) = \gamma(a)$ for all $t \in [0,1]$, since $\gamma(a) = \gamma(b)$. Therefore H is an Ω-homotopy of γ to the point $\gamma(a)$. ■

An immediate consequence of Exercise 11.28 is the following proposition, which establishes in particular that $\mathbb{B}$ and any other open disc in $\mathbb{C}$ is homotopically simply connected.

Proposition 11.29. *Every convex open set in $\mathbb{C}$ is homotopically simply connected.*

We now gather together the foremost equivalent statements encountered in these notes thus far, along with a few that will soon be found to be equivalent.

Theorem 11.30. *Let $\Omega \subseteq \mathbb{C}$ be a nonempty open set. Then the following statements are equivalent.*

1. $\overline{\mathbb{C}} \setminus \Omega$ *is connected.*
2. *Every closed curve in* Ω *is* Ω*-homologous to zero.*
3. Ω *is homologically simply connected.*
4. *For every closed path* γ *in* Ω *and every* $f \in \mathcal{A}(\Omega)$,
$$\oint_\gamma f = 0.$$
5. *Every analytic function on* Ω *has a primitive on* Ω.
6. *Every nonvanishing* $f \in \mathcal{A}(\Omega)$ *has an analytic logarithm.*
7. *Every nonvanishing* $f \in \mathcal{A}(\Omega)$ *has an analytic nth root for all* $n \in \mathbb{N}$.
8. *Every nonvanishing* $f \in \mathcal{A}(\Omega)$ *has an analytic square root.*
9. *If* Ω *is connected with* $\Omega \neq \mathbb{C}$*, then* Ω *is conformally equivalent to* $\mathbb{B}$.
10. *If* Ω *is connected, then* Ω *is homeomorphic to* $\mathbb{B}$.
11. Ω *is homotopically simply connected.*
12. *Every closed path in* Ω *is* Ω*-homotopic to a point.*
13. *Every harmonic function on* Ω *has a harmonic conjugate.*
14. *Every* $f \in \mathcal{A}(\Omega)$ *can be uniformly approximated by polynomials on compact subsets of* Ω.

Proof. That (1), (2), (3), (4), (5), (6), and (7) are equivalent is the statement of the Second Cauchy Theorem in §6.5. It is clear that (7) implies (8).

(8) ⇒ (9): Suppose (8). Suppose Ω is connected with $\Omega \neq \mathbb{C}$. Since Ω is a nonempty open subset of $\mathbb{C}$ by hypothesis, the Riemann Mapping Theorem implies that there exists an analytic bijection $\Omega \to \mathbb{B}$. Therefore Ω is conformally equivalent to $\mathbb{B}$.

(9) ⇒ (10): Suppose (9). Suppose Ω is connected. If $\Omega \neq \mathbb{C}$, then Ω is conformally equivalent to $\mathbb{B}$, which immediately implies that Ω is homeomorphic to $\mathbb{B}$. If $\Omega = \mathbb{C}$, then define $h : \mathbb{C} \to \mathbb{B}$ by
$$h(z) = \frac{z}{1 + |z|}.$$
It is shown in Exercise 11.32 below that h is a homeomorphism. Therefore if Ω is connected, then it is homeomorphic to $\mathbb{B}$.

(10) ⇒ (11): Suppose (10). Let $\gamma : [a, b] \to \Omega$ be a closed curve in Ω. Since γ is continuous, γ^* must lie in some component Ω_0 of Ω, and since Ω_0 is a region in $\mathbb{C}$, by (10) there exists a homeomorphism $h : \Omega_0 \to \mathbb{B}$. Now, $h \circ \gamma$ is a closed curve in $\mathbb{B}$, and since $\mathbb{B}$ is homotopically simply connected by Proposition 11.29, there is a homotopy $H : [a, b] \times [0, 1] \to \mathbb{B}$ of $f \circ \gamma$ to the point $f(\gamma(a))$. Now the function $h^{-1} \circ H : [a, b] \times [0, 1] \to \Omega_0$ is continuous, and is easily verified to be a homotopy in Ω of γ to the point $\gamma(a)$. Thus every closed curve in Ω is Ω-homotopic to a point.

(11) ⇒ (12): Suppose (11). Then every closed curve in Ω is Ω-homotopic to a point, which immediately implies that every closed path in Ω is Ω-homotopic to a point.

(12) ⇒ (3): Suppose (12). Let $\gamma : [a, b] \to \Omega$ be a closed path in Ω, and let $z \in \mathbb{C} \setminus \Omega$. By hypothesis γ is Ω-homotopic to the point $\gamma(a)$, and so by Proposition 11.27 the constant function $t \mapsto \gamma(a)$ (which we denote by $\gamma(a)$) and γ are Ω-homologous; that is,

$$\mathrm{wn}(\gamma, z) = \mathrm{wn}(\gamma(a), z) = 0.$$

We conclude that $\mathrm{wn}(\gamma, z) = 0$ for every closed path γ in Ω and every $z \in \mathbb{C} \setminus \Omega$, and therefore Ω is homologically simply connected. ■

The proof of Theorem 11.30 so far has established the equivalency of the first dozen statements in the list. The next proposition will bring (13) into the fold, whereas (14) is secured by a variant of Runge's Theorem presented in Exercise 11.39 in the next section. At this juncture we now see that homological and homotopical simple connectedness are equivalent concepts, and henceforth we will use the term **simply connected** to reference either.

Proposition 11.31. *Let $\Omega \subseteq \mathbb{C}$ be an open set. Then every harmonic function on Ω has a harmonic conjugate if and only if Ω is simply connected.*

Exercise 11.32 (AN5.2.2). Show that $h : \mathbb{C} \to \mathbb{B}$ given by

$$h(z) = \frac{z}{1 + |z|}$$

is a homeomorphism.

Solution. It is clear that h is continuous on the basis of the usual laws of limits, and we also have $h(0) = 0$. To show that h is surjective, let $w \in \mathbb{B}$ with $w \neq 0$, so that $w = re^{i\theta}$ for some $r \in [0, 1)$ and $\theta \in \mathbb{R}$. In general we have

$$h(\rho e^{i\theta}) = \frac{\rho}{1 + \rho} e^{i\theta},$$

and so we only need to find ρ such that $\rho/(1 + \rho) = r$. Solving for ρ gives $\rho = r/(1 - r)$, which is a real number, and hence choosing $z = re^{i\theta}/(1 - r)$ we obtain

$$h(z) = \frac{\frac{r}{1-r} e^{i\theta}}{1 + \frac{r}{1-r}} = \frac{re^{i\theta}}{(1 - r) + r} = re^{i\theta} = w.$$

Next we show that h is injective. Suppose $h(z) = h(w)$, where $z = re^{i\alpha}$ and $w = \rho e^{i\beta}$. Then $|h(z)| = |h(w)|$, so that

$$\frac{r}{1 + r} = \frac{\rho}{1 + \rho}$$

and thus $r = \rho$. Now,

$$h(z) = h(w) \;\; \Rightarrow \;\; \frac{re^{i\alpha}}{1 + r} = \frac{re^{i\beta}}{1 + r} \;\; \Rightarrow \;\; e^{i\alpha} = e^{i\beta},$$

and therefore $z = w$.

Finally we find h^{-1} and verify that it is continuous. For each $re^{i\theta} \in \mathbb{C}$, where $r \in [0, \infty)$,

$$h(re^{i\theta}) = \frac{r}{1 + r} e^{i\theta} \;\; \Leftrightarrow \;\; h^{-1}\left(\frac{r}{1 + r} e^{i\theta}\right) = re^{i\theta} \;\; \Leftrightarrow \;\; h^{-1}(\rho e^{i\theta}) = \frac{\rho}{1 - \rho} e^{i\theta},$$

where $\rho \in [0, 1)$. This shows that $h^{-1} : \mathbb{B} \to \mathbb{C}$ is given by

$$h^{-1}(z) = \frac{z}{1 - |z|}$$

for all $z \in \mathbb{B}$, which clearly is continuous on $\mathbb{B}$. Therefore h is a homeomorphism. ∎

11.4 – Runge's Theorem

For any function f, let $\overline{P}(f)$ denote the set of poles of f that lie in the extended complex plane $\overline{\mathbb{C}}$.

Lemma 11.33. *Suppose $K \subseteq \Omega \subseteq \mathbb{C}$, with K compact and Ω open. If $f \in \mathcal{A}(\Omega)$, then there exists a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq \Omega \setminus K$ for each n and* u-lim $f_n = f$ *on K.*

Lemma 11.34. *Let $U, V \subseteq \mathbb{C}$ be open with $V \subseteq U$ and $\partial V \cap U = \varnothing$. If U_0 is a component of U such that $U_0 \cap V \neq \varnothing$, then $U_0 \subseteq V$.*

Proof. Suppose that U_0 is a component of U such that $U_0 \cap V \neq \varnothing$. Define two sets: $A = U_0 \cap V$ and $B = U_0 \cap (\mathbb{C} \setminus V)$, which are disjoint. Since the components of an open set in $\mathbb{C}$ are open, the set A is open.

Let $z \in B$, so $z \in U_0 \subseteq U$ and $z \notin V$. Since $U \cap \partial V = \varnothing$, we have $z \notin \partial V$ in addition to $z \notin V$, so that $z \notin V \cup \partial V = \overline{V}$, and hence $z \in U_0 \cap (\mathbb{C} \setminus \overline{V})$. Conversely,

$$z \in U_0 \cap (\mathbb{C} \setminus \overline{V}) \;\;\Rightarrow\;\; z \in U_0 \cap (\mathbb{C} \setminus V) \;\;\Rightarrow\;\; z \in B,$$

and so we find that $B = U_0 \cap (\mathbb{C} \setminus \overline{V})$, which is also an open set. Thus A and B are disjoint open sets such that $A \cup B = U_0$, and since U_0 is connected either $A = \varnothing$ or $B = \varnothing$. But $A \neq \varnothing$ by hypothesis, so $B = \varnothing$ must be the case and therefore $U_0 \subseteq V$. ∎

Lemma 11.35. *Let $K \subseteq \mathbb{C}$ be compact, and $S \subseteq \overline{\mathbb{C}} \setminus K$ such that $S \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus K$. If $\zeta \in \mathbb{C} \setminus K$, then there exists a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq S$ for each n and* u-lim $f_n = (z - \zeta)^{-1}$ *on K.*

Theorem 11.36 (Runge's Theorem). *Let $K \subseteq \mathbb{C}$ be compact, and $S \subseteq \overline{\mathbb{C}} \setminus K$ such that $S \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus K$. If $\Omega \supseteq K$ is open in $\mathbb{C}$ and $f \in \mathcal{A}(\Omega)$, then there exists a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq S$ for each n and* u-lim $f_n = f$ *on K.*

Proof. Suppose $\Omega \supseteq K$ is open in $\mathbb{C}$ and $f \in \mathcal{A}(\Omega)$. By Lemma 11.33 there exists a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq \Omega \setminus K$ for each n and u-lim $f_n = f$ on K. Fix $n \in \mathbb{N}$. Since f_n has no pole at ∞, its partial fraction decomposition may be written in the form

$$f_n(z) = c_{n0} + \sum_{k=1}^{m_n} \frac{c_{nk}}{z - \zeta_{nk}},$$

for constants $c_{nk}, \zeta_{nk} \in \mathbb{C}$ and $m_n \geq 0$ (if $m_n = 0$ then we take $f_n \equiv c_{n0}$). For each $1 \leq k \leq m_n$, ζ_{nk} is a pole of f_n, so that $\zeta_{nk} \in \Omega \setminus K \subseteq \mathbb{C} \setminus K$, and so by Lemma 11.35 there is a sequence of rational functions $(f_{nkj})_{j=1}^{\infty}$ such that $\overline{P}(f_{nkj}) \subseteq S$ for each j and

$$\text{u-}\lim_{j \to \infty} f_{nkj} = \frac{1}{z - \zeta_{nk}} := f_{nk}$$

on K. For each j define

$$g_{nj} = c_{n0} + \sum_{k=1}^{m_n} c_{nk} f_{nkj}.$$

Then $(g_{nj})_{j=1}^{\infty}$ is a sequence of rational functions such that

$$\overline{P}(g_{nj}) = \bigcup_{k=1}^{m_n} \overline{P}(f_{nkj}) \subseteq S$$

for each j and

$$\text{u-}\lim_{j\to\infty} g_{nj} = c_{n0} + \sum_{k=1}^{m_n} c_{nk} f_{nk} = f_n$$

on K.

For each $n \in \mathbb{N}$ let $j_n \in \mathbb{N}$ be such that

$$\left|g_{nj_n}(z) - f_n(z)\right| < \frac{1}{n}$$

for all $z \in K$. Now consider the sequence $(g_{nj_n})_{n\in\mathbb{N}}$, which is a sequence of rational functions such that $\overline{P}(g_{nj_n}) \subseteq S$ for each n. Fix $\epsilon > 0$. Let $M_1 \in \mathbb{N}$ such that $1/M_1 < \epsilon/2$. Also let $M_2 \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{2}$$

for all $n \geq M_2$ and $z \in K$. Choose $N = \max\{M_1, M_2\}$, and let $n \geq N$ and $z \in K$. Then

$$\left|g_{nj_n}(z) - f(z)\right| \leq \left|g_{nj_n}(z) - f_n(z)\right| + \left|f_n(z) - f(z)\right| < \frac{1}{n} + \frac{\epsilon}{2} \leq \frac{1}{M_1} + \frac{\epsilon}{2} < \epsilon.$$

This shows that u-lim $g_{nj_n} = f$ on K, and the proof is finished. ∎

Exercise 11.37 (AN5.2.5). Let $\Omega \subseteq \mathbb{C}$ be open, and let $(K_n)_{n=1}^{\infty}$ be the sequence of sets defined by (11.1). For each $n \in \mathbb{N}$, show that each component of $\overline{\mathbb{C}} \setminus K_n$ contains a component of $\overline{\mathbb{C}} \setminus \Omega$.

Solution. Let U be a component of $\overline{\mathbb{C}} \setminus K_n$, so U is a (nonempty) maximal connected set in the metric space $(\overline{\mathbb{C}} \setminus K_n, \bar{d}\,)$. Letting $A = A_{n,\infty}(0) \cup \{\infty\}$ we have

$$\overline{\mathbb{C}} \setminus K_n = A \cup \{z : |z - w| < 1/n \text{ for some } w \in \mathbb{C} \setminus \Omega\} = A \cup \bigcup_{w\in\mathbb{C}\setminus\Omega} B_{1/n}(w),$$

and thus

$$U \subseteq A \cup \bigcup_{w\in\mathbb{C}\setminus\Omega} B_{1/n}(w).$$

Let $z \in U$. Then either $z \in B_{1/n}(w)$ for some $w \in \mathbb{C} \setminus \Omega$, or $z \in A$. Since $B_{1/n}(w)$ and A are connected sets in $(\overline{\mathbb{C}} \setminus K_n, \bar{d}\,)$, we have either $B_{1/n}(w) \subseteq U$ or $A \subseteq U$ by Theorem 2.34. In the former instance we have $w \in U \cap (\overline{\mathbb{C}} \setminus \Omega)$, and in the latter instance we have $\infty \in U \cap (\overline{\mathbb{C}} \setminus \Omega)$. In either case U contains a point w in $\overline{\mathbb{C}} \setminus \Omega$. Let V be the component of $\overline{\mathbb{C}} \setminus \Omega$ that contains w. Then $V \neq \varnothing$ is a connected set in $\overline{\mathbb{C}} \setminus \Omega$, and since

$$V \subseteq \overline{\mathbb{C}} \setminus \Omega \subseteq \overline{\mathbb{C}} \setminus K_n \subseteq (\overline{\mathbb{C}}, \bar{d}\,),$$

by the remark following Theorem 2.27 it follows that V is a connected set in $\overline{\mathbb{C}} \setminus K_n$. Now, because U is a component of $\overline{\mathbb{C}} \setminus K_n$ and $U \cap V \neq \varnothing$, by Theorem 2.34 we conclude that $V \subseteq U$. ■

Exercise 11.38 (AN5.2.6a). Let $\Omega \subseteq \mathbb{C}$ be open, and let $S \subseteq \overline{\mathbb{C}}$ be such that $S \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus \Omega$. If $f \in \mathcal{A}(\Omega)$, then there is a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq S$ for each n and u-lim $f_n = f$ on compact subsets of Ω.

Solution. Fix $n \in \mathbb{N}$. By Exercise 11.37 each component of $\overline{\mathbb{C}} \setminus K_n$ contains a component of $\overline{\mathbb{C}} \setminus \Omega$. Thus $S \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus K_n$, since C contains a component C' of $\overline{\mathbb{C}} \setminus \Omega$ and $S \cap C' \neq \varnothing$. Let $S_n = S \cap (\overline{\mathbb{C}} \setminus K_n)$, so $S_n \subseteq \overline{\mathbb{C}} \setminus K_n$ and $S_n \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus K_n$. Since $K_n \subseteq \mathbb{C}$ is compact by Lemma 11.1, $\Omega \supseteq K_n$ is open in $\mathbb{C}$, and $f \in \mathcal{A}(\Omega)$, by Runge's Theorem there exists a sequence $(g_{nk})_{k=1}^{\infty}$ of rational functions such that $\overline{P}(g_{nk}) \subseteq S_n \subseteq S$ for each k, and

$$\text{u-}\lim_{k\to\infty} g_{nk} = f$$

on K_n.

For each $n \in \mathbb{N}$ choose $k_n \in \mathbb{N}$ such that

$$|g_{nk_n}(z) - f(z)| < \frac{1}{n}$$

for all $z \in K_n$. Defining $f_n = g_{nk_n}$ for each n, we construct a sequence (f_n) of rational functions with $\overline{P}(f_n) \subseteq S$ for each n.

Let $K \subseteq \Omega$ be compact. Fix $\epsilon > 0$. By Lemma 11.1 there is some m_1 such that $K \subseteq K_n$ for all $n \geq m_1$. Letting m_2 be such that $1/m_2 < \epsilon$, choose $m = \max\{m_1, m_2\}$. Fix $n \geq m$. For all $z \in K_n$ we have

$$|f_n(z) - f(z)| = |g_{nk_n}(z) - f(z)| < \frac{1}{n} \leq \frac{1}{m} \leq \frac{1}{m_2} < \epsilon,$$

and since $n \geq m \geq m_1$ implies $K \subseteq K_n$, it follows that $|f_n(z) - f(z)| < \epsilon$ for all $z \in K$. Therefore u-lim $f_n = f$ on compact subsets of Ω. ■

Exercise 11.39 (AN5.2.6b). Let $\Omega \subseteq \mathbb{C}$ be open. Show that Ω is simply connected if and only if for each $f \in \mathcal{A}(\Omega)$ there is a sequence (f_n) of polynomial functions such that u-lim $f_n = f$ on compact subset of Ω.

Solution. Suppose Ω is simply connected, and let $f \in \mathcal{A}(\Omega)$. By Theorem 11.30 $\overline{\mathbb{C}} \setminus \Omega$ is connected, so $S = \{\infty\}$ is a set in $\overline{\mathbb{C}}$ such that $S \cap C \neq \varnothing$ for each component C of $\overline{\mathbb{C}} \setminus \Omega$, since of necessity $C = \overline{\mathbb{C}} \setminus \Omega$ is the only component. By Exercise 11.38 there is a sequence (f_n) of rational functions such that $\overline{P}(f_n) \subseteq \{\infty\}$ for each n and u-lim $f_n = f$ on compact subsets of Ω. Thus each f_n is an entire rational function, so there exist polynomial functions p_n and q_n such that $f_n = p_n/q_n$ and $q_n(z) \neq 0$ for all $z \in \mathbb{C}$. By the Fundamental Theorem of Algebra $q_n \equiv c_n$ for some $c_n \in \mathbb{C}_*$, whence $f_n = c_n^{-1} p_n$ obtains and (f_n) is seen to be a sequence of polynomial functions.

Now suppose that for each $f \in \mathcal{A}(\Omega)$ there is a sequence (f_n) of polynomial functions such that u-lim $f_n = f$ on compact subset of Ω. Fix $f \in \mathcal{A}(\Omega)$, and let γ be a closed path in Ω. Let (f_n) be a sequence of polynomial functions that converges uniformly to f on compact subsets of

Ω. Since any polynomial function has a primitive on Ω, by the Fundamental Theorem of Path Integrals we have

$$\oint_\gamma f_n = 0$$

for each n. Now, $\gamma^* \subseteq \Omega$ is compact and (f_n) converges uniformly to f on γ^*, so that

$$\oint_\gamma f = \lim_{n\to\infty} \oint_\gamma f_n = \lim_{n\to\infty} (0) = 0$$

by Proposition 3.37. We conclude that

$$\oint_\gamma f = 0$$

for every $f \in \mathcal{A}(\Omega)$ and closed path γ in Ω, and therefore Ω is simply connected by Theorem 11.30. ∎

11.5 – Extending Conformal Maps to the Boundary

Proposition 11.40. *Suppose $\Omega \subseteq \mathbb{C}$ is open, f is a homeomorphism on Ω, and (z_n) is a sequence in Ω. Then (z_n) has a limit point in Ω if and only if $(f(z_n))$ has a limit point in $f(\Omega)$.*

Proposition 11.41. *Suppose $f : \Omega \to \mathbb{B}$ is a conformal equivalence. If (z_n) is a sequence in Ω such that $z_n \to \zeta \in \partial\Omega$, then $|f(z_n)| \to 1$.*

Definition 11.42. *A point $\zeta \in \partial\Omega$ is **simple** if for every sequence (z_n) in Ω for which $z_n \to \zeta$, there exists a curve $\gamma : [0, 1] \to \Omega \cup \{\zeta\}$ and a strictly increasing sequence (t_n) in $[0, 1)$ such that $t_n \to 1$, $\gamma(t_n) = z_n$ for each n, and $\gamma([0, 1)) \subseteq \Omega$.*

Theorem 11.43. *Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected region, and let $\zeta \in \partial\Omega$ be simple. If $f : \Omega \to \mathbb{B}$ is a conformal equivalence, then f has continuous extension to $\Omega \cup \{\zeta\}$.*

For the following theorem recall that an analytic function $f : \Omega \to \mathbb{C}$ is a conformal map if its derivative is nonvanishing. Such a function is not necessarily a conformal equivalence of Ω onto $f(\Omega)$ even if the set Ω is as specified in the Riemann Mapping Theorem, since it may not be injective on its domain (though it will be locally injective).

Theorem 11.44. *Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected region, and let $f : \Omega \to \mathbb{B}$ be a surjective conformal map. If $\zeta_1, \zeta_2 \in \partial\Omega$ are distinct simple points and f has continuous extension to $\Omega \cup \{\zeta_1, \zeta_2\}$, then $f(\zeta_1) \neq f(\zeta_2)$.*

Theorem 11.45. *Let $\Omega \subseteq \mathbb{C}$ be a bounded simply connected region such that every boundary point is simple. If $f : \Omega \to \mathbb{B}$ is a conformal equivalence, then f extends to a homeomorphism $\overline{\Omega} \to \overline{\mathbb{B}}$.*

Exercise 11.46 (AN5.3.1). Let $\Omega \subseteq \mathbb{C}$ be a nonempty, bounded, simply connected region such that every boundary point is simple. Prove that the Dirichlet problem is solvable for Ω; that is, if $u_0 : \partial\Omega \to \mathbb{R}$ is continuous, then u_0 has a continuous extension $u : \overline{\Omega} \to \mathbb{R}$ that is harmonic on Ω.

Solution. Let $u_0 : \partial\Omega \to \mathbb{R}$ be continuous. By Theorem 11.30 the region Ω is conformally equivalent to $\mathbb{B}$, so there exists an analytic bijection $\Omega \to \mathbb{B}$ which, by Theorem 11.45, extends to a homeomorphism $\overline{\Omega} \to \overline{\mathbb{B}}$. Let $\varphi : \overline{\mathbb{B}} \to \overline{\Omega}$ be the inverse of this homeomorphism. Since $\varphi(\partial\mathbb{B}) = \partial\Omega$, we see that $\varphi_0 = \varphi|_{\partial\mathbb{B}} : \partial\mathbb{B} \to \partial\Omega$ is a continuous function, and hence

$$g_0 = u_0 \circ \varphi_0 : \partial\mathbb{B} \to \partial\Omega \to \mathbb{R}$$

is continuous. By Theorem 9.9 there exists a function $g : \overline{\mathbb{B}} \to \mathbb{R}$ that is continuous on $\overline{\mathbb{B}}$ and harmonic on $\mathbb{B}$, with $g|_{\partial\mathbb{B}} = g_0$. Define $u : \overline{\Omega} \to \mathbb{R}$ by $u = g \circ \varphi^{-1}$, which is continuous on $\overline{\Omega}$. Since $\varphi^{-1}|_{\partial\Omega} = \varphi_0^{-1}$ and $\varphi_0^{-1}(\partial\Omega) = \partial\mathbb{B}$, we find that $u|_{\partial\Omega} = g_0 \circ \varphi_0^{-1} = u_0$.

It remains to show that u is harmonic on Ω. Since $\mathbb{B}$ is simply connected, by Theorem 11.30 the function g has a harmonic conjugate h on $\mathbb{B}$, so that $\psi = g + ih$ is analytic on $\mathbb{B}$, and then $\psi \circ \varphi^{-1} : \Omega \to \mathbb{C}$ is analytic. Now, for any $z \in \Omega$,

$$\big(\mathrm{Re}(\psi \circ \varphi^{-1})\big)(z) = \mathrm{Re}[\psi(\varphi^{-1}(z))] = \mathrm{Re}[g(\varphi^{-1}(z)) + ih(\varphi^{-1}(z))] = g(\varphi^{-1}(z)) = u(z),$$

which shows that $\mathrm{Re}(\psi \circ \varphi^{-1}) = u$ and therefore u is harmonic on Ω by Theorem 4.28. ■

Exercise 11.47 (AN5.3.3). Let $\Omega \subseteq \mathbb{C}$ be a nonempty, bounded, simply connected region such that every boundary point is simple. Show that every nonvanishing continuous function on $\overline{\Omega}$ has a continuous logarithm.

Solution. Let $f : \overline{\Omega} \to \mathbb{C}$ be a nonvanishing continuous function, and let

$$R = \{x + iy : x, y \in [0, 1]\}.$$

Since $R^\circ \notin \{\varnothing, \mathbb{C}\}$ is a simply connected region in $\mathbb{C}$, Theorem 11.30 implies that there exists a conformal equivalence $\varphi_0 : R^\circ \to \mathbb{B}$, and then by Theorem 11.45 the function φ_0 extends to a homeomorphism $\varphi : R \to \overline{\mathbb{B}}$. Also by Theorem 11.30 there is a conformal equivalence $\psi_0 : \Omega \to \mathbb{B}$ which, again by Theorem 11.45, extends to a homeomorphism $\psi : \overline{\Omega} \to \overline{\mathbb{B}}$. Now,

$$f \circ \psi^{-1} \circ \varphi : R \to \overline{\mathbb{B}} \to \overline{\Omega} \to \mathbb{C}_*$$

is continuous on the rectangle R, and so has a continuous logarithm h on R by Exercise 6.16. Thus

$$(f \circ \psi^{-1} \circ \varphi)(z) = e^{h(z)}$$

for all $z \in R$. Define $\xi : R \to \overline{\Omega}$ by $\xi = \psi^{-1} \circ \varphi$, which is a homeomorphism such that $f(\xi(z)) = e^{h(z)}$ for all $z \in R$. Let $\lambda = h \circ \xi^{-1}$, which is a continuous function. Then

$$e^{\lambda(z)} = e^{h(\xi^{-1}(z))} = f(\xi(\xi^{-1}(z))) = f(z)$$

for all $z \in \overline{\Omega}$, and therefore λ is a continuous logarithm for f on $\overline{\Omega}$. ■

If the function f in the solution to Exercise 11.47 is given to be analytic on Ω, then by Exercise 6.17 it follows that the continuous logarithm for f will likewise be analytic on Ω.

12

Entire and Meromorphic Functions

12.1 – Infinite Products

Given a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, the associated **infinite product** is the ordered formal product

$$\prod_{n=1}^{\infty} z_n = z_1 z_2 z_3 \cdots , \tag{12.1}$$

and the associated sequence of **partial products** is $(p_n)_{n=1}^{\infty}$, where

$$p_n = \prod_{k=1}^{n} z_k$$

is called the ***n*th partial product**. We say the infinite product (12.1) is **convergent** if $(p_n)_{n=1}^{\infty}$ converges so some $p \in \mathbb{C}$, in which case we write

$$p = \prod_{n=1}^{\infty} z_n;$$

that is,

$$\prod_{n=1}^{\infty} z_n = \lim_{n\to\infty} \prod_{k=1}^{n} z_k,$$

and thus an ordered formal product is identified with the complex number obtained by "multiplying all its factors." If an infinite product is not convergent, then it is **divergent**. The symbol $\prod$ will often be used to represent $\prod_{n=1}^{\infty}$, just as $\sum$ represents $\sum_{n=1}^{\infty}$

Proposition 12.1. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, and let $(p_n)_{n=1}^{\infty}$ be the associated sequence of partial products. If $\lim p_n = p \in \mathbb{C}_*$, then $\lim z_n = 1$.*

Proof. Suppose $\lim p_n = p \in \mathbb{C}_*$. Then $z_n \neq 0$, and hence $p_n \neq 0$, for all n (otherwise there exists some $k \in \mathbb{N}$ such that $p_k = 0$ for all $n \geq k$, in which case $\lim p_n = 0$). As a result we may write $z_n = p_n/p_{n-1}$ for all $n \geq 2$, and therefore

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{p_n}{p_{n-1}} = \frac{p}{p} = 1$$

as desired. ■

Proposition 12.2. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_*$. Then $\prod_{n=1}^{\infty} z_n$ converges in $\mathbb{C}_*$ if and only if $\sum_{n=1}^{\infty} \operatorname{Log}(z_n)$ converges in $\mathbb{C}$.*

Proof. Suppose $\prod z_n$ converges in $\mathbb{C}_*$, which is to say $\prod z_n = p$ for some $p \in \mathbb{C}_*$, and thus $\lim p_n = p$ with $p_n \neq 0$ for all n. Choose $\theta \in \mathbb{R}$ such that $p \in \mathbb{C} \setminus R_\theta$. By Proposition 6.4, $\log_\theta$ is continuous at p and therefore

$$\lim_{n\to\infty} \log_\theta(p_n) = \log_\theta(p)$$

by Theorem 2.20. For each $n \in \mathbb{N}$ let

$$s_n = \sum_{k=1}^{n} \operatorname{Log}(z_n).$$

Since $\exp(s_n) = p_n$ and $\exp(\log_\theta p_n) = p_n$, we have $s_n = \log_\theta p_n + 2\pi i m_n$ for some $m_n \in \mathbb{Z}$ by Theorem 4.42(7). Now, Log is continuous on $\mathbb{C} \setminus R_{-\pi}$, which contains 1, and since $(z_n)_{n=1}^{\infty}$ converges to 1, by Proposition 12.1 it follows that

$$\lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} \operatorname{Log}(z_n) = \operatorname{Log}(1) = 0,$$

and thus

$$\begin{aligned}\lim_{n\to\infty} \big[(\log_\theta p_n + 2\pi i m_n) - (\log_\theta p_{n-1} + 2\pi i m_{n-1})\big]&\\ = \lim_{n\to\infty} \big[(\log_\theta p_n - \log_\theta p_{n-1}) + 2\pi i (m_n - m_{n-1})\big] &= 0.\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} (\log_\theta p_n - \log_\theta p_{n-1}) = \log_\theta p - \log_\theta p = 0,$$

and so

$$\lim_{n\to\infty} 2\pi i (m_n - m_{n-1}) = 0.$$

Since $m_n \in \mathbb{Z}$ for all n, there must be some k such that $m_n - m_{n-1} = 0$ for all $n \geq k$, and hence there is some $m \in \mathbb{Z}$ such that $m_n = m$ for $n \geq k$. Therefore,

$$\sum_{k=1}^{\infty} \operatorname{Log}(z_n) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (\log_\theta p_n + 2\pi i m) = \log_\theta p + 2\pi i m,$$

and so the infinite series converges in $\mathbb{C}$.

Conversely, suppose that $\sum_{n=1}^{\infty} \operatorname{Log} z_n$ converges in $\mathbb{C}$. Thus, letting s_n denote the nth partial sum, $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{C}$. Now, since the exponential function is continuous at s, by Theorem 4.42(3) and Theorem 2.20,

$$\lim_{n\to\infty} \prod_{k=1}^{n} z_k = \lim_{n\to\infty} \prod_{k=1}^{n} \exp\big(\operatorname{Log}(z_k)\big) = \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \operatorname{Log}(z_k)\right) = \lim_{n\to\infty} \exp(s_n) = \exp(s).$$

Thus

$$\prod_{n=1}^{\infty} z_n = e^s,$$

which shows that the infinite product converges in $\mathbb{C}_*$. ■

Let $f(x) = e^x - x - 1$ for all $x \in \mathbb{R}$, which is increasing on $[0, \infty)$ since $f'(x) = e^x - 1 \geq 0$ for all $x \geq 0$. Noting that $f(0) = 0$, it follows that $f \geq 0$ on $[0, \infty)$. That is, $x + 1 \leq e^x$ for all $x \geq 0$. We use this fact in the proof of the next proposition.

Proposition 12.3. *Let $a_n \in [0, \infty)$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Since $a_n \geq 0$ for each n, we have $a_n + 1 \leq \exp(a_n)$ for all n, and hence

$$\sum_{k=1}^{n} a_k \leq \prod_{k=1}^{n}(1 + a_k) \leq \prod_{k=1}^{n} \exp(a_k) = \exp\left(\sum_{k=1}^{n} a_k\right) \tag{12.2}$$

for all n. Suppose $\sum a_n = s$ for some $s \in \mathbb{R}$. Let

$$p_n = \prod_{k=1}^{n}(1 + a_k).$$

Since $1 + a_n \geq 1$ for all n, the sequence (p_n) is monotone increasing, and also it is bounded since

$$p_n \leq \exp\left(\sum_{k=1}^{n} a_k\right) \leq \exp\left(\sum_{k=1}^{\infty} a_k\right) = e^s$$

for all n by (12.2). Thus (p_n) converges in $\mathbb{R}$ by the Monotone Convergence Theorem, which immediately implies that $\prod(1 + a_n)$ converges.

Conversely, suppose that $\prod(1 + a_n) = p$ for some $p \in \mathbb{R}$. Let

$$s_n = \sum_{k=1}^{n} a_k.$$

Since $a_n \geq 0$ for all n, the sequence (s_n) is monotone increasing, and also it is bounded since

$$s_n \leq \prod_{k=1}^{n}(1 + a_k) \leq p$$

for all n by (12.2). Therefore (s_n) converges in $\mathbb{R}$, which is to say $\sum a_n$ converges. ■

Definition 12.4. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. The infinite product $\prod_{n=1}^{\infty}(1 + z_n)$ is said to be* ***absolutely convergent*** *if $\prod_{n=1}^{\infty}(1 + |z_n|)$ is convergent.*

As the next proposition makes clear, an absolutely convergent infinite product is also convergent, and moreover the value of the infinite product is invariant under arbitrary rearrangements. Recall that a rearrangement of an infinite sum or product is defined to be a permutation of its terms.

Proposition 12.5. *Suppose $\prod_{n=1}^{\infty}(1 + z_n)$ is absolutely convergent. Then the following hold.*

1. *$\prod_{n=1}^{\infty}(1 + z_n)$ is convergent.*
2. *For any permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the rearrangement $\prod_{n=1}^{\infty}(1 + z_{\sigma(n)})$ is absolutely convergent with*

$$\prod_{n=1}^{\infty}(1 + z_{\sigma(n)}) = \prod_{n=1}^{\infty}(1 + z_n). \tag{12.3}$$

Proof.
Proof of Part (1). Suppose that $\prod(1+z_n)$ is absolutely convergent. By Proposition 12.3 the series $\sum|z_n|$ is convergent, so that $\lim|z_n|=0$ and hence there exists some n_0 such that $z_n\in\mathbb{B}$ for all $n\geq n_0$. From Example 6.6 we have

$$\operatorname{Log}(1+z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n$$

for $z\in\mathbb{B}$, and thus $\operatorname{Log}(1+z)=zh(z)$ with

$$h(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^{n-1} \tag{12.4}$$

for $z\in\mathbb{B}$. The set $\{|h(z_n)|:n\in\mathbb{N}\}$ is bounded since

$$\lim_{z\to 0}h(z)=\lim_{z\to 0}\left(1-\frac{z}{2}+\frac{z^2}{3}-\frac{z^3}{4}+\cdots\right)=1$$

and $z_n\to 0$ as $n\to\infty$. Let $M\in(0,\infty)$ be such that $|h(z_n)|\leq M$ for all n.

Fix $\epsilon>0$. Since $\sum|z_n|$ converges, by the Cauchy Criterion there exists some integer n_1 such that

$$\sum_{k=n}^{m}|z_k|<\frac{\epsilon}{M}$$

for all $m\geq n\geq n_1$. Choose $N=\max\{n_0,n_1\}$, and suppose $m\geq n\geq N$. Then

$$\left|\sum_{k=n}^{m}\operatorname{Log}(1+z_k)\right|=\left|\sum_{k=n}^{m}z_kh(z_k)\right|\leq\sum_{k=n}^{m}|z_k||h(z_k)|\leq M\sum_{k=n}^{m}|z_k|<M\cdot\frac{\epsilon}{M}=\epsilon,$$

and therefore the series

$$\sum_{n=n_0}^{\infty}\operatorname{Log}(1+z_n)=\sum_{n=1}^{\infty}\operatorname{Log}(1+z_{n+n_0-1})$$

converges in $\mathbb{C}$ by the Cauchy Criterion. Now, $1+z_{n+n_0-1}\in B_1(1)\subseteq\mathbb{C}_*$ for all $n\in\mathbb{N}$, and so by Proposition 12.2 the infinite product

$$\prod_{n=1}^{\infty}(1+z_{n+n_0-1})=\prod_{n=n_0}^{\infty}(1+z_n)$$

converges in $\mathbb{C}_*$. Since

$$\prod_{n=1}^{\infty}(1+z_n)=\prod_{n=1}^{n_0-1}(1+z_n)\cdot\prod_{n=n_0}^{\infty}(1+z_n),$$

it follows that $\prod(1+z_n)$ is convergent.

Proof of Part (2). Let $\sigma:\mathbb{N}\to\mathbb{N}$ be a permutation. By hypothesis $\prod(1+|z_n|)$ converges, and so $\sum|z_n|$ converges by Proposition 12.3. It is immediate that $\sum|z_n|$ is absolutely convergent, and so has value that is invariant under rearrangements, and thus $\sum|z_{\sigma(n)}|$ converges since $\sum|z_{\sigma(n)}|=\sum|z_n|$. Now, Proposition 12.3 implies that $\prod(1+|z_{\sigma(n)}|)$ converges, and therefore $\prod(1+z_{\sigma(n)})$ is absolutely convergent. It remains to verify (12.3), omitted at present. ■

We consider now infinite products of functions. Let S be a subset of $\mathbb{C}$, and let $(f_n)_{n=1}^\infty$ be a sequence of complex-valued functions defined on S. Define

$$p_n = \prod_{k=1}^{n} f_n$$

for each n. The infinite product

$$\prod_{n=1}^{\infty} f_n,$$

also written $\prod f_n$, is said to **converge pointwise** to $p : S \to \mathbb{C}$ on S if $\lim p_n(z) = p(z)$ for all $z \in S$, which is to say

$$\prod_{n=1}^{\infty} f_n(z) = p(z)$$

for each $z \in S$. We say $\prod f_n$ **converges uniformly** to p on S if the sequence $(p_n)_{n=1}^\infty$ converges uniformly to p on S.

One last definition is in order before the statement of the next proposition, so as to dispel any ambiguity.

Definition 12.6. *Let $(f_n)_{n=1}^\infty$ be a sequence of complex-valued functions defined on $S \subseteq \mathbb{C}$. The infinite product $\prod_{n=1}^\infty (1 + f_n)$ is said to be **absolutely convergent** on S if $\prod_{n=1}^\infty (1 + |f_n|)$ is pointwise convergent on S.*[15]

Proposition 12.7. *Let $(g_n)_{n=1}^\infty$ be a sequence of bounded complex-valued functions defined on $S \subseteq \mathbb{C}$. Suppose $\sum_{n=1}^\infty |g_n|$ converges uniformly on S. Then the following hold.*

1. *The infinite product $\prod_{n=1}^\infty (1+g_n)$ is absolutely convergent on S, and also converges uniformly on S to the function $f : S \to \mathbb{C}$ given by*

$$f(z) = \prod_{n=1}^{\infty} (1 + g_n(z)).$$

2. *If $z \in S$, then $f(z) = 0$ if and only if $1 + g_n(z) = 0$ for some $n \in \mathbb{N}$.*

Proof.
Proof of Part (1). Fix $z \in S$. The series $\sum |g_n(z)|$ is convergent since $\sum |g_n|$ is given to converge uniformly on S, and so $\prod(1 + |g_n(z)|)$ is convergent by Proposition 12.3. It follows that $\prod(1 + |g_n|)$ is pointwise convergent on S, and therefore $\prod(1 + g_n)$ is absolutely convergent on S.

Next, let σ_n denote the nth partial sum of $(|g_n|)_{n=1}^\infty$:

$$\sigma_n = \sum_{k=1}^{n} |g_k|.$$

By hypothesis $(|g_n|)_{n=1}^\infty$ converges uniformly to some $\sigma : S \to \mathbb{C}$, and so there exists some N such that $|\sigma_n(z) - \sigma(z)| < 1/4$ for all $n \geq N - 1$ and $z \in S$, and hence

$$|\sigma(z)| - \frac{1}{4} < \sigma_n(z) < |\sigma(z)| + \frac{1}{4}.$$

[15] Another term that may be used is "pointwise absolutely convergent," but it is unneeded here.

Fix $n \geq N$ and $z \in S$. Since $\sigma_n(z) = \sigma_{n-1}(z) + |g_n(z)|$, we have

$$|\sigma(z)| - \frac{1}{4} < \sigma_{n-1}(z) + |g_n(z)| < |\sigma(z)| + \frac{1}{4}; \tag{12.5}$$

and since $n - 1 \geq N - 1$, we have

$$|\sigma(z)| - \frac{1}{4} < \sigma_{n-1}(z) < |\sigma(z)| + \frac{1}{4}. \tag{12.6}$$

Multiplying (12.6) by -1 and adding it to (12.5) then gives $|g_n(z)| < 1/2$ for all $n \geq N$ and $z \in S$.

Fix $\epsilon > 0$. Let function h be as given by (12.4). For $n \geq N$ and $z \in S$ we have, for any $k \geq 0$,

$$\left| \frac{(-1)^k}{k+1} [g_n(z)]^k \right| \leq |g_n(z)|^k \leq \frac{1}{2^k},$$

and since $\sum_{k=0}^{\infty} 2^{-k}$ converges in $\mathbb{R}$, it follows by the Direct Comparison Test that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} [g_n(z)]^k$$

converges absolutely in $\mathbb{C}$, and so also converges. Now, for all $n \geq N$ and $z \in S$,

$$|h(g_n(z))| = \left| \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} [g_n(z)]^k \right| \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Next, since $\sum |g_n(z)|$ converges uniformly on S, by the Cauchy Criterion for uniform convergence[16] there exists some N' such that, for all $m > n \geq N'$ and $z \in S$,

$$\sum_{k=n+1}^{m} |g_k(z)| < \frac{\epsilon}{2}.$$

Choose $M = \max\{N, N'\}$, let $m, n \geq M$ with $m > n$, and let $z \in S$. For all $k \geq n+1$ we have $g_k(z) \in B_{1/2}(0) \subseteq \mathbb{B}$, so that

$$\operatorname{Log}(1 + g_k(z)) = g_k(z) h(g_k(z))$$

is defined in $\mathbb{C}$, and hence

$$\begin{aligned} \left| \sum_{k=n+1}^{m} \operatorname{Log}(1 + g_k(z)) \right| &= \left| \sum_{k=n+1}^{m} g_k(z) h(g_k(z)) \right| \leq \sum_{k=n+1}^{m} |g_k(z)||h(g_k(z))| \\ &\leq 2 \sum_{k=n+1}^{m} |g_k(z)| < 2 \cdot \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

It follows by the Cauchy Criterion for uniform convergence that the series

$$\sum_{n=N}^{\infty} \operatorname{Log}(1 + g_n)$$

is uniformly convergent on S. The remainder of the proof is outlined very scantily in [AN], and is omitted here.

[16]See Rudin's *Principles of Mathematical Analysis*, page 147.

Proof of Part (2). Fix $z \in S$, and suppose that $f(z) = 0$; that is,

$$\prod_{n=1}^{\infty}(1 + g_n(z)) = 0.$$

Now, from Part (1),

$$\sum_{n=N}^{\infty} \operatorname{Log}(1 + g_n(z)) = \sum_{n=1}^{\infty} \operatorname{Log}(1 + g_{n+N-1}(z))$$

converges in $\mathbb{C}$, and since $1 + g_{n+N-1}(z) \neq 0$ for all $n \in \mathbb{N}$, by Proposition 12.2

$$\prod_{n=1}^{\infty}(1 + g_{n+N-1}(z))$$

converges in $\mathbb{C}_*$, which is to say $\prod_{n=N}^{\infty}(1 + g_n(z))$ is a nonzero complex number. Since

$$\prod_{n=1}^{\infty}(1 + g_n(z)) = \prod_{n=1}^{N-1}(1 + g_n(z)) \cdot \prod_{n=N}^{\infty}(1 + g_n(z)),$$

with $\prod_{n=1}^{\infty} = 0$ and $\prod_{n=N}^{\infty} \neq 0$, it follows that $\prod_{n=1}^{N-1} = 0$. That is, $1 + g_n(z) = 0$ for some $1 \leq n \leq N-1$, and therefore $1 + g_n(z) = 0$ for some $n \in \mathbb{N}$.

The proof of the converse is trivial. ∎

Theorem 12.8. *Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{A}(\Omega)$. If $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on compact subsets of Ω, then*

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{12.7}$$

defines a function in $\mathcal{A}(\Omega)$. Moreover, if $z \in \Omega$, then $f(z) = 0$ if and only if $f_n(z) = 0$ for some $n \in \mathbb{N}$.

Proof. Suppose $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on compact subsets of Ω. Let $K \subseteq \Omega$ be compact. For each n, the analyticity of $f_n - 1$ on Ω implies the boundedness of $f_n - 1$ on K. Thus, since $\sum |f_n - 1|$ converges uniformly on K, by Proposition 12.7(1) it follows that $\prod f_n$ converges uniformly on K. That is, if p_n is the nth partial product of $\prod f_n$, then $(p_n)_{n=1}^{\infty}$ converges uniformly on compact subsets of Ω to the function $\Omega \to \mathbb{C}$ given by $z \mapsto \lim p_n(z)$, which is precisely the function f given by (12.7). By Theorem 4.30 we conclude that $f : \Omega \to \mathbb{C}$ is analytic on Ω.

Next, fix $z \in \Omega$. Let $K \subseteq \Omega$ be any compact set containing z. By Proposition 12.7(1) and the findings of the preceding paragraph, $\prod f_n|_K$ converges uniformly on K to $f|_K$, and then by Proposition 12.7(2), $f|_K(z) = 0$ iff $f_n|_K(z) = 0$ for some $n \in \mathbb{N}$. Therefore $f(z) = 0$ iff $f_n(z) = 0$ for some $n \in \mathbb{N}$. ∎

Exercise 12.9 (AN6.1.1). Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{A}(\Omega)$ such that $\sum_{n=1}^\infty |f_n - 1|$ converges uniformly on compact subsets of Ω, and no f_n is identically zero on any component of Ω. For f as defined by (12.7), prove that

$$\operatorname{ord}(f,z) = \sum_{n=1}^\infty \operatorname{ord}(f_n,z)$$

for each $z \in \Omega$, where as usual we take $\operatorname{ord}(f,z) = 0$ if $f(z) \neq 0$.

Solution. Fix $z \in \Omega$. If $f(z) \neq 0$, then $f_n(z) \neq 0$ for all $n \in \mathbb{N}$ by Theorem 12.8, and so $\operatorname{ord}(f_n,z) = 0$ for all n. Hence

$$\sum_{n=1}^\infty \operatorname{ord}(f_n,z) = 0 = \operatorname{ord}(f,z).$$

Suppose $f(z) = 0$. We have $|f_n(z) - 1| \to 0$ as $n \to \infty$ since $\sum |f_n(z) - 1|$ is convergent, so that $f_n(z) \to 1$ as $n \to \infty$, and hence there exists some N such that $f_n(z) \neq 0$ for all $n \geq N+1$. Since $(f_{N+n})_{n=1}^\infty$ is a sequence in $\mathcal{A}(\Omega)$ and $\sum_{n=1}^\infty |f_{N+n} - 1|$ converges uniformly on compact subsets of Ω, by Theorem 12.8

$$g(w) = \prod_{n=1}^\infty f_{N+n}(w) = \prod_{n=N+1}^\infty f_n(w)$$

defines a function in $\mathcal{A}(\Omega)$. Also by the same theorem $g(z) \neq 0$ since $f_{N+n}(z) \neq 0$ for all $n \in \mathbb{N}$. Now,

$$f(w) = \prod_{n=1}^\infty f_n(w) = \prod_{n=1}^N f_n(w) \cdot \prod_{n=N+1}^\infty f_n(w) = g(w) \prod_{n=1}^N f_n(w). \tag{12.8}$$

For each $1 \leq n \leq N$ let $k_n = \operatorname{ord}(f_n,z)$, which by Proposition 5.15 is an integer since f_n is not identically zero on the component of Ω containing z, and so

$$f_n(w) = (w-z)^{k_n} \varphi_n(w)$$

for some $\varphi_n \in \mathcal{A}(\Omega)$ with $\varphi_n(z) \neq 0$. Setting $h(w) = g(w) \prod_{n=1}^N \varphi_n(w)$, from (12.8) we obtain

$$f(w) = (w-z)^{k_1 + \cdots + k_N} h(w),$$

where $h \in \mathcal{A}(\Omega)$ with $h(z) \neq 0$. Therefore

$$\operatorname{ord}(f,z) = \sum_{n=1}^N k_n = \sum_{n=1}^N \operatorname{ord}(f_n,z) = \sum_{n=1}^\infty \operatorname{ord}(f_n,z),$$

the last equality due to $f_n(z) \neq 0$ for $n \geq N+1$ implying $\operatorname{ord}(f_n,z) = 0$ for $n \geq N+1$. ■

Exercise 12.10 (AN6.1.2). Show that $-\ln(1-x) = x + x^2 g(x)$ for all $x \in (-1,1)$, where $g(x) \to 1/2$ as $x \to 0$. Use this to show the following, where $(a_n)_{n=1}^\infty$ is a sequence in $\mathbb{R} \setminus \{1\}$.

(a) Suppose $\sum a_n$ converges. Then $\prod (1-a_n)$ converges to a nonzero limit iff $\sum a_n^2$ converges.
(b) Suppose $\sum a_n^2$ converges. Then $\prod (1-a_n)$ converges to a nonzero limit iff $\sum a_n$ converges.

Solution. Fix $x \in (-1, 1)$, so that $1 - x \in (0, 2) \subseteq B_1(1)$. By Proposition 6.2,

$$\operatorname{Log}(1-x) = \ln|1-x| + i \arg_{-\pi}(1-x) = \ln(1-x),$$

and so by Example 6.6,

$$\ln(1-x) = \operatorname{Log}(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-x)^n = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Thus

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + x^2 \sum_{n=0}^{\infty} \frac{x^n}{n+2} = x + x^2 g(x),$$

where $g : \mathbb{B} \to \mathbb{C}$ given by

$$g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+2} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^n}{n+2}$$

is analytic on $\mathbb{B}$ by Proposition 4.31(1), so that

$$\lim_{z \to 0} g(z) = g(0) = \frac{1}{2}$$

as desired.

Proof of (a). Suppose that $\prod(1 - a_n)$ converges to a nonzero limit. Since $(1 - a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{C}_*$, by Proposition 12.2 the series $\sum \operatorname{Log}(1 - a_n)$ converges in $\mathbb{C}$. The convergence of $\sum a_n$ implies that $\lim a_n = 0$, and so there exists some $N \in \mathbb{N}$ such that $a_n \in (-1, 1)$ for all $n \geq N$. Then

$$\sum_{n=N}^{\infty} \operatorname{Log}(1-a_n) = \sum_{n=N}^{\infty} \ln(1-a_n) = \sum_{n=N}^{\infty} \big[-a_n - g(a_n)a_n^2 \big], \tag{12.9}$$

and since $\sum_{n=N}^{\infty} a_n$ as well as the series at right in (12.9) converge, it follows that $\sum_{n=N}^{\infty} g(a_n)a_n^2$ also converges. Now, $\lim g(a_n) = 1/2$, and so we may assume N to be sufficiently large that $g(a_n) > 1/4$ for all $n \geq N$, so that

$$0 \leq \frac{1}{4}a_n^2 \leq g(a_n)a_n^2$$

holds for all $n \geq N$, and the Direct Comparison Test implies that $\sum_{n=N}^{\infty} a_n^2/4$ is convergent. Therefore $\sum a_n^2$ converges.

For the converse, suppose that $\sum a_n^2$ converges. Taking N to be as before, an application of the Direct Comparison Test will easily show that $\sum_{n=N}^{\infty} g(a_n)a_n^2$ converges. Then, since $\sum_{n=N}^{\infty} a_n$ also converges, it follows that the series in (12.9) converge. Thus $\prod_{n=N}^{\infty}(1 - a_n)$ converges in $\mathbb{C}_*$ by Proposition 12.2, and therefore $\prod(1 - a_n)$ converges to a nonzero limit.[17]

Proof of (b). Suppose that $\prod(1 - a_n)$ converges to a nonzero limit. As before, there exists some N such that the series in (12.9) converge. Now, the hypothesized convergence of $\sum a_n^2$ may be

[17] Note that we require the hypothesis that $a_n \neq 1$ for all $n \in \mathbb{N}$ (missing in [AN]) to draw the final desired conclusion, for otherwise nothing prevents one among $a_1, \ldots, a_{N-1}$ equalling 1 so that $\prod(1 - a_n) = 0$!

used to show that $\sum_{n=N} g(a_n)a_n^2$ also converges, for we may assume N to be sufficiently large that

$$0 \leq g(a_n)a_n^2 \leq a_n^2$$

for all $n \geq N$. It follows that

$$\sum_{n=N}^{\infty} \left[g(a_n)a_n^2 + (-a_n - g(a_n)a_n^2)\right] = \sum_{n=N}^{\infty}(-a_n)$$

converges, and therefore so does $\sum a_n$.

The converse, in which we suppose $\sum a_n$ and $\sum a_n^2$ converge and show that $\prod(1 - a_n)$ converges to a nonzero limit, has already been done in the proof of (a). ■

Exercise 12.11 (AN6.1.3a). Determine whether or not the infinite product

$$\prod_{n=1}^{\infty}(1 - 2^{-n}) \tag{12.10}$$

is convergent.

Solution. Since the series $\sum_{n=1}^{\infty} 2^{-n}$ is convergent, by Proposition 12.3 the infinite product

$$\prod_{n=1}^{\infty}(1 + 2^{-n}) = \prod_{n=1}^{\infty}\left(1 + |-2^{-n}|\right)$$

is convergent, and hence

$$\prod_{n=1}^{\infty}(1 - 2^{-n}) = \prod_{n=1}^{\infty}\left(1 + (-2^{-n})\right)$$

is absolutely convergent. Therefore (12.10) is convergent by Proposition 12.5. ■

Exercise 12.12 (AN6.1.3b). Determine whether or not the infinite product

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{n+1}\right)$$

is convergent.

Solution. For each $n \in \mathbb{N}$,

$$\prod_{k=1}^{n}\left(1 - \frac{1}{k+1}\right) = \prod_{k=1}^{n}\frac{k}{k+1} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n}\cdot\frac{n}{n+1} = \frac{1}{n+1},$$

and so

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{n+1}\right) = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1 - \frac{1}{k+1}\right) = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Exercise 12.13 (AN6.1.3c). Determine whether or not the infinite product

$$\prod_{n=1}^{\infty}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

is convergent.

Solution. For $n = 1$ we have

$$1 + \frac{(-1)^n}{\sqrt{n}} = 1 + \frac{(-1)^1}{\sqrt{1}} = 1 + (-1) = 0,$$

and therefore the infinite product converges to 0. ■

Exercise 12.14 (AN6.1.3d). Determine whether or not the infinite product

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{n^2}\right) \tag{12.11}$$

is convergent.

Solution. Since the series $\sum_{n=1}^{\infty} n^{-2}$ is convergent, by Proposition 12.3 the infinite product

$$\prod_{n=1}^{\infty}(1 + n^{-2}) = \prod_{n=1}^{\infty}\big(1 + |-n^{-2}|\big)$$

is convergent, and hence

$$\prod_{n=1}^{\infty}(1 - n^{-2}) = \prod_{n=1}^{\infty}\big(1 + (-n^{-2})\big)$$

is absolutely convergent. Therefore (12.11) is convergent by Proposition 12.5. ■

Exercise 12.15 (AN6.1.5a). Show that the function

$$f(z) = \prod_{n=1}^{\infty}(1 + a^n z)$$

is entire for any $a \in \mathbb{B}$.

Solution. Fix $a \in \mathbb{B}$, and let $f_n(z) = 1 + a^n z$ for each $n \in \mathbb{N}$, so that $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}(\mathbb{C})$. Let $K \subseteq \mathbb{C}$ be compact, and let $\alpha = \max_{z\in K}|z|$. For each n,

$$\|f_n - 1\|_K = \sup_{z\in K}|f_n(z) - 1| = \sup_{z\in K}|a|^n|z| \le \alpha|a|^n,$$

and since the series $\sum \alpha|a|^n$ is convergent, it follows by the Weierstrass M-Test that $\sum |f_n - 1|$ converges uniformly on K. Since $\sum |f_n - 1|$ converges uniformly on compact subsets of $\mathbb{C}$, Theorem 12.8 implies that $f \in \mathcal{A}(\mathbb{C})$. Therefore f is entire. ■

For the next exercise we must entertain a new definition. Given a two-tailed sequence $(z_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$, the associated infinite product is the ordered formal product

$$\prod_{n\in\mathbb{Z}} z_n = \cdots z_{-3}z_{-2}z_{-1}z_0z_1z_2z_3\cdots \tag{12.12}$$

We say (12.12) is **convergent** if the infinite products

$$\prod_{n=1}^{\infty} z_n \quad \text{and} \quad \prod_{n=0}^{\infty} z_{-n}$$

both converge, in which case we define

$$\prod_{n\in\mathbb{Z}} z_n = \prod_{n=0}^{\infty} z_{-n} \cdot \prod_{n=1}^{\infty} z_n.$$

Exercise 12.16 (AN6.1.5b). Show that the function

$$p(z) = \prod_{n\in\mathbb{Z},n\neq 0} (1 - z/n)e^{z/n}$$

is entire.

Solution. Define

$$f(z) = \prod_{n=1}^{\infty}(1 - z/n)e^{z/n} \quad \text{and} \quad g(z) = \prod_{n=1}^{\infty}(1 + z/n)e^{-z/n}. \tag{12.13}$$

We start by showing that f is entire. For each $n \in \mathbb{N}$ let $f_n(z) = (1 - z/n)e^{z/n}$, so that $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}(\mathbb{C})$. Let $K \subseteq \mathbb{C}$ be compact, and let $\alpha = \sup_{z\in K} |z|$. For n sufficiently large we have

$$|e^{z/n}| = e^{\mathrm{Re}(z)/n} \leq 2, \tag{12.14}$$

and so

$$\begin{aligned}\|f_n - 1\|_K &= \left\|(1 - z/n)e^{z/n} - 1\right\|_K = \left\|(1 - z/n - e^{-z/n})e^{z/n}\right\|_K \\ &\leq \sup_{z\in K}\left|1 - \frac{z}{n} - e^{-z/n}\right| |e^{z/n}| \leq 2\sup_{z\in K}\left|1 - \frac{z}{n} - \sum_{k=0}^{\infty}\frac{(-z/n)^k}{k!}\right| \\ &\leq \sup_{z\in K}\sum_{k=1}^{\infty}\frac{2}{(k+1)!}\left(\frac{|z|}{n}\right)^{k+1} \leq \sum_{k=1}^{\infty}\left(\frac{\alpha}{n}\right)^{k+1}.\end{aligned}$$

Let $N \in \mathbb{N}$ be sufficiently large that, for all $n \geq N$, (12.14) holds and $\alpha/n < 1/2$. Then for all $n \geq N$,

$$\|f_n - 1\|_K \leq \sum_{k=1}^{\infty}\left(\frac{\alpha}{n}\right)^{k+1} = \sum_{k=0}^{\infty}\frac{\alpha^2}{n^2}\left(\frac{\alpha}{n}\right)^k = \frac{\alpha^2}{n^2}\cdot\frac{1}{1-\alpha/n} \leq \frac{2\alpha^2}{n^2},$$

and since

$$\sum_{n=N}^{\infty}\frac{2\alpha^2}{n^2}$$

is a convergent series, by the Weierstrass M-Test it follows that $(|f_n - 1|)_{n=N}^{\infty}$ converges uniformly on K, and hence $(|f_n - 1|)_{n=1}^{\infty}$ also converges uniformly on K. Therefore $f \in \mathcal{A}(\mathbb{C})$ by Theorem 12.8. A similar argument will show that $g \in \mathcal{A}(\mathbb{C})$ as well, and hence both infinite products in (12.13) are convergent for all $z \in \mathbb{C}$. Then, by definition,

$$p(z) = \prod_{n\in\mathbb{Z},n\neq 0}(1 - z/n)e^{z/n} = \prod_{n=1}^{\infty}(1 - z/n)e^{z/n} \cdot \prod_{n=1}^{\infty}(1 + z/n)e^{-z/n} = f(z)g(z)$$

for all $z \in \mathbb{C}$, and therefore $p = fg$ is entire. ∎

Exercise 12.17 (AN6.1.5c). Show that the function

$$f(z) = \prod_{n=2}^{\infty} \left[1 + \frac{z}{n(\ln n)^2}\right]$$

is entire.

Solution. For $n \geq 2$ let

$$f_n(z) = 1 + \frac{z}{n(\ln n)^2},$$

so that $(f_n)_{n=2}^{\infty}$ is a sequence in $\mathcal{A}(\mathbb{C})$. Let $K \subseteq \mathbb{C}$ be compact, and let $\alpha = \max_{z \in K} |z|$. Then

$$\|f_n - 1\|_K = \sup_{z \in K} \frac{z}{n(\ln n)^2} \leq \frac{M}{n(\ln n)^2}$$

for all $n \geq 2$. Now, making the substitution $u = \ln x$, we obtain

$$\int_2^{\infty} \frac{M}{x(\ln x)^2} dx = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{M}{u^2} du = \lim_{t \to \infty} \left[-\frac{M}{u}\right]_{\ln 2}^{\ln t} = \frac{M}{\ln 2},$$

which shows that the integral is convergent, and hence by the Integral Test in §9.4 of the *Calculus Notes* we conclude that the series

$$\sum_{n=2}^{\infty} \frac{M}{n(\ln n)^2}$$

is convergent. By the Weierstrass M-Test it follows that $\sum_{n=2}^{\infty} |f_n - 1|$ converges uniformly on K, and therefore $f \in \mathcal{A}(\mathbb{C})$ by Theorem 12.8. ■

12.2 – The Weierstrass Factorization Theorem

Definition 12.18. *Let $E_0 : \mathbb{C} \to \mathbb{C}$ be given by $E_0(z) = z-1$, and for $m \in \mathbb{N}$ define $E_m : \mathbb{C} \to \mathbb{C}$ by*

$$E_m(z) = (1-z)\exp\left(\sum_{k=1}^{m} \frac{z^k}{k}\right).$$

Lemma 12.19. *For all $z \in \overline{\mathbb{B}}$,*

$$|E_m(z) - 1| \le |z|^{m+1}.$$

Definition 12.20. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_*$ such that $\lim |z_n| = +\infty$. The **canonical product** associated with $(z_n)_{n=1}^{\infty}$ is*

$$E(z) = \prod_{n=1}^{\infty} E_m(z/z_n),$$

where $m = \min\{n \in \mathbb{W} : E \in \mathcal{A}(\mathbb{C})\}$.

Not every sequence $(z_n)_{n=1}^{\infty}$ of nonzero complex numbers such that $\lim |z_n| \to +\infty$ has a canonical product.

Proposition 12.21. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}_*$ such that $\lim |z_n| = +\infty$, and set*

$$f(z) = \prod_{n=1}^{\infty} E_{m_n}(z/z_n).$$

Then there is a sequence $(m_n)_{n=1}^{\infty}$ in $\mathbb{W}$ such that f is entire, in which case the following hold.

1. *$Z(f) = \{z_n : n \in \mathbb{N}\}$.*
2. *For each $z \in Z(f)$, $\operatorname{ord}(f, z) = \operatorname{card}\{n \in \mathbb{N} : z_n = z\}$.*
3. *If $g(z) = z^m f(z)$ for $m \in \mathbb{N}$, then g is entire with $Z(g) = Z(f) \cup \{0\}$ and $\operatorname{ord}(g, 0) = m$.*

Proof. Let $m_n = n - 1$ for all $n \in \mathbb{N}$. Let $K \subseteq \mathbb{C}$ be compact, and choose $r > 0$ such that $K \subseteq B_r(0)$. Finally, let $N \in \mathbb{N}$ be such that $r/|z_n| < 1/2$ for all $n \ge N$. Since $z/z_n \in B_{1/2}(0)$ for all $z \in K$ and $n \ge N$, by Lemma 12.19 we obtain

$$\|E_{m_n}(z/z_n) - 1\|_K = \sup_{z \in K} |E_{m_n}(z/z_n) - 1| \le \sup_{z \in K} \left|\frac{z}{z_n}\right|^{m_n+1} \le \left(\frac{r}{|z_n|}\right)^n < \frac{1}{2^n}$$

for all $n \ge N$. Since $\sum_{n=N}^{\infty} 2^{-n}$ converges, by the Weierstrass M-Test the series

$$\sum_{n=N}^{\infty} |E_{m_n}(z/z_n) - 1|$$

converges uniformly on K, and hence so too does $\sum |E_{m_n}(z/z_n) - 1|$. Since $\sum |E_{m_n}(z/z_n) - 1|$ converges uniformly on compact subset of $\mathbb{C}$, by Theorem 12.8 we conclude that $f \in \mathcal{A}(\mathbb{C})$; that is, f is entire if we choose $(m_n)_{n=1}^{\infty} = (n-1)_{n=1}^{\infty}$.

Also by Theorem 12.8, $f(z) = 0$ if and only if $E_{m_n}(z/z_n) = 0$ for some $n \in \mathbb{N}$. It is clear that

$$E_{m_n}(z/z_n) = 0 \quad \Leftrightarrow \quad z/z_n = 1 \quad \Leftrightarrow \quad z = z_n,$$

and therefore $Z(f) = \{z_n : n \in \mathbb{N}\}$.

That $\text{ord}(f, z)$ equals the cardinality of the set $\{n \in \mathbb{N} : z_n = z\}$ for each $z \in Z(f)$ is especially easy to verify by use of the formula in Exercise 12.9. (Note that the set must be finite for any $z \in \mathbb{C}$ since $|z_n| \to +\infty$.)

The last part of the proposition is clear. ■

The following statement of the Weierstrass Factorization Theorem avoids the "each z_n is repeated as often as its multiplicity" bunkum that is found in the literature. Provision for multiplicities is built explicitly into the formula for f.

Theorem 12.22 (**Weierstrass Factorization Theorem**). *Let $f \not\equiv 0$ be an entire function, and for $n \in \mathbb{N} \cup \{\infty\}$ let $Z(f, \mathbb{C}_*) = \{z_j : 1 \le j \le n \text{ and } j \ne \infty\}$. Then*

$$f(z) = e^{g(z)} z^{\text{ord}(f,0)} \prod_{j=1}^{n} \left(\prod_{k=1}^{\text{ord}(f,z_j)} E_{m_{jk}}(z/z_j) \right)$$

for some entire function g and whole numbers m_{jk}.

Proof. First suppose $Z = Z(f, \mathbb{C}_*)$ is finite, so that $Z = \{z_1, \ldots, z_n\}$ for some $n \in \mathbb{N}$. Let $\ell_j = \text{ord}(f, z_j)$ and $\ell = \text{ord}(f, 0)$. We will assume that $f(0) = 0$ so that $\ell \in \mathbb{N}$. By Proposition 5.7 there exists some $h \in \mathcal{A}(\mathbb{C})$ such that $h(z_j) \ne 0$ for each j, $h(0) \ne 0$, and

$$f(z) = h(z) z^{\ell} \prod_{j=1}^{n} (z - z_j)^{\ell_j}$$

for all $z \in \mathbb{C}$. Since h is nonvanishing on the simply connected set $\mathbb{C}$, Theorem 11.30 implies that h has an analytic logarithm. Thus there exists some $\hat{g} \in \mathcal{A}(\mathbb{C})$ such that $h = e^{\hat{g}}$. Now, for each j,

$$(z - z_j)^{\ell_j} = (-z_j)^{\ell_j}(1 - z/z_j)^{\ell_j} = (-z_j)^{\ell_j} E_0^{\ell_j}(z/z_j),$$

and so

$$f(z) = e^{\hat{g}(z)} z^{\ell} \prod_{j=1}^{n} (-z_j)^{\ell_j} \cdot \prod_{j=1}^{n} E_0^{\ell_j}(z/z_j).$$

If $\alpha = \prod_{j=1}^{n} (-z_j)^{\ell_j}$, then $\alpha = e^{\text{Log}(\alpha)}$ since $\alpha \ne 0$, and so if we define $g(z) = \hat{g}(z) + \text{Log}(\alpha)$ we obtain

$$f(z) = e^{g(z)} z^{\ell} \prod_{j=1}^{n} E_0^{\ell_j}(z/z_j) = e^{g(z)} z^{\ell} \prod_{j=1}^{n} \left(\prod_{k=1}^{\ell_j} E_0(z/z_j) \right)$$

as desired. The argument is little altered if $f(0) \ne 0$, in which case $\ell = 0$.

Next suppose that Z is infinite, so that $Z = \{z_j : j \in \mathbb{N}\}$. Assume the elements of Z to be indexed such that $|z_j| \le |z_{j+1}|$ for all j, and set $\ell = \text{ord}(f, 0)$. Since $f \not\equiv 0$, by the Identity Theorem Z has no limit point in $\mathbb{C}$, and so $Z \not\subseteq \overline{B}_r(0)$ for all $r > 0$ by Theorem 2.39. This implies that $(z_j)_{j=1}^{\infty}$ is a sequence in $\mathbb{C}_*$ such that $\lim |z_j| = +\infty$. Consider the sequence $(w_j)_{j=1}^{\infty}$ that arises from $(z_j)_{j=1}^{\infty}$ by repeating each zero z_j for f according to its order ℓ_j, which we may write as

$$\left((z_j)_{k=1}^{\ell_j} \right)_{j=1}^{\infty}.$$

By Proposition 12.21 there exist whole numbers m_j such that the function

$$\varphi(z) = z^\ell \prod_{j=1}^{\infty} E_{m_j}(z/w_j) = z^\ell \prod_{j=1}^{\infty}\left(\prod_{k=1}^{\ell_j} E_{m_{jk}}(z/z_j)\right)$$

is entire, with $Z(\varphi) = Z(f)$, $\operatorname{ord}(\varphi, z_j) = \ell_j$ for all $j \in \mathbb{N}$, and $\operatorname{ord}(\varphi, 0) = \ell$. (Note that $Z(f) = Z$ iff $\ell = 0$.) Define

$$h(z) = \frac{f(z)}{\varphi(z)}$$

for all $z \in \mathbb{C} \setminus Z(\varphi)$, where $Z(\varphi)$ consists only of isolated points. Fix $\zeta \in Z(\varphi)$. Since $\operatorname{ord}(\varphi, \zeta) = p = \operatorname{ord}(f, \zeta)$, there exist functions $f_0, \varphi_0 \in \mathcal{A}(\mathbb{C})$ that are nonzero at ζ, with

$$f(z) = (z-\zeta)^p f_0(z) \quad \text{and} \quad \varphi(z) = (z-\zeta)^p \varphi_0(z);$$

and then

$$\lim_{z\to\zeta} h(z) = \lim_{z\to\zeta} \frac{(z-\zeta)^p f_0(z)}{(z-\zeta)^p \varphi_0(z)} = \lim_{z\to\zeta} \frac{f_0(z)}{\varphi_0(z)} = \frac{f_0(\zeta)}{\varphi_0(\zeta)} \in \mathbb{C}_*.$$

This shows that h has a removable singularity at ζ, and by defining $h(\zeta) = f_0(\zeta)/\varphi_0(\zeta)$ for each $\zeta \in Z(\varphi)$, we find by Corollary 4.22 that h is entire. Also h is nonvanishing on $\mathbb{C}$, so that it has analytic logarithm g by Theorem 11.30. Now, $h(z) = f(z)/\varphi(z)$ holds for all $z \in \mathbb{C}$, with $h(z) = e^{g(z)}$. We obtain at last

$$f(z) = e^{g(z)}\varphi(z),$$

which is the desired result. ∎

Theorem 12.23. *Let Ω be a proper open subset of $\overline{\mathbb{C}}$, let $Z = \{\zeta_n : n \in \mathbb{N}\}$ be a set of distinct points in Ω having no limit point in Ω, and let $(m_n)_{n=1}^\infty$ be a sequence in $\mathbb{N}$. Then there exists some $f \in \mathcal{A}(\Omega)$ such that $Z(f) = Z$ and $\operatorname{ord}(f, \zeta_n) = m_n$ for each n.*

Theorem 12.24. *If h is meromorphic on $\Omega \subseteq \mathbb{C}$, then there exist $f, g \in \mathcal{A}(\Omega)$ such that $h = f/g$.*

Proof. Suppose h is meromorphic on $\Omega \subseteq \mathbb{C}$. Clearly if $P(h) = \varnothing$, so that $h \in \mathcal{A}(\Omega)$, then we may simply set $f = h$ and $g = 1$. Suppose $P(h) \neq \varnothing$. Each point in $P(h)$ is isolated from the others, so it is clear that $P(h)$ is a set of distinct points in Ω having no limit points in Ω. For each $\zeta_n \in P(h)$ let $m_n = \operatorname{ord}(h, \zeta_n)$, so (m_n) is a sequence in $\mathbb{N}$. By Theorem 12.23 there exists $g \in \mathcal{A}(\Omega)$ such that $Z(g) = P(h)$ with $\operatorname{ord}(g, \zeta_n) = m_n$ for each n. Define $f : \Omega \setminus P(h) \to \mathbb{C}$ by $f = gh$. At each $\zeta_n \in P(h)$ the function h has a pole of order equal to the order of the zero that h has there, and so f has a removable singularity at ζ_n. Thus f may be extended to a function $f \in \mathcal{A}(\Omega)$, and moreover we have $f/g = hg/g = h$ as desired. ∎

Exercise 12.25 (AN6.2.1a). Find the canonical product associated with $(2^n)_{n=1}^\infty$.

Solution. We must find the smallest whole number m such that

$$E(z) = \prod_{n=1}^{\infty} E_m(z/2^n) = \prod_{n=1}^{\infty}(1-z)\exp\left[\sum_{k=1}^{m}\frac{(z/2^n)^k}{k}\right]$$

defines an entire function.

For each $n \in \mathbb{N}$ let $f_{m,n}(z) = E_m(z/2^n)$. Let $K \subseteq \mathbb{C}$ be compact. Choose some $r > 0$ such that $K \subseteq B_r(0)$, and let N be such that $r/2^n \leq 1$ for all $n \geq N$. Then $|z/2^n| < r/2^n \leq 1$ for all $z \in K$ and $n \geq N$, and so by Lemma 12.19 we have

$$\|f_{m,n} - 1\|_K = \sup_{z \in K} |E_m(z/2^n) - 1| \leq \sup_{z \in K} \left|\frac{z}{2^n}\right|^{m+1} \leq \left(\frac{r}{2^n}\right)^{m+1}$$

for all $n \geq N$. Thus if we choose $m = 0$, then for all $n \geq N$ we have

$$\|f_{0,n} - 1\|_K \leq \frac{r}{2^n},$$

and since $\sum_{n=N}^{\infty} r/2^n$ converges, by the Weierstrass M-Test it follows that $\sum_{n=N}^{\infty} |f_{0,n} - 1|$ converges uniformly on K, and hence so too does $\sum |f_{0,n} - 1|$. Therefore, by Theorem 12.8, the function

$$f(z) = \prod_{n=1}^{\infty} f_{0,n}(z) = \prod_{n=1}^{\infty} E_0(z/2^n) = \prod_{n=1}^{\infty}(1 - z/2^n)$$

is entire.

We conclude that the smallest whole number m for which E is entire is $m = 0$, and so $\prod(1 - z/2^n)$ is the canonical product associated with $(2^n)_{n=1}^{\infty}$. ■

Exercise 12.26 (AN6.2.1b). Find the canonical product associated with $(n^b)_{n=1}^{\infty}$, where $b > 0$.

Solution. We must find the smallest whole number m such that

$$E(z) = \prod_{n=1}^{\infty} E_m(z/n^b)$$

defines an entire function.

For each $n \in \mathbb{N}$ let $f_{m,n}(z) = E_m(z/n^b)$. Let $K \subseteq \mathbb{C}$ be compact. Choose some $r > 0$ such that $K \subseteq B_r(0)$, and let N be such that $r/n^b \leq 1$ for all $n \geq N$. Then $|z/n^b| < r/n^b \leq 1$ for all $z \in K$ and $n \geq N$, and so by Lemma 12.19 we have

$$\|f_{m,n} - 1\|_K = \sup_{z \in K} |E_m(z/n^b) - 1| \leq \sup_{z \in K} \left|\frac{z}{n^b}\right|^{m+1} \leq \left(\frac{r}{n^b}\right)^{m+1}$$

for all $n \geq N$. The series $\sum(r/n^b)^{m+1}$ converges if and only if $b(m + 1) > 1$, and so we choose

$$m = \min\{k \in \mathbb{W} : k > 1/b - 1\}. \tag{12.15}$$

Then, since $\sum_{n=N}^{\infty}(r/n^b)^{m+1}$ converges, by the Weierstrass M-Test it follows that $\sum_{n=N}^{\infty} |f_{m,n} - 1|$ converges uniformly on K, and hence so too does $\sum |f_{m,n} - 1|$. Therefore

$$f(z) = \prod_{n=1}^{\infty} f_{m,n}(z) = \prod_{n=1}^{\infty} E_m(z/n^b)$$

is entire by Theorem 12.8.

We conclude that the smallest whole number m for which E is entire is that given by (12.15), in which case $\prod E_m(z/n^b)$ is the canonical product associated with $(n^b)_{n=1}^{\infty}$. ■

Exercise 12.27 (AN6.2.1c). Find the canonical product associated with $(n \ln^2 n)_{n=2}^{\infty}$.

Solution. We must find the smallest whole number m such that

$$E(z) = \prod_{n=2}^{\infty} E_m\left(\frac{z}{n\ln^2 n}\right)$$

defines an entire function.

For each $n \in \mathbb{N}$ let $f_{m,n}(z) = E_m(z/n\ln^2 n)$. Let $K \subseteq \mathbb{C}$ be compact. Choose some $r > 0$ such that $K \subseteq B_r(0)$, and let N be such that $r/n\ln^2 n \leq 1$ for all $n \geq N$. Then $|z/n\ln^2 n| < r/n\ln^2 n \leq 1$ for all $z \in K$ and $n \geq N$, and so by Lemma 12.19 we have

$$\|f_{m,n} - 1\|_K = \sup_{z\in K} |E_m(z/n\ln^2 n) - 1| \leq \sup_{z\in K} \left|\frac{z}{n\ln^2 n}\right|^{m+1} \leq \left(\frac{r}{n\ln^2 n}\right)^{m+1}$$

for all $n \geq N$. In Exercise 12.17 it was shown that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges; thus if we choose $m = 0$, then for all $n \geq N$ we have

$$\|f_{0,n} - 1\|_K \leq \frac{r}{n\ln^2 n},$$

and by the Weierstrass M-Test it follows that $\sum_{n=N}^{\infty} |f_{0,n} - 1|$ converges uniformly on K, and hence so too does $\sum |f_{0,n} - 1|$. Therefore, by Theorem 12.8, the function

$$f(z) = \prod_{n=2}^{\infty} f_{0,n}(z) = \prod_{n=2}^{\infty} E_0\left(\frac{z}{n\ln^2 n}\right) = \prod_{n=2}^{\infty}\left(1 - \frac{z}{n\ln^2 n}\right) \tag{12.16}$$

is entire.

We conclude that the smallest whole number m for which E is entire is $m = 0$, and so the product at right in (12.16) is the canonical product associated with the given sequence. ■

Exercise 12.28 (AN6.2.2). Construct a function $f \in \mathcal{A}(\mathbb{B})$ such that f has no proper analytic extension to a region $\Omega \supset \mathbb{B}$.

Solution. For each $n \in \mathbb{N}$ let

$$A_n = \left\{\left(1 - \frac{1}{n}\right)\exp\left(\frac{2\pi i}{n}(k-1)\right) : 1 \leq k \leq n\right\},$$

and define $A = \bigcup_{n=1}^{\infty} A_n$. Then A is a countably infinite subset of $\mathbb{B}$ with no limits points in $\mathbb{B}$, but such that every point on $\partial\mathbb{B}$ is a limit point. By Theorem 12.23 there exists some $f \in \mathcal{A}(\mathbb{B})$ such that $Z(f) = A$, and $\operatorname{ord}(f, a) = 1$ for every $a \in A$. Now, suppose Ω is a region such that $\Omega \supset \mathbb{B}$, and suppose there is some $\hat{f} \in \mathcal{A}(\Omega)$ such that $\hat{f}|_{\mathbb{B}} = f$. Since Ω is connected it must contain some point $b \in \partial\mathbb{B}$, and then since b is a limit point of $Z(f)$ and $Z(f) \subseteq Z(\hat{f})$, we conclude that $Z(\hat{f})$ has a limit point in Ω and hence $\hat{f} \equiv 0$ by the Identity Theorem. This implies that $f \equiv 0$, which yields the contradiction $Z(f) \neq A$. Therefore f has no proper analytic extension to a region containing $\mathbb{B}$. ■

12.3 – Mittag-Leffler's Theorem

Theorem 12.29 (**Mittag-Leffler's Theorem**). *Let $\Omega \subseteq \mathbb{C}$ be open, and let*

$$P = \{p_j : j \in J\} \subseteq \Omega$$

be a set with no limit point in Ω and an at-most countable index set J. For each $j \in J$ define the rational function

$$S_j(z) = \sum_{k=1}^{n_j} \frac{a_{jk}}{(z-p_j)^k}.$$

Then there is a function f that is meromorphic on Ω with $P(f) = P$ and, for each $j \in J$, has Laurent series representation

$$S_j(z) + \sum_{k=0}^{\infty} c_{jk}(z-p_j)^k$$

on any annulus of analyticity $A_{s_1,s_2}(p_j)$.

Theorem 12.30. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $P = \{p_j : j \in J\} \subseteq \Omega$ be a set with no limit point in Ω and an at-most countable index set J. For each $j \in J$ fix $n_j \in \mathbb{W}$ and let $a_{0j}, \dots, a_{n_j j} \in \mathbb{C}$. Then there exists some $f \in \mathcal{A}(\Omega)$ such that, for each $j \in J$,*

$$\frac{f^{(k)}(b_j)}{k!} = a_{kj}$$

for all $0 \le k \le n_j$.

Given an open set $\Omega \subseteq \mathbb{C}$, the set $\mathcal{A}(\Omega)$ can be shown to be a ring (defined in §1.1 of the *Linear Algebra Notes*) under the usual operations of function addition and function multiplication. Thus, as with any ring, we may entertain some notion of division. If functions $f, g \in \mathcal{A}(\Omega)$ are such that $f = gq$ on Ω for some $q \in \mathcal{A}(\Omega)$, then we say g **divides** f and write $g|f$.

12.4 – The Genus and Order of Entire Functions

12.5 – Hadamard's Factorization Theorem

12.6 – The Picard Theorems

13

The Fourier Transform

13.1 – Functions of Moderate Decrease

Definition 13.1. *The* ***Fourier transform*** *of $f : \mathbb{R} \to \mathbb{C}$ is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx,$$

provided the integral exists for all $\xi \in \mathbb{R}$.

The Fourier transform of a function f may be denoted by $f^\frown$ as well as $\hat{f}$, as demonstrated in the statement of the following proposition.

Proposition 13.2. *Let $a \in \mathbb{R}_*$. If $T : \mathbb{R} \to \mathbb{R}$ is given by $T(x) = ax$, then*

$$(f \circ T)^\frown = \frac{1}{|a|}(\hat{f} \circ T^{-1})$$

provided the Fourier transform of f exists.

Proof. Suppose $\hat{f}$ exists. By Proposition 7.51 and Theorem 3.18,

$$\begin{aligned}
(f \circ T)^\frown(\xi) &= \int_{-\infty}^{\infty} (f \circ T)(x) e^{-2\pi i x \xi} dx = \lim_{r \to \infty} \int_{-r}^{r} f(ax) e^{-2\pi i x \xi} dx \\
&= \lim_{r \to \infty} \int_{-ar}^{ar} \frac{1}{a} f(x) e^{-2\pi i x \xi / a} dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi / a} dx \\
&= \frac{1}{|a|} \hat{f}(\xi / a) = \frac{1}{|a|} (\hat{f} \circ T^{-1})(\xi),
\end{aligned}$$

noting that $\pm ar \to \pm\infty$ if $a > 0$, and $\pm ar \to \mp\infty$ if $a < 0$. ∎

Example 13.3. Show that $e^{-\pi x^2}$ is its own Fourier transform. That is, if $f(x) = e^{-\pi x^2}$ then $\hat{f}(\xi) = e^{-\pi \xi^2}$.

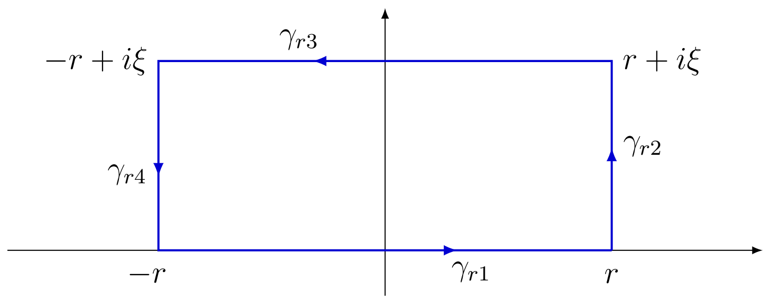

FIGURE 24.

Solution. Fix $\xi \in \mathbb{R}$. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^{-\pi z^2}$. From elementary analysis it is known that

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1.$$

For $r > 0$ define the closed rectangular path $\gamma_r = \gamma_{r1} * \gamma_{r2} * \gamma_{r3} * \gamma_{r4}$, where

$$\gamma_{r1}(t) = t, \quad t \in [-r, r]$$
$$\gamma_{r2}(t) = r + it, \quad t \in [0, \xi]$$
$$\gamma_{r3}(t) = i\xi - t, \quad t \in [-r, r]$$
$$\gamma_{r4}(t) = i\xi - it - r, \quad t \in [0, \xi],$$

shown in Figure 24. By Proposition 7.51,

$$\lim_{r\to\infty} \int_{\gamma_{r1}} f = \lim_{r\to\infty} \int_{-r}^{r} e^{-\pi t^2} dt = \int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1,$$

and

$$\lim_{r\to\infty} \int_{\gamma_{r3}} f = -\lim_{r\to\infty} \int_{-r}^{r} e^{-\pi(i\xi - t)^2} dt = -e^{\pi\xi^2} \lim_{r\to\infty} \int_{-r}^{r} e^{-\pi t^2} e^{2\pi i t\xi} dt = -e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{2\pi i t\xi} dt.$$

Next,

$$\int_{\gamma_{r2}} f = i \int_0^{\xi} e^{-\pi(r+it)^2} dt = ie^{-\pi r^2} \int_0^{\xi} e^{\pi t^2} e^{-2\pi i r t} dt,$$

so

$$\left| \int_{\gamma_{r2}} f \right| = e^{-\pi r^2} \left| \int_0^{\xi} e^{\pi t^2} e^{-2\pi i r t} dt \right| \le e^{-\pi r^2} \int_0^{\xi} \left| e^{\pi t^2} e^{-2\pi i r t} \right| dt = e^{-\pi r^2} \int_0^{\xi} e^{\pi t^2} dt \le e^{-\pi r^2}$$

by Theorem 3.16(5), whereupon the Squeeze Theorem implies that

$$\lim_{r\to\infty} \left| \int_{\gamma_{r2}} f \right| = 0,$$

and hence

$$\lim_{r\to\infty} \int_{\gamma_{r2}} f = 0.$$

Finally,

$$\int_{\gamma_{r4}} f = -i\int_0^\xi e^{-\pi(i\xi - it - r)^2}dt = -ie^{-\pi r^2}e^{2\pi i r\xi}\int_0^\xi e^{\pi(\xi - t)^2}e^{-2\pi i r t}dt,$$

and so

$$\lim_{r\to\infty}\int_{\gamma_{r4}} f = 0$$

obtains by another application of the Squeeze Theorem.

Since f is an entire function and γ_r is a closed path, Theorem 11.30 implies that $\oint_{\gamma_r} f = 0$ for all $r > 0$. Thus

$$0 = \lim_{r\to\infty}\oint_{\gamma_r} f = \lim_{r\to\infty}\left(\int_{\gamma_{r1}} f + \int_{\gamma_{r2}} f + \int_{\gamma_{r3}} f + \int_{\gamma_{r4}} f\right) = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi t^2}e^{2\pi i t\xi}dt,$$

and therefore

$$\int_{-\infty}^{\infty} e^{-\pi t^2}e^{2\pi i t\xi}dt = e^{-\pi\xi^2}.$$

From this we obtain

$$\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i x\xi}dx = e^{-\pi\xi^2}$$

with the change of variable $x = -t$, and since $\xi \in \mathbb{R}$ is arbitrary we are done. ■

Example 13.4. We now find the Fourier transform of $g(x) = e^{-\pi a x^2}$ for any $a > 0$. Let $f(x) = e^{-\pi x^2}$, and let $T(x) = x\sqrt{a}$. Then $g = f \circ T$, and by Proposition 13.2 and Example 13.3 we have

$$\hat{g}(\xi) = (f \circ T)\widehat{\ }(\xi) = \frac{1}{\sqrt{a}}(\hat{f} \circ T^{-1})(\xi) = \frac{1}{\sqrt{a}}\hat{f}\big(\xi/\sqrt{a}\big) = \frac{1}{\sqrt{a}}e^{-\pi\xi^2/a}.$$

That is,

$$\big(e^{-\pi a x^2}\big)\widehat{\ }(\xi) = \frac{1}{\sqrt{a}}e^{-\pi\xi^2/a}$$

for all $a > 0$ and $\xi \in \mathbb{R}$. ■

A function $f : \mathbb{R} \to \mathbb{C}$ is of **moderate decrease** if f is continuous and there exist constants $\epsilon, A \in \mathbb{R}_+$ such that

$$|f(x)| \le \frac{A}{1 + |x|^{1+\epsilon}}. \tag{13.1}$$

for all $x \in \mathbb{R}$. If (13.1) holds for all x in some unbounded interval $I \subseteq \mathbb{R}$, then we say f is of **moderate decrease on** I.

Example 13.5. For fixed $c > 1$ let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{|x|^c}{e^{|x|} - 1}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Since

$$\lim_{x\to\infty}\frac{(x^2+1)x^c}{e^x - 1} = 0,$$

there exists some $x_0 > 0$ such that

$$|f(x)| = \frac{x^c}{e^x - 1} \leq \frac{1}{x^2+1}$$

for all $x \geq x_0$. Now, by L'Hôpital's Rule,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^c}{e^x - 1} = \lim_{x\to 0^+} \frac{cx^{c-1}}{e^x} = 0$$

since $c - 1 > 0$, so f is continuous on $[0, x_0]$, and thus so is $x \mapsto (x^2+1)f(x)$. It follows there is some $M > 0$ for which $(x^2+1)|f(x)| \leq M$ for all $x \in [0, x_0]$, and hence

$$|f(x)| \leq \frac{M}{x^2+1}.$$

Letting $A = M + 1$, we then have $|f(x)| \leq A/(1+x^2)$ for all $x \in [0, \infty)$. But f is an even function, and so $|f(x)| \leq A/(1+x^2)$ holds for all $x \in \mathbb{R}$. This shows that f is of moderate decrease for any choice of constant $c > 1$. ■

Proposition 13.6. *If $f : \mathbb{R} \to \mathbb{C}$ is of moderate decrease, then f has a Fourier transform.*

Proof. Suppose $f : \mathbb{R} \to \mathbb{C}$ is of moderate decrease, so (13.1) holds for some $\epsilon, A > 0$. Fix $\xi \in \mathbb{R}$. We have

$$\left|f(x)e^{-2\pi i x\xi}\right| = |f(x)| \leq \frac{A}{1+|x|^{1+\epsilon}}$$

for all $x \in \mathbb{R}$, and since

$$\int_{-\infty}^{\infty} \frac{A}{1+|x|^{1+\epsilon}} dx$$

converges by the p-Test for Integrals, the Comparison Test for integrals of real-valued functions implies that

$$\int_{-\infty}^{\infty} \left|f(x)e^{-2\pi i x\xi}\right| dx$$

converges. Therefore

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi} dx$$

converges by Theorem 7.53. That is, $\hat{f}(\xi)$ exists in $\mathbb{C}$ for all $\xi \in \mathbb{R}$, and we conclude that f has a Fourier transform. ■

Definition 13.7. *For $a \in \mathbb{R}_+$ let $\mathfrak{F}_a$ be the collection of functions f that are analytic on*

$$S_a = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < a\}$$

and for which there is some $A \in \mathbb{R}_+$ such that

$$|f(z)| \leq \frac{A}{(\operatorname{Re} z)^2 + 1}$$

for all $z \in S_a$. Also define

$$\mathfrak{F} = \bigcup_{a \in \mathbb{R}_+} \mathfrak{F}_a.$$

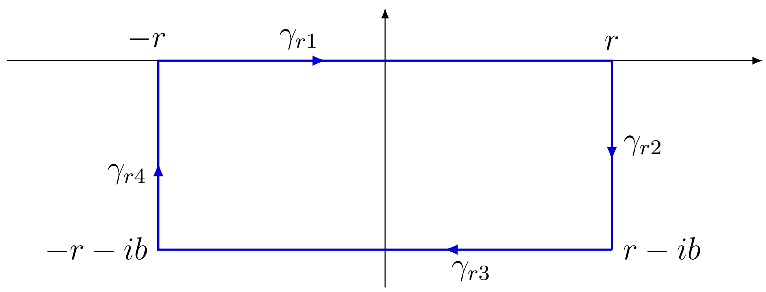

FIGURE 25.

Theorem 13.8. *If $f \in \mathfrak{F}$, then there exists some $B \in \mathbb{R}$ such that*

$$\left|\hat{f}(\xi)\right| \leq Be^{-2\pi b|\xi|} \tag{13.2}$$

for all $\xi \in \mathbb{R}$ and $b \in [0, a)$.

Proof. Suppose $f \in \mathfrak{F}$. Then $f \in \mathfrak{F}_a$ for some $a \in \mathbb{R}_+$, and there exists some $A \in \mathbb{R}_+$ for which

$$|f(x+iy)| \leq \frac{A}{x^2+1} \tag{13.3}$$

for all $x + iy \in S_a$. Set $B = \pi A$. We first note that

$$\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\,dx \leq \int_{-\infty}^{\infty} \frac{A}{x^2+1}dx = \pi A = Be^{-2\pi(0)|\xi|}$$

for any $\xi \in \mathbb{R}$, so the statement of the theorem is seen to be true in the special case when $b = 0$. Also

$$\left|\hat{f}(0)\right| = \left|\int_{-\infty}^{\infty} f(x)\,dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\,dx \leq \int_{-\infty}^{\infty} \frac{A}{x^2+1}dx = \pi A = Be^{-2\pi b|0|}$$

for any $b \in [0, a)$, so the theorem holds when $\xi = 0$. We henceforth may assume $\xi, b \neq 0$.

Fix $b \in (0, a)$ and $\xi \in \mathbb{R}_+$. Let $g(z) = f(z)e^{-2\pi iz\xi}$. For $r > 0$ define the rectangular path $\gamma_r = \gamma_{r1} * \gamma_{r2} * \gamma_{r3} * \gamma_{r4}$, where

$$\begin{aligned}
\gamma_{r1}(t) &= t, \quad t \in [-r, r] \\
\gamma_{r2}(t) &= r - it, \quad t \in [0, b] \\
\gamma_{r3}(t) &= -t - ib, \quad t \in [-r, r] \\
\gamma_{r4}(t) &= -r - i(b-t), \quad t \in [0, b],
\end{aligned}$$

shown in Figure 25. We have

$$\begin{aligned}
\left|\int_{\gamma_{r2}} g\right| &= \left|\int_0^b -ig(r-it)\,dt\right| \leq \int_0^b \left|f(r-it)e^{-2\pi i(r-it)\xi}\right|dt = \int_0^b |f(r-it)|e^{-2\pi\xi t}dt \\
&\leq \int_0^b \frac{A}{r^2+1}e^{-2\pi\xi t}dt \leq \frac{A}{r^2}\int_0^b e^{-2\pi\xi t}dt = \frac{A}{2\pi\xi r^2}\left(1 - e^{-2\pi b\xi}\right),
\end{aligned}$$

and so $\int_{\gamma_{r2}} g \to 0$ as $r \to \infty$. Similarly, $\int_{\gamma_{r4}} g \to 0$ as $r \to \infty$. It is clear that

$$\lim_{r\to\infty} \int_{\gamma_{r1}} g = \hat{f}(\xi),$$

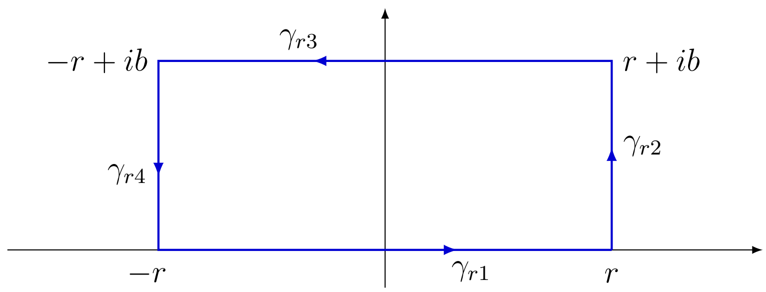

FIGURE 26.

and since $\oint_{\gamma_r} g = 0$ for all $r > 0$ by Theorem 11.30, we have

$$0 = \lim_{r\to\infty} \oint_{\gamma_r} g = \lim_{r\to\infty}\left(\int_{\gamma_{r1}} g + \int_{\gamma_{r2}} g + \int_{\gamma_{r3}} g + \int_{\gamma_{r4}} g\right) = \hat{f}(\xi) + \lim_{r\to\infty}\int_{\gamma_{r3}} g,$$

and hence

$$\begin{aligned}\hat{f}(\xi) &= -\lim_{r\to\infty}\int_{\gamma_{r3}} g = -\lim_{r\to\infty}\int_{-r}^{r} -f(-t-ib)e^{-2\pi i(-t-ib)\xi}\,dt \\ &= e^{-2\pi b\xi}\lim_{r\to\infty}\int_{-r}^{r} f(-t-ib)e^{2\pi it\xi}\,dt = e^{-2\pi b\xi}\int_{-\infty}^{\infty} f(-t-ib)e^{2\pi it\xi}\,dt.\end{aligned}$$

The last equality is justified by Proposition 7.51, for with (13.3) the last integral is easily shown to converge. Finally,

$$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(-t-ib)|e^{-2\pi b\xi}\,dt \le \int_{-\infty}^{\infty} \frac{Ae^{-2\pi b\xi}}{t^2+1}dt = \pi Ae^{-2\pi b\xi} = Be^{-2\pi b|\xi|}.$$

Now fix $\xi \in \mathbb{R}_-$. Define $\gamma_r = \gamma_{r1} * \gamma_{r2} * \gamma_{r3} * \gamma_{r4}$, where

$$\begin{aligned}\gamma_{r1}(t) &= t, \quad t \in [-r,r] \\ \gamma_{r2}(t) &= r + it, \quad t \in [0,b] \\ \gamma_{r3}(t) &= -t + ib, \quad t \in [-r,r] \\ \gamma_{r4}(t) &= -r + i(b-t), \quad t \in [0,b],\end{aligned}$$

shown in Figure 26. As before we find that $\int_{\gamma_{r2}} g, \int_{\gamma_{r4}} g \to 0$ and $\int_{\gamma_{r1}} g \to \hat{f}(\xi)$ as $r \to \infty$. Then

$$0 = \lim_{r\to\infty} \oint_{\gamma_r} g = \lim_{r\to\infty}\left(\int_{\gamma_{r1}} g + \int_{\gamma_{r2}} g + \int_{\gamma_{r3}} g + \int_{\gamma_{r4}} g\right) = \hat{f}(\xi) + \lim_{r\to\infty}\int_{\gamma_{r3}} g,$$

and hence

$$\hat{f}(\xi) = -\lim_{r\to\infty}\int_{\gamma_{r3}} g = e^{2\pi b\xi}\int_{-\infty}^{\infty} f(-t+ib)e^{2\pi it\xi}\,dt.$$

Finally,

$$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(-t+ib)|e^{2\pi b\xi}\,dt \le \int_{-\infty}^{\infty} \frac{Ae^{2\pi b\xi}}{t^2+1}dt = \pi Ae^{2\pi b\xi} = Be^{-2\pi b|\xi|}.$$

Thus $|\hat{f}(\xi)| \le Be^{-2\pi b|\xi|}$ for all $\xi \in \mathbb{R}$ and $b \in [0,a)$. ∎

In the course of proving Theorem 13.8 we also largely obtained the following result, with only the $\xi = 0$ case needing verification in each equation.

Proposition 13.9. *If* $f \in \mathfrak{F}_a$ *and* $b \in (0, a)$*, then*

$$\hat{f}(\xi) = e^{-2\pi b\xi} \int_{-\infty}^{\infty} f(-t - ib) e^{2\pi i t\xi} dt = e^{-2\pi b\xi} \int_{-\infty}^{\infty} f(t - ib) e^{-2\pi i t\xi} dt \tag{13.4}$$

for all $\xi \geq 0$*, and*

$$\hat{f}(\xi) = e^{2\pi b\xi} \int_{-\infty}^{\infty} f(-t + ib) e^{2\pi i t\xi} dt = e^{2\pi b\xi} \int_{-\infty}^{\infty} f(t + ib) e^{-2\pi i t\xi} dt \tag{13.5}$$

for all $\xi \leq 0$.

13.2 – Fourier Inversion and Poisson Summation Formulas

Theorem 13.10 (**Fourier Inversion Formula**). *If $f \in \mathfrak{F}$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi \tag{13.6}$$

for all $x \in \mathbb{R}$.

Proof. Suppose $f \in \mathfrak{F}$, so $f \in \mathfrak{F}_a$ for some $a \in \mathbb{R}_+$. Let $b \in (0, a)$. Fix $x \in \mathbb{R}$. By Proposition 13.6, $\hat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$, with $|\hat{f}(\xi)| \le Be^{-2\pi b|\xi|}$ by Theorem 13.8. This makes it clear that the integral

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

converges, and thus

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi = \int_{-\infty}^{0} \hat{f}(\xi)e^{2\pi i x\xi}d\xi + \int_{0}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

by Proposition 7.51(1). For $\xi > 0$ we find $\hat{f}(\xi)$ be to given by (13.4), so that by Fubini's Theorem and Example 7.52,

$$\begin{aligned}
\int_{0}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi &= \int_{0}^{\infty}\left(e^{-2\pi b\xi}\int_{-\infty}^{\infty} f(-t-ib)e^{2\pi i t\xi}dt\right)e^{2\pi i x\xi}d\xi \\
&= \int_{0}^{\infty}\int_{-\infty}^{\infty} f(-t-ib)e^{[-2\pi b+2\pi(t+x)i]\xi}\,dt\,d\xi \\
&= \int_{-\infty}^{\infty} f(-t-ib)\int_{0}^{\infty} e^{-[2\pi b-2\pi(t+x)i]\xi}\,d\xi\,dt \\
&= \int_{-\infty}^{\infty} f(-t-ib)\frac{1}{2\pi b-2\pi(t+x)i}dt \\
&= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(-t-ib)}{t+ib+x}dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t-ib)}{t-ib-x}dt \\
&= \frac{1}{2\pi i}\int_{\ell_1}\frac{f(z)}{z-x}dz,
\end{aligned} \tag{13.7}$$

where ℓ_1 is the line given by $\ell_1(t) = t - ib$, $t \in \mathbb{R}$.

For $\xi < 0$, $\hat{f}(\xi)$ is given by (13.5), and so

$$\begin{aligned}
\int_{-\infty}^{0} \hat{f}(\xi)e^{2\pi i x\xi}d\xi &= \int_{0}^{\infty}\left(e^{2\pi b\xi}\int_{-\infty}^{\infty} f(-t+ib)e^{2\pi i t\xi}dt\right)e^{2\pi i x\xi}d\xi \\
&= \int_{-\infty}^{\infty} f(-t+ib)\int_{0}^{\infty} e^{[2\pi b+2\pi(t+x)i]\xi}\,d\xi\,dt \\
&= \int_{-\infty}^{\infty} f(-t+ib)\frac{-1}{2\pi b+2\pi(t+x)i}dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t+ib)}{t+ib-x}dt = \frac{1}{2\pi i}\int_{\ell_2}\frac{f(z)}{z-x}dz,
\end{aligned} \tag{13.8}$$

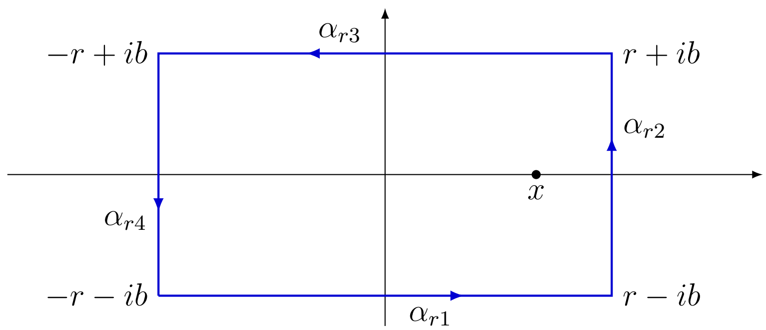

FIGURE 27.

where ℓ_2 is the line given by $\ell_2(t) = -t + ib$, $t \in \mathbb{R}$.

Now, let $\alpha_r = \alpha_{r1} * \alpha_{r2} * \alpha_{r3} * \alpha_{r4}$ be the rectangular path

$$\alpha_{r1}(t) = t - ib, \quad t \in [-r, r]$$
$$\alpha_{r2}(t) = r + it, \quad t \in [-b, b]$$
$$\alpha_{r3}(t) = -t + ib, \quad t \in [-r, r]$$
$$\alpha_{r4}(t) = -r - it, \quad t \in [-b, b]$$

for $r > |x|$, as in Figure 27. Let $h(z) = f(z)(z-x)^{-1}$, which is analytic on $S_a \setminus \{x\}$ and has a simple pole at x. By the Residue Theorem and Proposition 7.33,

$$\oint_{\alpha_r} h = 2\pi i \operatorname{res}(h, x) = 2\pi i \lim_{z \to x}(z - x)h(z) = 2\pi i \lim_{z \to x} f(z) = 2\pi i f(x).$$

It is straightforward to show that $\int_{\alpha_{r2}} h, \int_{\alpha_{r4}} h \to 0$ as $r \to \infty$, and also

$$\lim_{r \to \infty} \int_{\alpha_{r1}} h = \int_{\ell_1} \frac{f(z)}{z - x} dz \quad \text{and} \quad \lim_{r \to \infty} \int_{\alpha_{r3}} h = \int_{\ell_2} \frac{f(z)}{z - x} dz.$$

From (13.7) and (13.8) it follows that

$$\begin{aligned} 2\pi i f(x) &= \lim_{r \to \infty} \oint_{\alpha_r} h = \lim_{r \to \infty} \sum_{k=1}^{4} \int_{\alpha_{rk}} h = \int_{\ell_1} \frac{f(z)}{z - x} dz + \int_{\ell_2} \frac{f(z)}{z - x} dz \\ &= 2\pi i \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi + 2\pi i \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x \xi} d\xi \\ &= 2\pi i \int_{-\infty}^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi \end{aligned}$$

which readily yields (13.6). ■

The following technical lemma arguably has overly stringent hypotheses, nevertheless it will be well suited to the purpose of helping prove the Poisson Summation Formula below. The existence of the integrals in (13.9) is part of the conclusion of the lemma.

Lemma 13.11. *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be given by $\gamma(t) = t + ib$ for some $b \in \mathbb{R}$, and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of functions continuous on γ^*. Suppose there exists some $A \in \mathbb{R}_+$ such that*

$$|h_n(z)| \leq \frac{A}{(\operatorname{Re} z)^2 + 1}$$

for all $n \in \mathbb{N}$ and $z \in \gamma^*$. If $(h_n)_{n\in\mathbb{N}}$ converges pointwise on γ^* to a continuous function $h : \gamma^* \to \mathbb{C}$, then

$$\lim_{n\to\infty} \int_\gamma h_n = \int_\gamma h. \tag{13.9}$$

Proof. Suppose $(h_n)_{n\in\mathbb{N}}$ converges pointwise on γ^* to a continuous function $h : \gamma^* \to \mathbb{C}$. Let $u_n = \operatorname{Re} h_n$ and $v_n = \operatorname{Im} h_n$ for each n, and also set $u = \operatorname{Re} h$ and $v = \operatorname{Im} h$.

We now operate in the measure space $\mathfrak{M} = (\mathbb{R}, \overline{\mathcal{B}}(\mathbb{R}), \lambda)$, where $\overline{\mathcal{B}}(\mathbb{R})$ is the collection of Lebesgue measurable sets in $\mathbb{R}$, and $\lambda : \overline{\mathcal{B}}(\mathbb{R}) \to \overline{\mathbb{R}}$ is (complete) Lebesgue measure. Each function $u_n \circ \gamma : \mathbb{R} \to \mathbb{R}$ is continuous, and so $(u_n \circ \gamma)_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions on $(\mathbb{R}, \overline{\mathcal{B}}(\mathbb{R}))$. That is, each $u_n \circ \gamma$ is $(\overline{\mathcal{B}}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable. Now, $u_n \to u$ pointwise on γ^* by Proposition 2.16, and hence $u_n \circ \gamma \to u \circ \gamma$ pointwise on $\mathbb{R}$. Of course $u \circ \gamma : \mathbb{R} \to \mathbb{R}$ is Borel measurable on account of being continuous. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = A(t^2 + 1)^{-1}$. Clearly g is Borel measurable, and since g is nonnegative and $\int_{-\infty}^{\infty} g$ converges, Theorem 2.53 in [MT] implies that g is λ-integrable in $\mathfrak{M}$ with

$$\int_{\mathbb{R}} g \, d\lambda = \int_{-\infty}^{\infty} g.$$

Moreover, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$|(u_n \circ \gamma)(t)| = |u_n(\gamma(t))| \leq |h_n(\gamma(t))| = |h_n(t + ib)| \leq \frac{A}{t^2 + 1} = g(t), \tag{13.10}$$

which is to say $|u_n \circ \gamma| \leq g$ for all n. It follows by the Dominated Convergence Theorem in [MT] that $u \circ \gamma$ is λ-integrable with

$$\lim_{n\to\infty} \int_{\mathbb{R}} (u_n \circ \gamma) \, d\lambda = \int_{\mathbb{R}} (u \circ \gamma) \, d\lambda.$$

A nearly identical argument, only with every u and u_n replaced with v and v_n, leads to the conclusion that $v \circ \gamma$ is λ-integrable with

$$\lim_{n\to\infty} \int_{\mathbb{R}} (v_n \circ \gamma) \, d\lambda = \int_{\mathbb{R}} (v \circ \gamma) \, d\lambda.$$

Now, by definition,

$$\int_{\mathbb{R}} (h_n \circ \gamma) \, d\lambda := \int_{\mathbb{R}} \operatorname{Re}(h_n \circ \gamma) \, d\lambda + i \int_{\mathbb{R}} \operatorname{Im}(h_n \circ \gamma) \, d\lambda = \int_{\mathbb{R}} (u_n \circ \gamma) \, d\lambda + i \int_{\mathbb{R}} (v_n \circ \gamma) \, d\lambda,$$

and so

$$\lim_{n\to\infty} \int_{\mathbb{R}} (h_n \circ \gamma) \, d\lambda = \int_{\mathbb{R}} (u \circ \gamma) \, d\lambda + i \int_{\mathbb{R}} (v \circ \gamma) \, d\lambda := \int_{\mathbb{R}} (h \circ \gamma) \, d\lambda. \tag{13.11}$$

Next, the last inequality in (13.10) makes clear, by the Comparison Test for integrals of real-valued functions, that $\int_{-\infty}^{\infty} |h_n \circ \gamma|$ converges, and therefore $h_n \circ \gamma$ is λ-integrable in $\mathfrak{M}$, with

$$\int_{\mathbb{R}} (h_n \circ \gamma)\, d\lambda = \int_{-\infty}^{\infty} (h_n \circ \gamma) = \int_{\gamma} h_n \tag{13.12}$$

by Proposition 2.54 in [MT]. In addition, for each $t \in \mathbb{R}$ we have

$$|(h \circ \gamma)(t)| = \left| \lim_{n\to\infty} h_n(\gamma(t)) \right| = \lim_{n\to\infty} \left| h_n(\gamma(t)) \right| = \lim_{n\to\infty} \left| h_n(t + ib) \right| \le \frac{A}{t^2 + 1}, \tag{13.13}$$

and so

$$\int_{\mathbb{R}} (h \circ \gamma)\, d\lambda = \int_{-\infty}^{\infty} (h \circ \gamma) = \int_{\gamma} h$$

also obtains by Proposition 2.54 in [MT]. Finally, (13.12) and (13.13), together with (13.11), yields (13.9). ■

Theorem 13.12 (Poisson Summation Formula). *If $f \in \mathfrak{F}$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof. Suppose $f \in \mathfrak{F}$, so $f \in \mathfrak{F}_a$ for some $a \in \mathbb{R}_+$, and there exists some $A \in \mathbb{R}_+$ such that $|f(x + iy)| \le A(x^2 + 1)^{-1}$ for all $x + iy \in S_a$. Let $b \in (0, a)$. Define

$$h(z) = \frac{f(z)}{e^{2\pi iz} - 1}.$$

For each $n \in \mathbb{Z}$ we have, by L'Hôpital's Rule,

$$\lim_{z\to n} (z - n)h(z) = \lim_{z\to n} \frac{(z - n)f(z)}{e^{2\pi iz} - 1} = \lim_{z\to n} \frac{(z - n)f'(z) + f(z)}{2\pi i e^{2\pi iz}} = \frac{f(n)}{2\pi i e^{2\pi in}} = \frac{f(n)}{2\pi i},$$

so Theorem 7.10 implies that h has a simple pole at $n \in \mathbb{Z}$ if $f(n) \neq 0$, with

$$\operatorname{res}(h, n) = \frac{f(n)}{2\pi i}$$

by Proposition 7.33. If $f(n) = 0$, then

$$\lim_{z\to n} h(z) = \lim_{z\to n} \frac{f'(z)}{2\pi i e^{2\pi iz}} = \frac{f'(n)}{2\pi i}$$

by L'Hôpital's Rule, showing h has a removable singularity at n by Theorem 7.9, and hence $\operatorname{res}(h, n) = 0$ by Definition 7.7.

Now, let $\gamma_m = \gamma_{m1} * \gamma_{m2} * \gamma_{m3} * \gamma_{m4}$ be the path

$$\begin{aligned}
\gamma_{m1}(t) &= t - ib, \quad t \in \left[-m - \tfrac{1}{2}, -m - \tfrac{1}{2} \right] \\
\gamma_{m2}(t) &= m + \tfrac{1}{2} + it, \quad t \in [-b, b] \\
\gamma_{m3}(t) &= -t + ib, \quad t \in \left[-m - \tfrac{1}{2}, -m - \tfrac{1}{2} \right] \\
\gamma_{m4}(t) &= -m - \tfrac{1}{2} - it, \quad t \in [-b, b]
\end{aligned}$$

for $m \in \mathbb{N}$. By the Residue Theorem,

$$\sum_{k=1}^{4} \int_{\gamma_{m1}} h = \oint_{\gamma_m} h = 2\pi i \sum_{z \in S(h)} \operatorname{res}(h, z) \operatorname{wn}(\gamma_m, z) = 2\pi i \sum_{n=-m}^{m} \operatorname{res}(h, n) = \sum_{n=-m}^{m} f(n). \tag{13.14}$$

It turns out that $\int_{\gamma_{m2}} h, \int_{\gamma_{m4}} h \to 0$ as $m \to \infty$. Moreover, we have

$$\lim_{m\to\infty} \int_{\gamma_{m1}} h = \int_{\ell_1} \frac{f(z)}{e^{2\pi iz} - 1} dz \quad \text{and} \quad \lim_{m\to\infty} \int_{\gamma_{m3}} h = \int_{\ell_2} \frac{f(z)}{e^{2\pi iz} - 1} dz,$$

where ℓ_1 and ℓ_2 are the lines $\ell_1(t) = t - ib$ and $\ell_2(t) = -t + ib$ for $t \in \mathbb{R}$. Finally, since $|f(n)| \le A(n^2+1)^{-1}$ for all $n \in \mathbb{Z}$, Theorem 2.7 shows that the series

$$\sum_{n=0}^{\infty} f(n) \quad \text{and} \quad \sum_{n=1}^{\infty} f(-n)$$

both converge, and therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n) &= \sum_{n=0}^{\infty} f(n) + \sum_{n=1}^{\infty} f(-n) = \lim_{m\to\infty} \sum_{n=0}^{m} f(n) + \lim_{m\to\infty} \sum_{n=1}^{m} f(-n) \\ &= \lim_{m\to\infty} \left[\sum_{n=0}^{m} f(n) + \sum_{n=1}^{m} f(-n) \right] = \lim_{m\to\infty} \sum_{n=-m}^{m} f(n),\end{aligned}$$

the first equality according with the definition of a two-tailed series given in §7.1. Taking the limit as $m \to \infty$ in (13.14) thus yields

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\ell_1} \frac{f(z)}{e^{2\pi iz} - 1} dz + \int_{\ell_2} \frac{f(z)}{e^{2\pi iz} - 1} dz.$$

Now, for $z \in \ell_1^*$ we have $z = t - ib$ for some $t \in \mathbb{R}$, so that $|e^{2\pi iz}| = e^{2\pi b} > 1$. By Exercise 4.9,

$$\int_{\ell_1} h = \int_{\ell_1} f(z) e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz} dz = \int_{\ell_1} \lim_{k\to\infty} h_k(z)\, dz,$$

where

$$h_k(z) = \sum_{n=0}^{k} f(z) e^{-2\pi i(n+1)z}$$

for each $k \in \mathbb{N}$. Setting $B = \sum_{n=0}^{\infty} e^{-2\pi(n+1)b}$, we also have

$$|h_k(z)| \le |f(z)| \sum_{n=0}^{k} \left| e^{-2\pi i(n+1)z} \right| = |f(z)| \sum_{n=0}^{k} e^{-2\pi(n+1)b} \le \frac{AB}{(\operatorname{Re} z)^2 + 1}$$

for $z \in \ell_1^*$. Noting that $(h_k)_{k\in\mathbb{N}}$ converges pointwise on ℓ_1^* to the continuous function $h : \ell_1^* \to \mathbb{C}$, by Lemma 13.11 and Proposition 13.9 it follows that

$$\begin{aligned}\int_{\ell_1} h &= \lim_{k\to\infty} \int_{\ell_1} h_k = \lim_{k\to\infty} \sum_{n=0}^{k} \int_{\ell_1} f(z) e^{-2\pi i(n+1)z} dz = \sum_{n=0}^{\infty} \int_{\ell_1} f(z) e^{-2\pi i(n+1)z} dz \\ &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(t - ib) e^{-2\pi i(n+1)(t-ib)} dt = \sum_{n=0}^{\infty} e^{-2\pi b(n+1)} \int_{-\infty}^{\infty} f(t - ib) e^{-2\pi it(n+1)} dt\end{aligned}$$

$$= \sum_{n=0}^{\infty} \hat{f}(n+1) = \sum_{n=1}^{\infty} \hat{f}(n).$$

Next, $z \in \ell_2^*$ implies $z = t + ib$ for some $t \in \mathbb{R}$, so $|e^{2\pi iz}| = e^{-2\pi b} < 1$. By Exercise 4.9, Lemma 13.11, and Proposition 13.9,

$$\int_{\ell_2} h = -\int_{\ell_2} f(z) \sum_{n=0}^{\infty} e^{2\pi inz}\, dz = -\sum_{n=0}^{\infty} \int_{\ell_2} f(z) e^{2\pi inz}\, dz$$
$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(-t+ib) e^{2\pi in(-t+ib)}\, dt = \sum_{n=0}^{\infty} \hat{f}(-n).$$

Therefore

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\ell_1} h + \int_{\ell_2} h = \sum_{n=1}^{\infty} \hat{f}(n) + \sum_{n=0}^{\infty} \hat{f}(-n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

and the proof is done. ■

13.3 – The Paley-Wiener Theorem

Proposition 13.13. *Suppose $\hat{f}$ is the Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$, with*

$$\left|\hat{f}(\xi)\right| \leq Ae^{-2\pi a|\xi|}$$

for some constants $a, A \in \mathbb{R}_+$. Then for some $b \in (0, a)$ there exists an analytic function $g : S_b \to \mathbb{C}$ such that $f = g|_{\mathbb{R}}$.

Theorem 13.14 (**Paley-Wiener Theorem**)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and of moderate decrease on $\mathbb{R}$, and let $M \in \mathbb{R}_+$. Then there is an entire function g such that $g|_{\mathbb{R}} = f$ and $|g(z)| \leq Ae^{2\pi M|z|}$ for some $A \in \mathbb{R}_+$ if and only if $\hat{f}$ is supported in $[-M, M]$.*

14

The Gamma and Zeta Functions

14.1 – The Gamma Function

It is convenient to define the **right half-plane** and **left half-plane** of $\mathbb{C}$ to be the sets

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z \in \mathbb{R}_+\} \quad \text{and} \quad \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z \in \mathbb{R}_-\},$$

respectively.

The **gamma function** on $\mathbb{C}_+$ is the function $\Gamma : \mathbb{C}_+ \to \mathbb{C}$ defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

for all $z \in \mathbb{C}_+$.

Proposition 14.1. *The gamma function on $\mathbb{C}_+$ is analytic.*

Proposition 14.2. $\Gamma(z+1) = z\Gamma(z)$ *for all* $z \in \mathbb{C}_+$*, and* $\Gamma(n+1) = n!$ *for all* $n \in \mathbb{W}$*.*

Proposition 14.3. *There exists an analytic continuation* $\hat{\Gamma} : \mathbb{C} \setminus (\mathbb{Z} \setminus \mathbb{N}) \to \mathbb{C}$ *of* Γ *that is meromorphic on* $\mathbb{C}$ *and has simple poles on* $\mathbb{Z} \setminus \mathbb{N}$*, with* $\operatorname{res}(\hat{\Gamma}, -n) = (-1)^n/n!$ *for each* $n \in \mathbb{W}$*.*

As is traditional, the analytic extension $\hat{\Gamma}$ of the gamma function to the set $\mathbb{C} \setminus (\mathbb{Z} \setminus \mathbb{N})$ is again denoted by Γ and called the gamma function.

Lemma 14.4. *For all* $a \in (0,1)$*,*

$$\int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin \pi a}.$$

Proposition 14.5. *For all* $z \in \mathbb{C} \setminus \mathbb{Z}$*,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

with both functions having a simple pole at each $n \in \mathbb{Z}$*.*

Proof. Define $f(z) = \Gamma(z)\Gamma(1-z)$ and $g(z) = \pi/\sin(\pi z)$. It must be shown that $f(z) = g(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$, and $P(f) = P(g) = \mathbb{Z}$ with $\operatorname{ord}(f, n) = \operatorname{ord}(g, n) = 1$ for all $n \in \mathbb{Z}$. Since f and g are clearly analytic on $\mathbb{C} \setminus \mathbb{Z}$, to show $f = g$ on $\mathbb{C} \setminus \mathbb{Z}$ it is sufficient by Corollary 5.14 to show that $f(s) = g(s)$ for all $s \in (0, 1)$. ∎

Proposition 14.6. *Define the function* $1/\Gamma : \mathbb{C} \to \mathbb{C}$ *by*

$$(1/\Gamma)(z) = \begin{cases} 1/\Gamma(z), & z \in \mathbb{C} \setminus (\mathbb{Z} \setminus \mathbb{N}) \\ 0, & z \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

1. $1/\Gamma$ *is an entire function with* $Z(1/\Gamma) = \mathbb{Z} \setminus \mathbb{N}$ *and* $\operatorname{ord}(1/\Gamma, n) = 1$ *for each* $n \in \mathbb{Z} \setminus \mathbb{N}$.
2. *There exist constants* $A, B \in \mathbb{R}_+$ *such that*
$$\left|(1/\Gamma)(z)\right| \leq Ae^{B|z|\ln|z|}$$
for all $z \in \mathbb{C}$, *where we define the right-hand expression to be* 0 *when* $z = 0$.
3. *There exists* $B \in \mathbb{R}_+$ *such that, for each* $\epsilon > 0$, *there is some* $A_\epsilon \in \mathbb{R}_+$ *for which*
$$\left|(1/\Gamma)(z)\right| \leq A_\epsilon e^{B|z|^{1+\epsilon}}$$
for all $z \in \mathbb{C}$, *where we define the right-hand expression to be* 0 *when* $z = 0$.

The real number

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right),$$

known as **Euler's constant**, appears in the following product formula for $1/\Gamma$.

Theorem 14.7. *For all* $z \in \mathbb{C}$,

$$(1/\Gamma)(z) = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-z/n}.$$

Exercise 14.8 (SS6.3.1). Prove that

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z n!}{z(z+1)\cdots(z+n)}$$

for all $z \notin \mathbb{Z} \setminus \mathbb{N}$.

Solution. Fix $z \in \mathbb{C} \setminus (\mathbb{Z} \setminus \mathbb{N})$. We will prove that

$$(1/\Gamma)(z) = \lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n^z n!}.$$

For each $n \in \mathbb{N}$ let

$$f_n(z) = \sum_{k=1}^{n}\left[\ln(1 + z/k) - z/k\right] + \gamma z \quad \text{and} \quad g_n(z) = \sum_{k=1}^{n}\ln(1 + z/k) - z\ln n.$$

By Theorem 14.7,

$$(1/\Gamma)(z) = z\lim_{n\to\infty}\left(e^{\gamma z}\prod_{k=1}^{n}\left(1 + \frac{z}{k}\right)e^{-z/k}\right) = z\lim_{n\to\infty}\exp\left(\sum_{k=1}^{n}\ln\left[(1 + z/k)e^{-z/k}\right] + \gamma z\right)$$

$$= z \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \big[\ln(1+z/k) - z/k\big] + \gamma z\right) = z \lim_{n\to\infty} e^{f_n(z)},$$

while

$$\begin{aligned}
\lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n^z n!} &= \lim_{n\to\infty} \exp\big(\ln[z(z+1)\cdots(z+n)] - \ln(n^z n!)\big) \\
&= \lim_{n\to\infty} \exp\left(\ln z + \sum_{k=1}^{n} \ln(z+k) - z\ln n - \ln n!\right) \\
&= \lim_{n\to\infty} z \exp\left(\sum_{k=1}^{n} \ln(z+k) - z\ln n - \sum_{k=1}^{n} \ln k\right) \\
&= z \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \ln(1+z/k) - z\ln n\right) = z \lim_{n\to\infty} e^{g_n(z)}.
\end{aligned}$$

It remains only to show that $\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} g_n(z)$. Since $\lim_{n\to\infty} f_n(z)$ exists in $\mathbb{C}$, this will follow once

$$\lim_{n\to\infty} [f_n(z) - g_n(z)] = 0$$

is shown. However,

$$\begin{aligned}
\lim_{n\to\infty} [f_n(z) - g_n(z)] = 0 \quad &\Leftrightarrow \quad \lim_{n\to\infty} \left(\gamma z - \sum_{k=1}^{n} \frac{z}{k} + z\ln n\right) = 0 \\
&\Leftrightarrow \quad \lim_{n\to\infty} \left(\gamma - \sum_{k=1}^{n} \frac{1}{k} + \ln n\right) = 0 \\
&\Leftrightarrow \quad \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) = \gamma,
\end{aligned}$$

and so we are done. ■

Exercise 14.9 (SS6.3.10). The **Mellin transform** of a function $f : [0,\infty) \to \mathbb{C}$ is the function $\mathcal{M}(f)$ given by

$$\mathcal{M}(f)(z) = \int_0^\infty f(t) t^{z-1}\, dt.$$

Prove that

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t) t^{z-1}\, dt = \Gamma(z)\cos\Big(\frac{\pi}{2}z\Big) \tag{14.1}$$

and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t) t^{z-1}\, dt = \Gamma(z)\sin\Big(\frac{\pi}{2}z\Big) \tag{14.2}$$

for all $z \in (0,1) \times \mathbb{R}$.

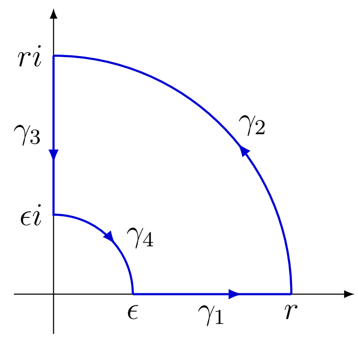

FIGURE 28.

Solution. Proposition 14.1 makes clear that $\Gamma(z)\cos\left(\frac{\pi}{2}z\right)$ and $\Gamma(z)\sin\left(\frac{\pi}{2}z\right)$ are analytic on $\Omega = (0,1) \times \mathbb{R}$, and an argument similar to that in the proof of the proposition will show that $\mathcal{M}(\cos)(z)$ and $\mathcal{M}(\cos)(z)$ are analytic on Ω. Thus, in light of the Identity Theorem, to show (14.1) and (14.2) hold on Ω, it suffices to show the equations hold on the interval $(0,1)$.

Fix $s \in (0,1)$. Recalling (6.1), define $f : \mathbb{C}_* \to \mathbb{C}$ by $f(z) = e^{-z}z^{s-1}$. For $0 < \epsilon < r < \infty$ let $\gamma = \gamma_1 * \gamma_2 * \overline{\gamma}_3 * \overline{\gamma}_4$, where

$$\begin{aligned}
\gamma_1(t) &= t, \quad t \in [\epsilon, r] \\
\gamma_2(t) &= re^{it}, \quad t \in [0, \pi/2] \\
\gamma_3(t) &= it, \quad t \in [\epsilon, r] \\
\gamma_4(t) &= \epsilon e^{it}, \quad t \in [0, \pi/2]
\end{aligned}$$

as in Figure 28. Since f is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and $\gamma^* \subseteq \mathbb{C} \setminus (-\infty, 0]$, Theorem 11.30 and Propositions 3.27 and 3.28 imply that

$$0 = \oint_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\overline{\gamma}_3} f + \int_{\overline{\gamma}_4} f = \int_{\gamma_1} f + \int_{\gamma_2} f - \int_{\gamma_3} f - \int_{\gamma_4} f \tag{14.3}$$

for all $0 < \epsilon < r < \infty$. As $\epsilon \to 0^+$ and $r \to \infty$ is it clear that

$$\int_{\gamma_1} f = \int_\epsilon^r e^{-t}t^{s-1}\,dt \to \int_0^\infty e^{-t}t^{s-1}\,dt = \Gamma(s).$$

Making the substitution $u = \pi/2 - t$ and recalling Exercise 7.58, we have

$$\begin{aligned}
\left|\int_{\gamma_2} f\right| &= \left|\int_0^{\pi/2} e^{-re^{it}}(re^{it})^{s-1} \cdot ire^{it}\,dt\right| \le \int_0^{\pi/2} \left|e^{-re^{it}}(re^{it})^{s-1} \cdot ire^{it}\right| dt \\
&= \int_0^{\pi/2} r^s e^{-r\cos t}\,dt = r^s \int_{\pi/2}^0 -e^{-r\sin u}du = r^s \int_0^{\pi/2} e^{-r\sin u}du \\
&\le r^s \cdot \frac{\pi}{2r}(1 - e^{-r}) = \frac{\pi}{2r^{1-s}}(1 - e^{-r}),
\end{aligned}$$

and so $\int_{\gamma_2} f \to 0$ as $r \to \infty$ since $0 < s < 1$.

Next,

$$\begin{aligned}\int_{\gamma_3} f &= \int_\epsilon^r e^{-it}(it)^{s-1}\cdot i\,dt = i^s\int_\epsilon^r e^{-it}t^{s-1}\,dt\\ &= i^s\left(\int_\epsilon^r \cos(t)t^{s-1}\,dt - i\int_\epsilon^r \sin(t)t^{s-1}\,dt\right),\end{aligned}$$

so

$$\int_{\gamma_3} f \to i^s\left(\int_0^\infty \cos(t)t^{s-1}\,dt - i\int_0^\infty \sin(t)t^{s-1}\,dt\right)$$

as $\epsilon\to 0^+$ and $r\to\infty$.

Finally,

$$\begin{aligned}\left|\int_{\gamma_4} f\right| &\le \mathcal{L}(\gamma_4)\sup_{z\in\gamma_4^*}|f(z)| = \frac{\pi}{\epsilon}\sup_{t\in[0,\pi/2]}|f(\gamma_4(t))| = \frac{\pi}{2}\epsilon\sup_{t\in[0,\pi/2]}\left|e^{-\epsilon e^{it}}(\epsilon e^{it})^{s-1}\right|\\ &= \frac{\pi}{2}\epsilon^s\sup_{t\in[0,\pi/2]}(e^{-\epsilon\cos t}) = \frac{\pi}{2}\epsilon^s,\end{aligned}$$

and so $\int_{\gamma_4} f\to 0$ as $\epsilon\to 0^+$.

Returning to (14.3), we find

$$0 = \lim_{r\to\infty}\lim_{\epsilon\to 0^+}\int_\gamma f = \Gamma(s) - i^s\left(\int_0^\infty \cos(t)t^{s-1}\,dt - i\int_0^\infty \sin(t)t^{s-1}\,dt\right),$$

which gives

$$\begin{aligned}\int_0^\infty \sin(t)t^{s-1}\,dt + i\int_0^\infty \cos(t)t^{s-1}\,dt &= i^{1-s}\Gamma(s) = e^{\frac{\pi}{2}(1-s)i}\Gamma(s)\\ &= \left[\cos\left((1-s)\frac{\pi}{2}\right) + i\sin\left((1-s)\frac{\pi}{2}\right)\right]\Gamma(s)\\ &= \Gamma(s)\sin\left(\frac{\pi}{2}s\right) + i\Gamma(s)\cos\left(\frac{\pi}{2}s\right).\end{aligned}$$

By equating imaginary parts, we see (14.1) holds for all $z\in(0,1)$; and by equating real parts, we see (14.2) holds for all $z\in(0,1)$. ■

14.2 – The Zeta Function

Define the set
$$\mathbb{C}_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}.$$
The Riemann **zeta function** on $\mathbb{C}_1$ is the function $\zeta : \mathbb{C}_1 \to \mathbb{C}$ given by
$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$
That the series truly does converge for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 1$ will be verified along the way to proving ζ is analytic on $\mathbb{C}_1$.

Proposition 14.10. *The Riemann zeta function on $\mathbb{C}_1$ is analytic.*

Proof. Fix $z \in \mathbb{C}_1$, so $z = x + iy$ with $x > 1$. For any $n \in \mathbb{N}$,
$$|n^{-z}| = \left|e^{-z \ln n}\right| = e^{\mathrm{Re}(-z \ln n)} = e^{-x \ln n} = n^{-x}$$
by Theorem 4.42, and since $\sum_{n=1}^{\infty} n^{-x}$ converges in $\mathbb{R}$, the Direct Comparison Test implies that $\sum_{n=1}^{\infty} n^{-z}$ converges in $\mathbb{C}$.

Define the sequence of analytic functions $(f_n : \mathbb{C}_1 \to \mathbb{C})_{n\in\mathbb{N}}$ by $f_n(z) = n^{-z}$ for each n. Clearly $\sum_{n=1}^{\infty} f_n$ converges pointwise to ζ on $\mathbb{C}_1$. For $1 < a < b < \infty$ let $S_{a,b} \subseteq \mathbb{C}_1$ be the vertical strip
$$S_{a,b} = \{z : \mathrm{Re}\, z \in [a, b]\}.$$
For each $n \in \mathbb{N}$,
$$\|f_n\|_{S_{a,b}} = \sup_{z\in S_{a,b}} |f_n(z)| = \sup_{z\in S_{a,b}} \left(n^{-\mathrm{Re}\, z}\right) \le \frac{1}{n^a},$$
and since $\sum_{n=1}^{\infty} n^{-a}$ converges in $\mathbb{R}$, the Weierstrass M-Test implies that $\sum_{n=1}^{\infty} f_n$ converges uniformly to ζ on $S_{a,b}$. Now, any compact set $K \subseteq \mathbb{C}_1$ may be contained in some $S_{a,b}$, so in fact $\sum_{n=1}^{\infty} f_n$ (i.e. the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums $s_n = \sum_{k=1}^{n} f_k$) converges uniformly to ζ on compact subsets of $\mathbb{C}_1$. By Theorem 4.30 it follows that ζ is analytic on $\mathbb{C}_1$. ■

In the proof of Proposition 14.10, observing that
$$(n^{-z})' = \left(e^{-z \ln n}\right)' = (-\ln n)e^{-z \ln n} = (-\ln n)n^{-z},$$
it is a further consequence of Theorem 4.30 that
$$\zeta^{(k)}(z) = \sum_{n=1}^{\infty} (-\ln n)^k \frac{1}{n^z}$$
for all $k \in \mathbb{N}$ and $z \in \mathbb{C}_1$.

Lemma 14.11. *Define the function ϑ by*
$$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}.$$

Then $\vartheta : \mathbb{R}_+ \to \mathbb{R}$ *with* $\vartheta(t) = t^{-1/2}\vartheta(t^{-1})$ *for all* $t \in \mathbb{R}_+$*. Moreover, there exists some* $\epsilon > 0$ *and* $C > 0$ *such that*

$$\forall t \in (0, \epsilon)\big[\vartheta(t) \leq Ct^{-1/2}\big] \quad \text{and} \quad \forall t \in [1, \infty)\big[|\vartheta(t) - 1| \leq Ce^{-\pi t}\big].$$

Proof. Clearly $\vartheta(t) \in \mathbb{R}$ for each $t \in \mathbb{R}_+$. Now, for each $t \in \mathbb{R}_+$ define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = e^{-\pi x^2 t}$. By Example 13.4,

$$\hat{f}_t(\xi) = \frac{1}{\sqrt{t}} e^{-\pi\xi^2/t},$$

and so by the Poisson Summation Formula we obtain

$$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} f_t(n) = \sum_{n\in\mathbb{Z}} \hat{f}_t(n) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{t}} e^{-\pi n^2/t} = \frac{1}{\sqrt{t}}\vartheta(1/t),$$

as was to be shown. ■

Proposition 14.12. *For all* $z \in \mathbb{C}_1$,

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{2}\int_0^\infty u^{z/2-1}[\vartheta(u) - 1]\,du.$$

Proposition 14.13. *The function* $\xi : \mathbb{C}_1 \to \mathbb{C}$ *given by*

$$\xi(z) = \pi^{-z/2}\Gamma(z/2)\zeta(z)$$

is analytic on $\mathbb{C}_1$*, and has an analytic continuation* $\xi : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ *with simple poles at* 0 *and* 1*. Moreover,*

$$\xi(z) = \xi(1 - z)$$

for all $z \in \mathbb{C} \setminus \{0, 1\}$.

Theorem 14.14. *There exists an analytic continuation* $\hat{\zeta} : \mathbb{C}\setminus\{1\} \to \mathbb{C}$ *of* ζ *that is meromorphic on* $\mathbb{C}$ *and has a simple pole at* 1.

Proposition 14.15.

1. *There exists a sequence of entire functions* $(\eta_n)_{n\in\mathbb{N}}$ *for which* $|\eta_n(z)| \leq |z|/n^{\operatorname{Re} z+1}$ *for all* $z \in \mathbb{C}$ *and* $n \in \mathbb{N}$*, and such that*

$$\sum_{n=1}^{N-1} \frac{1}{n^z} - \int_1^N \frac{1}{t^z}\,dt = \sum_{n=1}^{N-1} \eta_n(z)$$

   for any integer $N \geq 2$.
2. $\sum_{n=1}^\infty \eta_n$ *is analytic on* $\mathbb{C}_+$*, and moreover*

$$\zeta(z) = \sum_{n=1}^\infty \eta_n(z) + \frac{1}{z - 1}$$

   for all $z \in \mathbb{C}_+$.

Exercise 14.16 (SS6.3.15). Prove that

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt \tag{14.4}$$

for all $z \in \mathbb{C}_1$.

Solution. Fix $s \in (2, \infty)$. Define $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(t) = \begin{cases} t^{s-1} \displaystyle\sum_{n=1}^{k} e^{-nt}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

for all $k \in \mathbb{N}$, and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} \dfrac{t^{s-1}}{e^t - 1}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

Since $h_k(t) \to 0$ and $h(t) \to 0$ as $t \to 0$, the functions h_k and h are continuous on $\mathbb{R}$. Also, for $t \in (0, \infty)$,

$$\lim_{k\to\infty} h_k(t) = \lim_{k\to\infty} t^{s-1} \sum_{n=1}^{k} e^{-nt} = t^{s-1} \sum_{n=1}^{\infty} e^{-nt} = \frac{t^{s-1}}{e^t - 1} = h(t)$$

by Exercise 4.9, so that the sequence $(h_k)_{k\in\mathbb{N}}$ converges pointwise to h on $\mathbb{R}$. Now, by Example 13.5 there is some constant A such that

$$|h(t)| \le \frac{A}{t^2 + 1}$$

for all $t \in \mathbb{R}$, from which it is clear that $|h_k(t)| \le A(t^2 + 1)^{-1}$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$, and so if we define $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(t) = t$, then by Lemma 13.11 it follows that

$$\begin{aligned} \lim_{k\to\infty} \int_0^\infty t^{s-1} \sum_{n=1}^{k} e^{-nt}\, dt &= \lim_{k\to\infty} \int_{-\infty}^{\infty} h_k(t)\, dt = \lim_{k\to\infty} \int_\gamma h_k \\ &= \int_\gamma h = \int_{-\infty}^{\infty} h(t)\, dt = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt. \end{aligned}$$

Now, making the substitution $u = nt$, we have

$$\begin{aligned} \int_0^\infty e^{-nt} t^{s-1}\, dt &= \lim_{x\to\infty} \int_0^x e^{-nt} t^{s-1}\, dt = \lim_{x\to\infty} \int_0^{nx} \frac{e^{-u} u^{s-1}}{n^s}\, du \\ &= \frac{1}{n^s} \int_0^\infty e^{-u} u^{s-1}\, du = \frac{\Gamma(s)}{n^s}, \end{aligned}$$

and hence

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt = \lim_{k\to\infty} \sum_{n=1}^{k} \int_0^\infty e^{-nt} t^{s-1}\, dt = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{\Gamma(s)}{n^s} = \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \Gamma(s)\zeta(s).$$

Thus (14.4) holds for all $z \in (2, \infty)$.

Next, let $z \in \mathbb{C}_1$. Since $\operatorname{Re} z \in (1,\infty)$, by Exercise 4.53 we have

$$\int_0^1 \left|\frac{t^{z-1}}{e^t-1}\right| dt = \int_0^1 \frac{t^{\operatorname{Re} z-1}}{e^t-1}\,dt = \sum_{n=0}^{\infty} \frac{B_n}{n!(\operatorname{Re} z+n-1)} \in \mathbb{C}.$$

By Example 13.5 there is some $M > 0$ such that

$$\int_1^\infty \left|\frac{t^{z-1}}{e^t-1}\right| dt \le \int_1^\infty \left|\frac{t^{z}}{e^t-1}\right| dt = \int_1^\infty \frac{t^{\operatorname{Re} z}}{e^t-1}dt \le \int_1^\infty \frac{M}{t^2+1}\,dt \in \mathbb{C}.$$

So by comparison tests analogous to Theorem 7.53 we conclude that

$$\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt \quad \text{and} \quad \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

both converge in $\mathbb{C}$, and thus

$$F(z) := \int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt = \int_0^1 \frac{t^{z-1}}{e^t-1}\,dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt \tag{14.5}$$

converges in $\mathbb{C}$ for all $z \in \mathbb{C}_1$.

Define $\varphi : \mathbb{C}_1 \times (0,\infty) \to \mathbb{C}$ by

$$\varphi(z,t) = \frac{t^{z-1}}{e^t-1},$$

and for each $n \in \mathbb{N}$ let $F_n : \mathbb{C}_1 \to \mathbb{C}$ be given by

$$F_n(z) = \int_{1/n}^n \varphi(z,t)\,dt.$$

Since φ is continuous on $\mathbb{C}_1 \times [1/n, n]$, and $\varphi(\cdot,t) : \mathbb{C}_1 \to \mathbb{C}$ is analytic for each $t \in [1/n, n]$, Lemma 6.27 implies that F_n is analytic on $\mathbb{C}_1$. Fix $1 < a < b < \infty$, and define $S_{a,b} = [a,b] \times \mathbb{R}$ in $\mathbb{C}_1$. Let $\epsilon > 0$. The integral

$$\int_0^\infty \frac{t^{b-1}}{e^t-1}\,dt$$

is convergent since $b \in \mathbb{C}_1$, and so there must exist some $k \in \mathbb{N}$ such that

$$\int_0^{1/k} \frac{t^{b-1}}{e^t-1}dt < \frac{\epsilon}{2} \quad \text{and} \quad \int_k^\infty \frac{t^{b-1}}{e^t-1}dt < \frac{\epsilon}{2}.$$

Let $n \ge k$ and $z \in S_{a,b}$ be arbitrary, with $s = \operatorname{Re} z$. Since $s \le b$ and $1/k \ge 1/n$, we have

$$\begin{aligned}
|F(z) - F_n(z)| &= \left|\int_0^\infty \frac{t^{z-1}}{e^t-1}dt - \int_{1/n}^n \frac{t^{z-1}}{e^t-1}dt\right| = \left|\int_0^{1/n} \frac{t^{z-1}}{e^t-1}dt + \int_n^\infty \frac{t^{z-1}}{e^t-1}dt\right| \\
&\le \int_0^{1/n} \left|\frac{t^{z-1}}{e^t-1}\right| dt + \int_n^\infty \left|\frac{t^{z-1}}{e^t-1}\right| dt = \int_0^{1/n} \frac{t^{s-1}}{e^t-1}dt + \int_n^\infty \frac{t^{z-1}}{e^t-1}dt \\
&\le \int_0^{1/k} \frac{t^{b-1}}{e^t-1}dt + \int_k^\infty \frac{t^{b-1}}{e^t-1}dt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

and thus $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F on $S_{a,b}$. Now, any compact set $K \subseteq \mathbb{C}_1$ may be contained in some $S_{a,b}$, so $(F_n)_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{C}_1$ that converges uniformly to $F : \mathbb{C}_1 \to \mathbb{C}$ on compact subsets of $\mathbb{C}_1$. By Theorem 4.30, therefore, F is analytic on $\mathbb{C}_1$, and then by Proposition 14.6 it is clear that the function on the right-hand side of (14.4) is also analytic on $\mathbb{C}_1$. Recalling that ζ is analytic on $\mathbb{C}_1$ by Proposition 14.10, the Identity Theorem then implies that (14.4) holds for all $z \in \mathbb{C}_1$. ∎

Exercise 14.17 (SS6.3.16). Use the result of the previous exercise to show that $\zeta : \mathbb{C}_1 \to \mathbb{C}$ has analytic continuation to $\mathbb{C} \setminus \{1\}$ with simple pole at 1.

Solution. By Exercise 14.16 we have

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^1 \frac{t^{z-1}}{e^t - 1} dt + \frac{1}{\Gamma(z)} \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt \tag{14.6}$$

for all $z \in \mathbb{C}_1$. The second integral in (14.6) is readily seen to be an entire function since $1/\Gamma$ is entire and the function

$$\frac{t^{z-1}}{e^t - 1}$$

is of moderate decrease on $[1, \infty)$. By Exercise 4.53,

$$f(z) := \int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \sum_{n=0}^\infty \frac{B_n}{n!(z+n-1)} := \varphi(z) \tag{14.7}$$

for $z \in (1, \infty)$. The same argument that showed F given by (14.5) is analytic on $\mathbb{C}_1$ may be employed to show f is analytic on $\mathbb{C}_1$, whereas it is straightforward to show that φ is analytic on $\mathbb{C}_1$. Therefore (14.7) holds for all $z \in \mathbb{C}_1$ by the Identity Theorem. However, φ is in fact analytic on $\mathbb{C} \setminus S$ for $S = \{1, 0, -1, -2, \ldots\}$, and so serves as an analytic continuation of f to $\mathbb{C} \setminus S$. Returning to (14.6), it follows that

$$\zeta(z) = \frac{1}{\Gamma(z)} \sum_{n=0}^\infty \frac{B_n}{n!(z+n-1)} + \frac{1}{\Gamma(z)} \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt \tag{14.8}$$

is an analytic continuation of ζ to $\mathbb{C} \setminus S$. That the function at right in (14.8) has isolated singularities at each point in S is clear, and it remains only to show the points in $S \setminus \{1\}$ are removable singularities while there is a simple pole at 1.

We consider first the singularity at 1. Let $A = B'_{1/2}(1) \subseteq S$, and define $g_n : A \to \mathbb{C}$ by

$$g_n(z) = \frac{(z-1)B_n}{n!(z+n-1)}$$

for each $n \geq 0$. For $n \geq 2$, since

$$\begin{aligned} z \in A \quad &\Rightarrow \quad z + n - 1 \in B'_{1/2}(n) \quad \Rightarrow \quad |z + n - 1| > n - \frac{1}{2} \\ &\Rightarrow \quad \frac{1}{|z+n-1|} < \frac{2}{2n-1} < 1, \end{aligned}$$

we have

$$\|g_n\|_A = \sup_{z \in A} \left(\frac{|z-1||B_n|}{n!|z+n-1|} \right) \leq \frac{|B_n|}{2(n!)} \sup_{z \in A} \frac{1}{|z+n-1|} \leq \frac{|B_n|}{n!}.$$

In the remarks before Definition 4.49 it was determined that the series $\sum_{n=0}^\infty (B_n/n!)z^n$ converges on $B_{2\pi}(0)$, and in fact is absolutely convergent on $B_{2\pi}(0)$ by Proposition 4.3. Letting $z = 1$, it follows that $\sum_{n=0}^{\infty} |B_n/n!|$ converges in $\mathbb{R}$, and therefore

$$\sum_{n=0}^{\infty} g_n(z) = \sum_{n=0}^{\infty} \frac{(z-1)B_n}{n!(z+n-1)}$$

converges uniformly on A by the Weierstrass M-Test. Letting $s_n = \sum_{k=0}^{n} g_k$ for $n \geq 0$, it follows that the sequence $(s_n)_{n\in\mathbb{W}}$ converges uniformly on A. Now, 1 is a limit point of A, and

$$\lim_{z\to1} s_n(z) = \lim_{z\to1} \sum_{k=0}^{n} \frac{(z-1)B_k}{k!(z+k-1)} = \lim_{z\to1}\left[1 + \sum_{k=1}^{n} \frac{(z-1)B_k}{k!(z+k-1)}\right] = 1$$

for each $n \geq 0$. By Theorem 2.55,

$$\lim_{z\to1} \sum_{k=0}^{\infty} \frac{(z-1)B_k}{k!(z+k-1)} = \lim_{z\to1}\lim_{n\to\infty} s_n(z) = \lim_{n\to\infty}\lim_{z\to1} s_n(z) = \lim_{n\to\infty}(1) = 1,$$

and therefore

$$\lim_{z\to1} \frac{(z-1)\varphi(z)}{\Gamma(z)} = \lim_{z\to1} \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} \frac{(z-1)B_n}{n!(z+n-1)} = \frac{1}{\Gamma(1)}.$$

Since $1/\Gamma(1) \in \mathbb{C}_*$ by Proposition 14.6, we conclude by Theorem 7.10 that φ/Γ has a simple pole at 1.

Now let $k \in \{0, -1, -2, \ldots\}$. By Proposition 14.6 the function $1/\Gamma$ has a simple zero at k, so there exists some analytic function g such that $(1/\Gamma)(z) = (z-k)g(z)$ for all z near k, with $g(k) \neq 0$. By another application of Theorem 2.55,

$$\begin{aligned}\lim_{z\to k} \frac{\varphi(z)}{\Gamma(z)} &= \lim_{z\to k}(z-k)g(z) \sum_{n=0}^{\infty} \frac{B_n}{n!(z+n-1)} \\ &= \lim_{z\to k} g(z)\left[\frac{B_{1-k}}{(1-k)!} + \sum_{n\neq 1-k} \frac{(z-k)B_n}{n!(z+n-1)}\right] = \frac{B_{1-k}g(k)}{(1-k)!} \in \mathbb{C},\end{aligned}$$

and so φ/Γ has a removable singularity at k by Theorem 7.9. Defining

$$\zeta(k) = \lim_{z\to k} \frac{1}{\Gamma(z)}\left(\varphi(z) + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt\right)$$

for each $k \in \{0, -1, -2, \ldots\}$, we conclude that (14.8) is an analytic continuation of ζ to $\mathbb{C} \setminus \{1\}$, with a simple pole at 1. ■

Exercise 14.18 (SS6.4.2). Prove that

$$\zeta(z) = \frac{z}{z-1} - z\int_1^{\infty} \frac{t - [\![t]\!]}{t^{z+1}}\,dt \tag{14.9}$$

for all $z \in \mathbb{C}_+ \setminus \{1\}$, where $[\![\cdot]\!]$ is the greatest integer function.

Solution. Let $\Omega = \mathbb{C}_+ \setminus \{1\}$. For any $z \in \Omega$ and $t \in [1, \infty)$,

$$\left|\frac{t - [\![t]\!]}{t^{z+1}}\right| = \frac{|t - [\![t]\!]|}{t^{\operatorname{Re} z+1}} \leq \frac{1}{t^{\operatorname{Re} z+1}},$$

and since $\operatorname{Re} z + 1 > 1$ implies that $\int_{-\infty}^{\infty} t^{-(\operatorname{Re} z+1)}\,dt$ converges in $\mathbb{R}$, it follows by Theorem 7.53 that

$$F(z) = \int_1^\infty \frac{t - [\![t]\!]}{t^{z+1}}\,dt$$

converges in $\mathbb{C}$. Thus F is a complex-valued function on Ω.

For each $k \in \mathbb{N}$ define $\psi_k : \Omega \times [k, k+1] \to \mathbb{C}$ by

$$\psi_k(t) = \begin{cases} t - [\![t]\!], & t \in [k, k+1) \\ 1, & t = k+1. \end{cases}$$

Since $t - [\![t]\!]$ is the fractional part of t, we find that ψ_k is continuous on $[k, k+1]$, and hence

$$\varphi_k(z,t) := \frac{\psi_k(t)}{t^{z+1}}$$

is continuous on $\Omega \times [k, k+1]$. Now, for each $n \in \mathbb{N}$ let $G_n : \Omega \to \mathbb{C}$ be given by

$$G_n(z) = \sum_{k=1}^n \int_k^{k+1} \varphi_k(z,t)\,dt.$$

Because $\varphi_k(\cdot, t) : \Omega \to \mathbb{C}$ is analytic for each $t \in [k, k+1]$, Lemma 6.27 implies the function

$$z \mapsto \int_k^{k+1} \varphi_k(z,t)\,dt$$

is analytic on Ω for each k, and hence G_n itself is analytic on Ω. Now, define $F_n : \Omega \to \mathbb{C}$ by

$$F_n(z) = \sum_{k=1}^n \int_k^{k+1} \frac{t - [\![t]\!]}{t^{z+1}}\,dt.$$

In general $\int_a^b f = \int_a^b g$ for functions $f, g \in \mathcal{R}[a,b]$ such that $f(x) = g(x)$ for all but finitely many $x \in [a,b]$, so $F_n = G_n$ for all n, and hence (F_n) is likewise a sequence of analytic functions. With arguments similar to those employed in Example 14.16, we find that (F_n) converges uniformly to F on compact subsets of Ω. By Theorem 4.30, therefore, F is analytic on Ω, and then it is clear that the function at right in (14.9) is analytic on Ω.

It is already known that ζ is analytic on Ω, and so by the Identity Theorem the analysis will be finished if it can be shown that ζ equals the function at right in (14.9) on, say, the interval $(1, \infty)$. Fix $s \in (1, \infty)$. We have

$$\int_1^\infty \frac{1}{t^s}\,dt = \lim_{\tau\to\infty} \int_1^\tau t^{-s}\,dt = \lim_{\tau\to\infty} \left[\frac{1}{1-s} t^{1-s}\right]_1^\tau = \frac{1}{s-1}. \tag{14.10}$$

For each $k \in \mathbb{N}$ set $I_k = \int_k^{k+1} t^{-s}\,dt$. Then

$$\begin{aligned}\int_1^k \frac{1}{t^s}\,dt &= \sum_{n=1}^k \int_n^{n+1} \frac{t}{t^{s+1}}\,dt - I_k = \sum_{n=1}^k \int_n^{n+1} \frac{[\![t]\!] + (t - [\![t]\!])}{t^{s+1}}\,dt - I_k \\ &= \sum_{n=1}^k \int_n^{n+1} \frac{[\![t]\!]}{t^{s+1}}\,dt + \sum_{n=1}^k \int_n^{n+1} \frac{t - [\![t]\!]}{t^{s+1}}\,dt - I_k\end{aligned}$$

$$= \sum_{n=1}^{k} \int_{n}^{n+1} \frac{n}{t^{s+1}}\,dt + \int_{1}^{k+1} \frac{t - [\![t]\!]}{t^{s+1}}\,dt - I_k$$

$$= -\sum_{n=1}^{k} \frac{n}{s}\left[\frac{1}{(n+1)^s} - \frac{1}{n^s}\right] + \int_{1}^{k+1} \frac{t - [\![t]\!]}{t^{s+1}}\,dt - I_k,$$

whereupon Proposition 1.6 with $z_n = n/s$ and $w_n = 1/n^s$ yields

$$\int_{1}^{k} \frac{1}{t^s}\,dt = -\left[\frac{1}{s(k+1)^{s-1}} - \frac{1}{s} - \frac{1}{s}\sum_{n=1}^{k} \frac{1}{(n+1)^s}\right] + \int_{1}^{k+1} \frac{t - [\![t]\!]}{t^{s+1}}\,dt - I_k.$$

Recalling $s > 1$, letting $k \to \infty$ next gives

$$\int_{1}^{\infty} \frac{1}{t^s}\,dt = \frac{1}{s} + \frac{1}{s}\sum_{n=1}^{\infty} \frac{1}{(n+1)^s} + \int_{1}^{\infty} \frac{t - [\![t]\!]}{t^{s+1}}\,dt = \frac{\zeta(s)}{s} + \int_{1}^{\infty} \frac{t - [\![t]\!]}{t^{s+1}}\,dt,$$

noting in particular that $I_k \to 0$ as $k \to \infty$. Putting this into (14.10), we obtain

$$\frac{\zeta(s)}{s} + \int_{1}^{\infty} \frac{t - [\![t]\!]}{t^{s+1}}\,dt = \frac{1}{s-1},$$

which in turn becomes (14.9) with $z = s$. ∎

14.3 – The Zeros and Reciprocal of the Zeta Function

We start with a new expression for the Riemann zeta function $\zeta : \mathbb{C}_1 \to \mathbb{C}$ that connects it to the set $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$ of prime numbers.

Theorem 14.19. *For all $z \in \mathbb{C}_1$,*

$$\zeta(z) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}.$$

The zeros of the zeta function $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is a matter of great interest, with the **Riemann hypothesis** being but the foremost example. The strip

$$[0, 1] \times \mathbb{R} = \{x + iy : x \in [0, 1] \text{ and } y \in \mathbb{R}\}$$

mentioned in the hypothesis is known as the **critical strip**.

Conjecture (**Riemann Hypothesis**). *If $z \in [0, 1] \times \mathbb{R}$ is such that $\zeta(z) = 0$, then* $\mathrm{Re}(z) = \frac{1}{2}$.

In proving the second part of the next lemma we make use of the trigonometric identity

$$3 + 4\cos\theta + \cos 2\theta \geq 0 \tag{14.11}$$

The verification is straightforward:

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + (2\cos^2\theta - 1) = 2(\cos\theta + 1)^2 \geq 0$$

for all $\theta \in \mathbb{R}$.

Lemma 14.20.
1. *If $z \in \mathbb{C}_1$, then there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ such that*

$$\operatorname{Log}\zeta(z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} \frac{\ell^{-mz}}{m} = \sum_{n=1}^{\infty} \frac{c_n}{n^z}.$$

2. *If $s > 1$ and $t \in \mathbb{R}$, then*

$$\operatorname{Log}\left|\zeta^3(s)\zeta^4(s + it)\zeta(s + 2it)\right| \geq 0.$$

Proposition 14.21. *Let $Z(\zeta)$ be the set of zeros of $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$. The following hold.*
1. $Z(\zeta) \setminus ([0, 1] \times \mathbb{R}) = \{-2n : n \in \mathbb{N}\}$.
2. $Z(\zeta) \cap (\{1\} \times \mathbb{R}) = \varnothing$.

14.4 – The Chebyshev Functions

14.5 – The Prime Number Theorem